Math 156 Applied Honors Calculus II Review Sheet for Final Exam Fall 2011

You may use these integrals: $\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq -1)$, $\int \frac{dx}{x} = \ln x$, $\int e^x dx = e^x$, $\int \sin x\, dx = -\cos x$, $\int \cos x\, dx = \sin x$, $\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta)$, $\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))$, but all others should be derived.

1. **True or false?** Justify your answer with a reason or counterexample.

a) $\sum_{i=1}^{n^2} i = \sum_{i=1}^{n} i^2$

b) If $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f'(x_i)\Delta x = f(b) - f(a)$.

c) If the integral $\int_a^b f(x)\, dx$ is approximated by the right-hand Riemann sum and the number of intervals n is doubled, then the error in the approximation decreases by a factor of $\frac{1}{4}$.

d) If $f(0) = f(1) = g(0) = g(1) = 0$, then $\int_0^1 f(x)g''(x)dx = \int_0^1 f''(x)g(x)dx$.

e) $\int_0^\infty \frac{dx}{x^2}$ is a convergent improper integral.

f) A spring has natural length 10 cm. If 2 Joule of work is needed to stretch it from length 10 cm to 15 cm, then 4 Joules of work is needed to stretch it from length 10 cm to 20 cm.

g) The center of mass of the region $\{(x, y) : -1 \le x \le 1, 0 \le y \le \cosh x\}$ is $(\overline{x}, \overline{y}) = (0, \frac{1}{2})$.

h) If $f(x)$ is the pdf of a random variable with mean μ, then $f(x)$ has its maximum value at $x = \mu$.

i) If $f(x)$ is the pdf of a random variable with mean μ, then $\int_{-\infty}^{\infty} (x - \mu)f(x)dx = 0$.

j) A radioactive material has a half-life of 100 years. If a given sample has mass 1 kg, then there will be 0.25 kg remaining after 400 years.

k) If \$1000 is invested at 5% interest compounded continuously, then after 2 years the investment is worth more than \$1105.

l) If $y(t)$ is the solution of the differential equation $y' = 1 - y^2$ with initial condition $y(0) = 1$, then $\lim_{t\to 0} y(t) = 1$.

m) If a differential equation $y' = f(y)$ has a constant solution $y_1(t) = c$, and $y_2(t)$ is another solution with initial condition $y_2(0)$ sufficiently close to c, then $\lim_{t\to\infty} y_2(t) = c$.

n) If a differential equation $y' = f(y)$ is solved by Euler's method, and the step size h decreases by a factor of $\frac{1}{2}$, then the error in the numerical solution increases by a factor of $\frac{1}{2}$.

o) If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty} a_n b_n = 0$.

p) If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

q) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 0$

r) The ratio test can be used to show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

s) If the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = 1$, then it also converges for $x = -1$.

t) If the power series $\sum_{n=0}^{\infty} c_n (x-1)^n$ converges for $x = 2$, then it also converges for $x = \frac{1}{2}$.

u) If $f(x) = e^{-x^2}$, then $f^{(3)}(0) = 0$ and $f^{(6)}(0) = -6$. v) $\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

w) $2 < e < 3$ x) $\int_0^1 e^{-x^2} dx > \frac{2}{3}$ y) $1 = \frac{\pi}{2} - \frac{1}{3!}(\frac{\pi}{2})^3 + \frac{1}{5!}(\frac{\pi}{2})^5 - \frac{1}{7!}(\frac{\pi}{2})^7 + \cdots$

z) $\cosh^2 x - \sinh^2 x = 1$ aa) $\int \tanh x\, dx = \text{sech}^2 x$

bb) If $T_1(x)$ is the first degree Taylor polynomial of a function $f(x)$ at a point $x = a$, then $f(x)$ and $T_1(x)$ have the same slope at $x = a$.

cc) $0.895 \le e^{-0.1} \le 0.905$ dd) $\sqrt{1+x^2} = 1 + x^2 + \cdots$ ee) $\cosh ix = \cos x$

ff) $\log(-1) = \pi i$ gg) $\binom{6}{3} = 2$ hh) $\binom{10}{2} = \binom{10}{8}$ ii) $\sum_{n=0}^{k} \binom{k}{n}(-1)^n = 0$

