Math 156 Applied Honors Calculus II Final Exam Review Sheet Solutions Fall 2018

1. **True or False**

a) TRUE. $\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\cdots+\frac{n}{n}=\sum_{i=1}^{n}\frac{i}{n}=\frac{1}{n}\cdot\sum_{i=1}^{n}i=\frac{1}{n}\cdot\frac{n(n+1)}{2}=\frac{n+1}{2}$

b) TRUE. $\lim_{n\to\infty}\sum_{i=1}^{n}f'(x_i)\Delta x=\int_a^b f'(x)dx=f(b)-f(a)$ by FTC

c) FALSE. If $n$ is doubled, then $\Delta x$ decreases by a factor of $\frac{1}{2}$ and the error in the right-hand Riemann sum decreases by a factor of $\frac{1}{2}$, not $\frac{1}{4}$, as we saw in examples in class.

d) TRUE. Apply integration by parts. TO BE COMPLETED

e) FALSE. Since $\int_0^1\frac{dx}{x^2}$ diverges by $p-$test $\Rightarrow$ $\int_0^\infty\frac{dx}{x^2}$ diverges.

f) FALSE. When the spring is stretched from length 20 cm to 30 cm, the work done is $\int_{20-L}^{30-L}kxdx=\int_{20-20}^{30-20}kxdx=\int_0^{10}kxdx=\frac{1}{2}kx^2\Big|_0^{10}=50k=2\,\text{Joule}$. Then when the spring is stretched from length 30 cm to 40 cm, the work done is $\int_{30-L}^{40-L}kxdx=\int_{30-20}^{40-20}kxdx=\int_{10}^{20}kxdx=\frac{1}{2}kx^2\Big|_{10}^{20}=\frac{1}{2}k(400-100)=150k=3\cdot 50k=6\,\text{Joule}$.

g) FALSE. TO BE COMPLETED

h) FALSE. A counterexample is an exponential distribution, $f(x)$ attains its maximum value at $x=0$ rather than $\mu=\frac{1}{c}$. (However the statement is true for a normal distribution.)

i) TRUE. $\int_{-\infty}^{\infty}(x-\mu)f(x)dx=\int_{-\infty}^{\infty}xf(x)dx-\int_{-\infty}^{\infty}\mu f(x)dx=\mu-\mu\cdot 1=0$

j) FALSE. After 100 years the sample has mass $\frac{1}{2}$kg, and after 400 years the sample has mass $\left(\frac{1}{2}\right)^4=\frac{1}{16}$ kg.

k) TRUE. continuous compounding $\Rightarrow y(t)=y_0e^{rt}=1000e^{0.05t}\Rightarrow y(2)=1000e^{0.05\cdot 2}=1000e^{0.1}$; hence we need to show that $1105<1000e^{0.1}<1112$

<u>1st part</u> : we need to show that $1105<1000e^{0.1}$; recall that $e^x=1+x+\frac{1}{2}x^2+\cdots$, so $e^{0.1}=1+0.1+\frac{1}{2}(0.1)^2+\cdots=1+0.1+0.005+\cdots$; then summing the first three terms yields $e^{0.1}=1.105+\cdots$, and since the remainder is positive, we have $e^{0.1}>1.105$, thus $1000e^{0.1}>1000\cdot 1.105=1105$

<u>2nd part</u> : we need to show that $1000e^{0.1}<1112$; this is equivalent to showing that $e^{0.1}<1.112$; in this case let us consider $e^{-x}$; it is fairly easy to see that $e^{-x}>1-x$ for $x\neq 0$; for example, consider the graph of $e^{-x}$ and $1-x$; then let us set $x=0.1$, so that we have $e^{-0.1}>1-0.1$; this implies that $e^{-0.1}>0.9$; and this implies that $e^{0.1}<\frac{1}{0.9}=\frac{1}{\frac{9}{10}}=\frac{10}{9}=1.1111\cdots<1.112$; so we're done

l) FALSE. $y(t)=0$ is a constant solution of the differential equation, but it is unstable by looking at the phase plane.

m) FALSE. This is only true if the constant solution is stable.

n) FALSE. If the step size $\Delta t$ decreases, then the error also decreases.

o) FALSE. A counterexample is $a_n=\frac{1}{n}$, $b_n=n^2$.

p) FALSE. A counterexample is $a_n=\frac{1}{n^2}, b_n=\frac{1}{n}$, so that $\sum_{n=0}^{\infty}a_n$ converges, but $\sum_{n=0}^{\infty}b_n$ diverges.

q) FALSE.

