Math 289 – Winter 2007 – Problem Set 4

Due February 7, 2007

1.(10) The Albanian mathematical genius Le Chiffre has three secret codes (positive integers) which he uses to access his malicious accounts. James Bond kindly asked him to reveal the codes but Le Chiffre surprisingly denied his request. Later on, under some light pressure, Le Chiffre agreed to give James Bond one piece of information – a single positive integer. Is there any way James Bond can determine all three secret codes this way? (Is there any way to encode three positive integers of unknown length in to a single positive integer and decode them back?)

2.(10) 8 points are chosen on a circle. In how many ways can you join pairs of points by nonintersecting chords?
Extra credit (5) Generalize your formula for $2n$ points on a circle.

4.(15) Ten Math majors visit a Chinese restaurant and sit around a round table. During a false fire alarm they all have to leave the restaurant and leave their backpacks behind. When they return, all the backpacks are where they left them (just underneath each student's chair). To enhance the flow of a table discussion they want to change their seats a little but everybody wants to sit close to his/her own backpack (either the same seat as before or one of the two seats next). Certainly, they can't move the backpacks, they are not Physics majors! It does not take too long and a question: "In how many ways can we rearrange the seating?" arises. Find the answer! What about n Math majors?

4.(15) Let S be a set of positive integers such that for any $x, y \in S$

$$x > y \implies x - y \geq \frac{xy}{25}\,.$$

Find the maximal possible number of elements of the set S.

5.(20) Let $f \in C^1[a,b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}$, $\lambda > 0$, is such that

$$|f'(x)| \leq \lambda |f(x)| \qquad \text{for all } x \in [a,b].$$

Is it true that $f(x) = 0$ for all $x \in [a,b]$?

6.(25) Prove that every function of the form

$$f(x) = \frac{a_0}{2} + \cos x + \sum_{n=2}^{N} a_n \cos(nx)$$

with $|a_0| < 1$, has positive as well as negative values in the period $[0, 2\pi)$. Also, prove that the function

$$F(x) = \sum_{n=1}^{100} \cos\left(n^{\frac{3}{2}} x\right)$$

has at least 40 zeros in the interval $(0, 1000)$.