2. Evaluate the limit.

a) $1 + \frac{2011}{2012} + (\frac{2011}{2012})^2 + (\frac{2011}{2012})^3 + \cdots$ b) $\lim_{n\to\infty} \sum_{i=1}^{n} (1 + \frac{i}{n}) \cdot \frac{1}{n}$ c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+\frac{i}{n}} \cdot \frac{1}{n}$

d) $\lim_{x\to 0} \frac{\sin x}{x}$ e) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ f) $\lim_{n\to\infty} (1 + \frac{x}{n})^{2n}$ g) $\lim\limits_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ h) $\lim\limits_{h\to 0} \frac{(x+h)^4 - x^4}{h}$

i) $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ j) $\lim\limits_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ k) $\lim\limits_{h\to 0} \frac{1}{h} \int_0^h f(x)\,dx$ l) $\lim\limits_{h\to 0} \frac{1}{h^2} \int_0^h x f(x)\,dx$

3. Assume $nh = 1$. Prove: a) $\lim_{h\to 0} (1+h)^n = e$, b) $\lim_{h\to 0} \frac{e-(1+h)^n}{h} = \frac{e}{2}$. This answers a question in the notes about Euler's method applied to the differential equation $y' = y, y_0 = 1$.

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integration

4. Find the antiderivative.

a) $\int e^{-x} dx$ b) $\int xe^{-x} dx$ c) $\int e^{-x^2} dx$ d) $\int xe^{-x^2} dx$ g) $\int x \sin x\, dx$ h) $\int e^{-x} \sin x\, dx$

i) $\int \frac{dx}{4x^2}$ j) $\int \frac{x}{4+x^2}\, dx$ k) $\int \frac{dx}{4+x^2}$ l) $\int \frac{dx}{\sqrt{4+x^2}}$ m) $\int \frac{dx}{4-x^2}$ n) $\int \frac{dx}{4x-x^2}$ o) $\int \frac{dx}{\sqrt{4x-x^2}}$

p) $\int \sin^2 x\, dx$ q) $\int \sin^3 x\, dx$ r) $\int \sin^4 x\, dx$

5. Show that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$. (hint: substitute $u = \frac{\pi}{2} - x$)

6. Determine whether the integral converges or diverges. If it converges, find the value.

a) $\int_1^\infty \frac{dx}{x^2}$ b) $\int_1^\infty \frac{dx}{x}$ c) $\int_1^\infty \frac{dx}{x-1}$ d) $\int_0^1 \frac{dx}{x^2}$ e) $\int_0^1 \frac{dx}{\sqrt{x}}$ f) $\int_{-1}^1 \frac{dx}{x}$

7. A metal sphere of radius a has electric charge $q > 0$. Let r be the distance from the center of the sphere to a point in space. It is known from electromagnetic theory that the induced electric potential is $V(r) = \frac{q}{2a} \int_{-a}^{a} \frac{dx}{(r^2 - 2rx + a^2)^{1/2}}$.

a) Evaluate $V(r)$. Consider two cases: $0 \le r \le a$ and $r > a$.

b) Sketch the graph of $V(r)$ for $r \ge 0$.

8. An aquarium full of water is 2 m long, 0.5 m wide, and 1 m high. Find the work done in pumping the water out the top of the aquarium. If the width of the aquarium is doubled, is the work also doubled? If the height is doubled, is the work also doubled?

9. Two identical ions repel each other with force $F = -\frac{q^2}{r^2}$, where q is the ion charge and r is the distance between them. The negative sign indicates a repulsive force, and the magnitude increases as r decreases. (a) An ion is held fixed at $x = 0$. Find the work done in moving another ion from $x = 3$ to $x = 2$. (b) An ion is held fixed at $x = 1$. Find the work done in moving another ion from $x = 3$ to $x = 2$. (c) Two ions are held fixed at $x = 0$ and $x = 1$. Find the work done in moving a third ion from $x = 3$ to $x = 2$. (d) A metal rod of uniform charge density is held fixed on the interval $0 \le x \le 1$. The total charge on the rod is q. Find the work done in moving an ion from $x = 3$ to $x = 2$.

10. A cable hanging between two poles has the shape $y = \cosh x$, $-1 \le x \le 1$. a) Find the arclength of the cable. b) Find the surface area obtained by rotating the cable about the x-axis.