Method 1: draw a graph of $y=\frac{1}{x^2}$ for $x\geq 1$ and note that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a left-hand Riemann sum for $\int_1^\infty\frac{dx}{x^2}$, and hence the sum is larger than the integral.

Method 2: $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\int_1^\infty\frac{dx}{x^2}=\frac{-1}{x}\Big|_1^\infty=1$

r) FALSE.

Method 1: The error bound for a convergent alternating series says that $|s-s_n|<a_{n+1}$. If we set $n=1$, then $|s-s_1|<a_2\Rightarrow|s-1|<\frac{1}{2}\Rightarrow\frac{1}{2}<s<\frac{3}{2}$, so $s\neq 0$.

Method 2: In class we showed that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2\neq 0$.

s) FALSE.

$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n-1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})=\sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=1}^{\infty}\frac{1}{n+1}=\infty-\infty=0$

The 1st step is correct, but the 2nd step is incorrect; the original sum is equal to 1 (as shown on homework), so it is incorrect to write $1=\infty-\infty$.

t) FALSE. The AST cannot be used to show that a series diverges.

u) FALSE. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{n^2}{(n+1)^2}=1\Rightarrow$ the ratio test is inconclusive

v) FALSE. A counterexample is $\sum_{n=1}(-1)^nx^n$, which converges for $x=1$ by the AST, but diverges for $x=-1$ since it is the harmonic series in that case.

w1) TRUE. The radius of convergence is at least $R = 1$, so the interval of convergence is at least $0 < x \leq 2$, which contains $x = \frac{1}{2}$.

w2) FALSE. geometric series, $r = 2(x-1)$, converges for $-1 < r < 1 \Leftrightarrow -1 < 2(x-1) < 1 \Leftrightarrow -\frac{1}{2} < x-1 < \frac{1}{2} \Leftrightarrow \frac{1}{2} < x < \frac{3}{2}$, diverges at end points $x = \frac{1}{2}, \frac{3}{2}$

x) TRUE. $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$, then differentiating both sides yields $-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1} \quad \Rightarrow \quad \frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty}(-1)^n (n+1) x^n$, and we can check that the ioc is $-1 < x < 1$

y) FALSE. Don't try to find $f^{(3)}(0), f^{(6)}(0)$ directly. Instead use the Taylor series to derive them, since the general form of a Taylor series is $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{f^{(n)}(0)}{n!} \Rightarrow f^{(n)}(0) = n! \cdot c_n$. Since $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, then $e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots$. Then $c_3 = 0$, since there is no $x^3$ term, and hence $f^{(3)}(0) = 0$. Then $c_6 = -\frac{1}{6}$, thus $f^{(6)}(0) = 6! \cdot c_6 = 720 \cdot \left(-\frac{1}{6}\right) = -120$. Thus the statement is false.

z) TRUE.
part 1 : Since $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, then setting $x = 1$, we have $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$, and it is clear that $e > 2$.
part 2 : Setting $x = -1$ in the power series for $e^x$, we have $e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \cdots$, then using the error bound for a convergent alternating series we have $|e^{-1} - \frac{1}{2}| < \frac{1}{6} \Rightarrow \frac{1}{3} < e^{-1} < \frac{2}{3}$, then $e^{-1} > \frac{1}{3} \Rightarrow e < 3$.

aa) TRUE. $e^x = 1 + x + \frac{1}{2}x^2 + \cdots \Rightarrow e^{-x^2} = 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \cdots = 1 - x^2 + \frac{1}{2}x^4 - \cdots$ Next we note that $e^{-x^2} > 1 - x^2$ for $x > 0$; to show this we can sketch the graph of $e^{-x^2}$ and $1 - x^2$. To confirm the sketch, consider the function $f(x) = e^{-x^2} - (1 - x^2)$, note that $f(0) = 0$ and $f'(x) = -2xe^{-x^2} + 2x = 2x(1 - e^{-x^2}) > 0$ for $x > 0$, so $f(x)$ is an increasing function, and this implies $f(x) > 0$ for $x > 0$, and hence we have $e^{-x^2} > 1 - x^2$ for $x > 0$. Then we have $\int_0^1 e^{-x^2} dx > \int_0^1 (1 - x^2) dx = (x - \frac{x^3}{3})\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$.

bb) TRUE. Since the series is $\sin\frac{\pi}{2}$ (note that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$) and $\sin\frac{\pi}{2} = 1$.

cc) TRUE. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1$.

