Math 412 Second Exam

**Part I:** True or False (Total 20 points: 2 points for correct answer, -1 for incorrect).

1). The ring \( \mathbb{Z}_5[x]/(2x^3 + x^2) \) has 5³ elements.
2). Every finite field has \( p \) elements, for some prime integer \( p \).
3). The ring \( \mathbb{R}[x]/(x^2 + 1) \) is isomorphic to the field of complex numbers.
4). If \( p(x) \in F[x] \) divides \( f(x)g(x) \) in \( F[x] \), then \( p(x) \) divides either \( f(x) \) or \( g(x) \) (or both).\(^1\)
5). A polynomial \( f(x) \in \mathbb{Z}_{59}[x] \) is divisible by \( x^2 + 3x + 2 \) if and only if \( f(x) \) is divisible by both \( (x + 1) \) and \( (x + 2) \) in \( \mathbb{Z}_{59}[x] \).
6). There exists a non-constant polynomial \( f(x) \in \mathbb{Z}_5[x] \) such that all of the following four rings are fields: \( \mathbb{Z}_5[x]/(f(x)), \mathbb{Z}_5[x]/(f(x) + 1), \mathbb{Z}_5[x]/(f(x) + 2), \mathbb{Z}_5[x]/(f(x) + 3) \).
7). There is a monic polynomial in \( \mathbb{Q}[x] \) whose factorization into monic irreducibles over \( \mathbb{R}[x] \) has 15 distinct factors but whose factorization into monic irreducibles over \( \mathbb{C}[x] \) has 20 distinct factors.
8). Some non-zero elements of \( \mathbb{Q}(\sqrt{7}) \) do not have multiplicative inverses. (Here, \( \mathbb{Q}(\sqrt{7}) \) denotes the subring of \( \mathbb{R} \) consisting of elements of the form \( r + s\sqrt{7} \) where \( r \) and \( s \) are rational numbers).
9). The map \( \mathbb{Z}[x] \to \mathbb{Z} \) sending each polynomial to its constant term is a ring homomorphism.
10). The map \( \mathbb{Z}[x] \to \mathbb{Z} \) sending each polynomial to the coefficient of its highest degree term (and 0 to 0) is a ring homomorphism.

\(^1\)Here, and everywhere throughout this exam, \( F \) denotes a field.
Part II. Find examples or explain why none exist. (20 points)

a). A ring $R$ and two polynomials $f(x)$ and $g(x)$ in $R[x]$ such that the degree of $f(x)g(x)$ is strictly less than the degree of $f$ plus the degree of $g$.

b). Polynomials $g(x)$ and $f(x)$ in $\mathbb{Q}[x]$ such that the class $[f(x)]$ is a zero-divisor in $\mathbb{Q}[x]/(g(x))$.

c). A polynomial $f(x) \in \mathbb{Z}_{23}[x]$, and an element $a \in \mathbb{Z}_{23}[x]/(f(x))$ such that $a^3 = 0$ but $a^2 \neq 0$. 
Part III. Calculate (with justifications): (20 points)

a). Factorizations of \((x^2 - 5)(x^3 - x^2 + 5x - 5)\) in \(\mathbb{Q}[x]\), in \(\mathbb{R}[x]\), and in \(\mathbb{C}[x]\) into irreducible polynomials.

b). A factorization of the polynomial \(x^{11} - x \in \mathbb{Z}_{11}[x]\) completely into irreducible factors.

c). All solutions to the equation \(aT = 3\) in the ring \(\mathbb{Z}_5[x]/(x^3 + 3x + 3)\), where \(a = [2x + 1]\).
Part IV. Prove or disprove: (20 points)

a). Prove or disprove: The rings $\mathbb{Z}_2[x]/(x^2)$ and $\mathbb{Z}_2[x]/(x^2 + x)$ are isomorphic.

b). Prove or disprove: If $f(x)$ and $g(x)$ are relatively prime polynomials in $\mathbb{Q}[x]$, then $[f(x)]$ has a multiplicative inverse in $\mathbb{Q}[x]/(g(x))$. 
Part V. (20 points) This problem will help you prove the following

Theorem: There is a field with $p^n$ elements, or every prime $p$ and every positive integer $n$.

a). How many monic polynomials of degree 2 are there in $F[x]$?

b). How many of the polynomials you counted in part a) are reducible.

c). Prove that there is an irreducible polynomial of degree two in $F[x]$.

d). Using induction, prove the theorem stated above.
**Bonus** (Can also be turned in before noon tomorrow, for slightly less points.)

Let $p$ be a prime integer and let $g(x)$ be an irreducible polynomial in $\mathbb{Z}_p[x]$. Show that for every $[f(x)] \in \mathbb{Z}_p[x]/(g(x))$, there is an integer $k > 1$ such that $[f(x)]^k = [f(x)]$ in $\mathbb{Z}_p[x]/(g(x))$. (Hint: Can the set $[f(x)], [f(x)]^2, [f(x)]^3, \ldots$ be infinite?)