11. Sketch the region in the xy-plane and find the center of mass.

a) $\{(x,y) : 0 \le y \le x^2\ ,\ 0 \le x \le 2\}$ b) $\{(x,y) : x^2 \le y \le 4\ ,\ 0 \le x \le 2\}$

c) $\{(x,y) : -1 \le x \le 1,\ 0 \le y \le 1-x^2\}$ d) $\{(x,y) : 0 \le y \le \frac{1}{1+x^2}\ ,\ 0 \le x < \infty\}$

12. The lifetime of a light bulb is described by an exponential distribution with mean 1000 hours. Find the probability that the lightbulb: a) fails in the first 200 hours, b) lasts more than 800 hours.

13. Let $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ for $0 < x < 1$ and zero otherwise. Sketch the graph and show that $f(x)$ is a valid pdf.

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differential equations

14. Find the solution of the differential equation with initial condition $y(0) = y_0$. Sketch the solution for $t \ge 0$. Find $\lim_{t\to\infty} y(t)$.

a) $y' = -2y\ ,\ y_0 = 1$ b) $y' = 1 - 2y\ ,\ y_0 = 0$ c) $y' = 1 - y^2\ ,\ y_0 = 0$ d) $y' = -ty\ ,\ y_0 = 1$

15. Consider the differential equation $y'' = y$.

a) Show that $y(t) = c_1 e^t + c_2 e^{-t}$ is a solution, where c_1, c_2 are arbitrary constants.

b) Find the solution $y(t)$ subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

c) Repeat part (b) for initial conditions $y(0) = 0$, $y'(0) = 1$.

16. The cell count in a bacteria culture grows at a rate proportional to its size. After 30 minutes there are 200 cells and after 90 minutes there are 800 cells. (The answers below should be expressed as integers.)

a) Find the initial cell count. b) When will the cell count reach 6400?

17. Polonium-214 has a half-life of 1.4×10^{-4} s. If a sample has initial mass 40 mg, how long will it take for the mass to decay to 30 mg?

18. A tiger consumes 2500 calories per day and expends 20 calories per kg of its mass per day in daily activity. Assume that 1 kg of the tiger's mass is equivalent to 10,000 calories. Formulate a differential equation for the mass of the tiger as a function of time. Solve the equation and sketch the graph for $t \ge 0$. What value does the tiger's mass approach as time increases?

19. A thermometer at room temperature 70°F is placed in a patient's mouth. After one minute the thermometer reads 95°F and after two minutes it reads 100°F. Find the patient's temperature.

20. A common model for an epidemic assumes that the rate of spread of infection is proportional to the product of the number of people currently infected and the number of people not yet infected. In a town with 4000 inhabitants, if 10 people are infected at the beginning of the week and 20 people are infected at the end of the week, how long does it take for half the population to be infected?

21. Consider solving the differential equation $y' = 2y$ with initial condition $y_0 = 1$ by Euler's method. Let $t_n = nh = 1$. Make a table with the following entries; column 1: h (time step, take $h = 1, \frac{1}{2}, \frac{1}{4}$); column 2: u_n (numerical solution at time t_n given by Euler's method)

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series

22. Determine whether the series converges or diverges. Justify your answer.

a) $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ b) $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ c) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ d) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ e) $\sum\limits_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

23. Express the repeating decimal as a rational number (i.e. a ratio of two integers).

a) $0.11111111\ldots$ b) $0.1212121212\ldots$ c) $0.4999999999\ldots$

24. Find the sum of the series. a) $\sum\limits_{n=0}^{\infty} \frac{2^n}{n!}$ b) $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ c) $\sum\limits_{n=1}^{\infty} \frac{n}{3^n}$

25. It is known that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Use this to evaluate $\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

26. For each of the following series, find a bound for $|s - s_{10}|$. a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

27. Two students walk towards each other at 2 mi/hr starting from a separation of 20 miles. At the same time, a dog starts running back and forth between the students at 10 mi/hr. Let D be the total distance the dog has traveled when the students finally meet. Express D as an infinite series and find the sum of the series.