dd) FALSE. $\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{\cosh x} d\cosh x = \ln(\cosh x) \neq \operatorname{sech}^2 h$. Note that $(\tanh x)' = \operatorname{sech}^2 x$.

ee) TRUE. Note that $T_1(x) = f(a) + f'(a)(x - a)$, thus $T_1'(a) = f'(a)$.

ff) FALSE. In class we showed that the function defined by $f(x) = e^{-x^2}$ for $x \neq 0$ and $f(0) = 0$ has the property that $f^{(n)}(0) = 0$ for all $n$, yet $f(x) \neq 0$ for $x \neq 0$.

gg) TRUE. $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$, so $e^{-0.1} = 1 + (-0.1) + \frac{1}{2}(-0.1)^2 + \cdots = 1 - 0.1 + 0.005 - \cdots$, so $|e^{-0.1} - 0.9| \leq 0.005$, so $0.895 \leq e^{-0.1} \leq 0.905$.

hh) FALSE. Using the binomial series $(1 + x)^k = 1 + kx + \cdots$, replace $x$ with $x^2$, then $(1 + x^2)^k = 1 + kx^2 + \cdots$, and then setting $k = \frac{1}{2}$, we have $\sqrt{1 + x^2} = (1 + x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \cdots$.

ii) TRUE. Since $\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i\,\sin x + \cos x - i\,\sin x}{2} = \cos x$

jj) TRUE.
part 1 : $\int_0^{\pi/2} \sin^2\theta d\theta = \int_0^{\pi/2} (\frac{1}{2} - \frac{1}{2}\cos 2\theta) d\theta = (\frac{1}{2}\theta - \frac{1}{2} \cdot \frac{1}{2}\sin 2\theta)\Big|_0^{\pi/2} = \frac{\pi}{4}$
part 2 : $\int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2} (\frac{1}{2} + \frac{1}{2}\cos 2\theta) d\theta = (\frac{1}{2}\theta + \frac{1}{2} \cdot \frac{1}{2}\sin 2\theta)\Big|_0^{\pi/2} = \frac{\pi}{4}$

kk) TRUE. Since $e^{\pi i} = \cos\pi + i\,\sin\pi = -1 \quad \Rightarrow \quad \pi i = \log(-1)$. (Actually, more rigorously, $\log(-1) = (2k+1)\pi i$, where $k$ is an integer.)

ll) TRUE. $\binom{8}{3} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$

mm) FALSE. $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

nn) TRUE. $\binom{10}{2} = \frac{10!}{2! \cdot 8!} = \frac{10!}{8! \cdot 2!} = \binom{10}{8}$

oo) TRUE. $\binom{7}{3} + \binom{7}{4} = \frac{7!}{3! \cdot 4!} + \frac{7!}{4! \cdot 3!} = \frac{2 \cdot 7!}{3! \cdot 4!} = \frac{8 \cdot 7!}{4 \cdot 3! \cdot 4!} = \frac{8!}{4! \cdot 4!} = \binom{8}{4}$

pp) TRUE. $(a + b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$, then setting $a = 1, b = -1$ yields $\sum_{n=0}^{k} \binom{k}{n}(-1)^n = (1 + (-1))^k = 0^k = 0$

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Question 2 Solution

a) geometric series : sum $= \frac{1}{1 - \frac{2016}{2017}} = \frac{2017}{2017 - 2016} = 2017$

b) It equals $\int_0^1 (1 + x) dx = \left(x + \frac{x^2}{2}\right)\Big|_0^1 = \frac{3}{2}$ (change $\frac{i}{n} \to x, \ \frac{1}{n} \to dx$)

c) It equals $\int_0^1 \frac{1}{1+x} dx = \ln(1 + x)|_0^1 = \ln 2$

d) By L'Hospital's rule, $\lim\limits_{x\to 0}\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{(\sin x)'}{(x)'} = \lim\limits_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$.

e) By L'Hospital's rule, $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} = \lim\limits_{x\to 0}\frac{(1-\cos x)'}{(x^2)'} = \lim\limits_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}$.

f) Note that $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^{2n} = (\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^{n})^2 = (e^x)^2 = e^{2x}$

g) By L'Hospital's rule $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim\limits_{x\to 0}\frac{(\sqrt{1+x}-1)'}{x'} = \lim\limits_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

h) $\lim\limits_{h\to 0}\frac{(x+h)^4-x^4}{h} = \lim\limits_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h} = \lim\limits_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim\limits_{h\to 0}(4x^3+6x^2h+4xh^2+h^3) = 4x^3$ (one can use L'hospital rule, note that '$h$' is variable here, regard $x$ as constant)

i) By L'Hospital's rule, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f'(x+h)}{1} = f'(x)$.

j) Using L'Hospital's rule twice, $\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = \lim_{t\to 0}\frac{f''(x+h)+f''(x-h)}{2} = f''(x)$.

k) $\lim\limits_{h\to 0}\frac{\int_0^h f(x)dx}{h} \xrightarrow{\text{L'Hospital Rule}} \lim\limits_{h\to 0}\frac{(\int_0^h f(x)dx)'}{(h)'} = \lim\limits_{h\to 0}\frac{f(h)}{1} = f(0)$

l) $\lim\limits_{h\to 0}\frac{\int_0^h xf(x)dx}{h^2} \xrightarrow{\text{L'Hospital Rule}} \lim\limits_{h\to 0}\frac{(\int_0^h xf(x)dx)'}{(h^2)'} = \lim\limits_{h\to 0}\frac{hf(h)}{2h} = \lim\limits_{h\to 0}\frac{f(h)}{2} = \frac{f(0)}{2}$

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**integration**

Question 3 Solution

a) TO BE COMPLETED

b) TO BE COMPLETED

c) $e^x = \sum\limits_{n=0}^{\infty}\frac{1}{n!}x^n \Rightarrow e^{-x^2} = \sum\limits_{n=0}^{\infty}\frac{1}{n!}(-x^2)^n = \sum\limits_{n=0}^{\infty}\frac{1}{n!}(-1)^n x^{2n} \Rightarrow \int e^{-x^2}dx = \int\sum\limits_{n=0}^{\infty}\frac{1}{n!}(-1)^n x^{2n}dx = \sum\limits_{n=0}^{\infty}\frac{1}{n!}(-1)^n\frac{1}{2n+1}x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots$.

d) TO BE COMPLETED

e) integration by parts : $\int x\sin x dx = \int x(-1)d\cos x = -\int xd\cos x = -x\cos x + \int\cos x dx = -x\cos x + \sin x = \sin x - x\cos x$

f) use integration by parts twice

once : $\int e^{-x}\sin x dx = \int e^{-x}(-1)d\cos x = -e^{-x}\cos x + \int\cos x de^{-x} = -e^{-x}\cos x - \int e^{-x}\cos x dx$

twice : $\int e^{-x}\cos x dx = \int e^{-x}d\sin x = e^{-x}\sin x - \int\sin x de^{-x} = e^{-x}\sin x + \int e^{-x}\sin x dx$

$\Rightarrow \int e^{-x}\sin x dx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x dx \quad\Rightarrow\quad 2\int e^{-x}\sin x = -e^{-x}\cos x - e^{-x}\sin x$

$\Rightarrow \int e^{-x}\sin x = \frac{1}{2}\left(-e^{-x}\cos x - e^{-x}\sin x\right) = -\frac{1}{2}e^{-x}(\cos x + \sin x)$

g) $\int\frac{dx}{4x^2} = -\frac{1}{4x}$

h) $\int\frac{x}{4+x^2}dx = \int\frac{\frac{1}{2}}{4+x^2}dx^2 = \int\frac{\frac{1}{2}}{4+x^2}d(4+x^2) = \frac{1}{2}\ln(4+x^2)$

If you do not like the above way, instead use a change of variable $u = 4+x^2$, $du = 2xdx \quad\Rightarrow\quad dx = \frac{1}{2x}du$.

$\int\frac{x}{4+x^2}dx = \int\frac{x}{u}\cdot\frac{1}{2x}du = \int\frac{1}{2u}du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(4+x^2)$

i) For this type of problem, one needs to use a trig substituion, $x = 2\tan\theta$, $dx = 2(1+\tan^2\theta)d\theta$

$\int\frac{dx}{4+x^2} = \int\frac{2(1+\tan^2\theta)}{4+4\tan^2\theta}d\theta = \int\frac{1}{2}d\theta = \frac{1}{2}\theta = \frac{1}{2}\arctan\frac{x}{2}$

j) One can use the formula $1+\sinh^2\theta = \cosh^2\theta$, and the change of variable $x = 2\sinh\theta$, $dx = 2\cosh\theta d\theta$.