28. Winning a game of ping-pong requires a lead of two points, i.e. if the final score is tied, you must score two consecutive points in order to win the game. Suppose your probability of scoring a point is p, where $0 < p < 1$. If the final score is tied, find the probability you will eventually win the game. Evaluate for $p = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$. Interpret.

29. Start with the closed interval $[0, 1]$. Remove the open interval $(\frac{1}{3}, \frac{2}{3})$. That leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Remove the middle third of each of those. That leaves four intervals. Remove the middle third of each of those. Continue the process indefinitely. The Cantor set is the set of all points remaining after all the intervals have been removed.

a) Show that the total length of all the intervals removed is 1.

b) Show that, nonetheless, the Cantor set contains infinitely many numbers.

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power series, Taylor series

30. Find the radius of convergence, interval of convergence, and sum of the power series.

a) $\sum\limits_{n=0}^{\infty} x^n$ b) $\sum\limits_{n=0}^{\infty} \frac{x^n}{2^n}$ c) $\sum\limits_{n=0}^{\infty} (x-1)^n$ d) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ e) $\sum\limits_{n=1}^{\infty} nx^n$

31. Find the power series representation for $f(x) = \frac{1}{1-x}$ about $x = \frac{1}{2}$.

32. Find the Taylor series for $\sinh x$ and $\cosh x$ about $x = 0$.

33. Compute the first three nonzero terms in the power series for $\sin^2 x + \cos^2 x$ about $x = 0$ by squaring the power series for $\sin x$ and $\cos x$ and adding the results.

34. Find the Taylor series of $f(x) = e^{-x^2}$ about $x = 0$. Sketch $f(x)$, $T_0(x)$, $T_1(x)$, $T_2(x)$ in a neighborhood of $x = 0$. Label each curve.

35. Let $f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$

Evaluate the following limits and then sketch the graph of $f(x)$.

a) $\lim\limits_{x\to\infty} f(x)$ b) $\lim\limits_{x\to 0^+} f(x)$ c) $\lim\limits_{x\to 0^+} f'(x)$ d) $\lim\limits_{x\to 0^+} f''(x)$

36. Find an approximate value for $\sqrt{10}$ which is accurate to within 0.005.

37. Use the Taylor series for $f(x) = \ln(1+x)$ about $x = 0$ to evaluate $\ln\frac{3}{2}$ to within 10^{-3}.

38. Find the first two nonzero terms in the Taylor series for $f(x)$ about $x = 0$.

a) $\tan x$ b) $e^{-x}\sin x$ c) $(1-\cos x)/x$

39. The Bernoulli numbers B_n are defined by $\frac{x}{e^x-1} = \sum\limits_{n=0}^{\infty} B_n \frac{x^n}{n!}$. Find B_0, B_1, B_2.

40. Show that the following functions satisfy $f(0) = 0$ and $f'(0) = 1$. Find $f''(0)$ in each case. If the functions are graphed in a neighborhood of $x = 0$, in what order do they appear (from top to bottom)? a) x b) $\sin x$ c) $\ln(1+x)$ d) $e^x - 1$

41. Recall the power series for the Bessel function of order zero, $J_0(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

a) Evaluate $\int\limits_0^1 J_0(x)\,dx$ using 2 terms in the series. Find an upper bound for the error.

b) Show that $J_0(x)$ satisfies the differential equation $xy'' + y' + xy = 0$.

42. Define $f(t) = \sum\limits_{n=0}^{\infty} t^n$. We know that $f(t) = \frac{1}{1-t}$ for $|t| < 1$, but now we'll derive this a different way.

a) Show that $f(t)$ satisfies the differential equation $y' = y^2$ with initial condition $y(0) = 1$.

b) Solve the differential equation for $f(t)$ by separation of variables.

43. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges, but $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges.
(hints: sketch the graph of each integrand, express the first integral as an alternating series)

44. Use the 1st order Taylor approximation for $\cos x$ about $x = 0$ to show that $|\cos\frac{\pi}{5} - 1| \le \frac{1}{2}(\frac{\pi}{5})^2$. Derive a more accurate result using the 3rd degree Taylor approximation.

45. Recall the error function, $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Find the first three terms in the Taylor series of $\mathrm{erf}(x)$ about $x = 0$.

46. a) expand $\dfrac{a}{a+b}$ in powers of $\dfrac{a}{b}$ b) expand $\sqrt{R^2 - r^2}$ in powers of $\dfrac{r}{R}$
In each case find the first three nonzero terms.

47. Show that $f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \cdots$. This is an alternative form of the Taylor series commonly used in numerical analysis.

48. The equation $\dfrac{x^2}{(1+\epsilon)^2} + y^2 = 1$ defines an ellipse in the xy-plane (assume $0 \le \epsilon < 1$).

a) Find the intercepts on the x-axis and y-axis. Sketch the ellipse.

b) Let $A(\epsilon)$ be the area of the ellipse. Express $A(\epsilon)$ as a definite integral.

c) Find the first 2 nonzero terms in the power series expansion of $A(\epsilon)$ about $\epsilon = 0$.

49. The gravitational potential energy function due to a pair of point masses m_1, m_2 located at x_1, x_2 is $V(x) = \frac{Gm_1}{|x-x_1|} + \frac{Gm_2}{|x-x_2|}$, where G is the gravitational constant. For $x \to \infty$, the potential energy function can be approximated by $V(x) \approx \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \cdots$, where $a, b, c, \ldots$ are constants that depend on m_1, m_2 and x_1, x_2. Find the values of a, b, c. Are any of the results familiar? (hint: set $y = 1/x$ and expand $V(x)$ in powers of y.)

50. The Lennard-Jones potential energy function, $V(r) = V_0((\frac{r_0}{r})^{12} - 2(\frac{r_0}{r})^6)$, describes the interaction between two particles (e.g. atoms, molecules), where V_0 and r_0 are positive constants, and $r \geq 0$ is the distance between the particles.

a) Find $\lim_{r\to 0} V(r), \lim_{r\to\infty} V(r)$.

b) Show that $V(r)$ has a minimum at $r = r_0$.

c) Sketch the graph of $V(r)$ for $r \geq 0$.

d) Find $T_2(r)$, the quadratic Taylor approximation for $V(r)$ at $r = r_0$.

e) Find the work done in separating two particles from $r = r_0$ to $r = \infty$ (this corresponds to dissociating a molecule). The force is given by $f(r) = -V'(r)$.

51. Use the 2nd degree Taylor approximation of $\sqrt{1+x^2}$ at $x = 0$ to approximate $\int_0^1 \sqrt{1+x^2}\,dx$. Find an upper bound for the error.

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binomial series

52. Recall the binomial expansion, $(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$, where $k \geq 1$ is an integer.

a) Show that $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$.

b) Explain the connection between the formula in (a) and Pascal's triangle below.

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

c) Fill in the next two rows of the triangle. Use this to expand $(a+b)^6$.

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complex numbers, polar coordinates

53. Express the complex number in Cartesian form $x + iy$ and polar form $re^{i\theta}$. Plot each number in the complex plane. a) $1+i$ b) $(1+i)^2$ c) $(1+i)^3$ d) $\frac{1}{1+i}$ e) $\sqrt{1+i}$

54. Compute $(1+i)^6$ two ways, using: (a) binomial expansion , (b) polar form.

55. Find the roots of the equation. Plot the roots in the complex plane.

a) $z^2 + 2z - 2 = 0$ b) $z^2 + 2z + 2 = 0$ c) $z^2 = 1$ d) $z^3 = 1$ e) $z^4 = 1$ f) $e^z = 1$

56. Show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

57. a) Use integration by parts to find the antiderivatives $\int e^{ax}\cos bx\,dx$, $\int e^{ax}\sin bx\,dx$.

b) Show that $e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx$ and $\int e^{(a+ib)x}dx = \frac{a-ib}{a^2+b^2}e^{(a+ib)x}$.

c) Take the real and imaginary parts in (b) to rederive the formulas obtained in (a).

58. Derive the following results using Euler's formula, $e^{ix} = \cos x + i\sin x$.

a) $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ b) $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

Derive the following formulas using a) and b).

c) $\frac{d}{dx}\cos = -\sin x$ d) $\frac{d}{dx}\sin = \cos x$

e) $\sin^2 x + \cos^2 x = 1$ f) $\sin 2x = 2\sin x\cos x$ g) $\cos 2x = \cos^2 x - \sin^2 x$