$\int\frac{dx}{\sqrt{4+x^2}} = \int\frac{2\cosh\theta}{\sqrt{4+4\sinh^2\theta}}d\theta = \int\frac{2\cosh\theta}{\sqrt{4\cosh^2\theta}}d\theta = \int\frac{2\cosh\theta}{2\cosh\theta}d\theta = \int d\theta = \theta = \text{arcsinh}\frac{x}{2} = \sinh^{-1}\frac{x}{2}$

k) Using partial fractions

$\int\frac{dx}{4-x^2} = \int\frac{dx}{(2+x)(2-x)} = \int\left(\frac{1/4}{2+x} + \frac{1/4}{2-x}\right)dx = \int\frac{1/4}{2+x}dx + \int\frac{1/4}{2-x}dx = \frac{1}{4}\ln|2+x| - \frac{1}{4}\ln|2-x|$

l) partial fractions again

$\int\frac{dx}{4x-x^2} = \int\frac{dx}{x(4-x)} = \int\left(\frac{1/4}{x} + \frac{1/4}{4-x}\right)dx = \int\frac{1/4}{x}dx + \int\frac{1/4}{4-x}dx = \frac{1}{4}\ln|x| - \frac{1}{4}\ln|4-x|$

m) TO BE COMPLETED

n) using integration by parts once and using the equality $\sin^2 x + \cos^2 x = 1$

$\int\sin^2 x dx = \int\sin x\cdot\sin x dx = \int\sin x(-1)d\cos x = -\sin x\cdot\cos x + \int\cos x d\sin x = -\sin x\cdot\cos x + \int\cos^2 x dx = -\sin x\cdot\cos x + \int(1-\sin^2 x)dx = -\sin x\cdot\cos x + \int 1dx - \int\sin^2 x dx \quad\Rightarrow\quad 2\int\sin^2 x dx = -\sin x\cdot\cos x + x \Rightarrow \int\sin^2 x dx = -\frac{1}{2}\sin x\cos x + \frac{x}{2} = \frac{x}{2} - \frac{1}{4}\sin 2x$

A simple way, using $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2\cos x\sin x$, $(\cos 2x)^2 = (1+\cos 4x)/2$

thus $\int\sin^2 x dx = \int\frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{\sin 2x}{4}$

o) using the equality $\sin^2 x + \cos^2 x = 1$

$\int\sin^3 x dx = \int\sin^2 x\cdot\sin x dx = \int(1-\cos^2 x)(-1)d\cos x = -\int(1-\cos^2 x)d\cos x = -\int 1d\cos x + \int\cos^2 x d\cos x = -\cos x + \frac{1}{3}\cos^3 x$

p) using integration by parts will be too complicated, using $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2\cos x \sin x$, $(\cos 2x)^2 = (1 + \cos 4x)/2$

Thus $\int \sin^4 x = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1-2\cos 2x+\cos^2 2x}{4} dx = \int \frac{1-2\cos 2x+1-\sin^2 2x}{4} dx$
$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x - \frac{1}{4}\sin^2 2x\right) dx = \frac{x}{2} - \frac{1}{4}\sin 2x - \frac{1}{8}\int \sin^2 2x d(2x)$
$\xrightarrow{using\ 3\ i) x\to 2x} \frac{x}{2} - \frac{1}{4}\sin 2x - \frac{1}{8}\left(\frac{2x}{2} - \frac{\sin 4x}{4}\right) = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

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Question 4 Solution

a) $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\, dx$ TO BE COMPLETED

b) $\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx$ TO BE COMPLETED

c) $\int_{-\infty}^{\infty} (x-1)^2 \frac{1}{2\sqrt{2\pi}} e^{\frac{-(x-1)^2}{8}}\, dx$ TO BE COMPLETED

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Question 5 Solution

Using variable substitution $u = \frac{\pi}{2} - x$, $du = -dx$

$\int_{\pi/2}^{0} \frac{\sin(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}(-1)du = -\int_{\pi/2}^{0} \frac{\cos u}{\cos u + \sin u} du = \int_0^{\pi/2} \frac{\cos u}{\cos u + \sin u} du$

both $u$ and $x$ are integral variables changing from 0 to $\frac{1}{2}$, one may replace them with $\theta$, thus $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta} d\theta = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta} d\theta$ on one hand, one the other hand $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta} d\theta + \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta} d\theta = \int_0^{\pi/2} \left(\frac{\sin\theta}{\sin\theta+\cos\theta} + \frac{\cos\theta}{\sin\theta+\cos\theta}\right) d\theta = \int_0^{\pi/2} 1 d\theta = \frac{\pi}{2}$.

Thus $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta} d\theta = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta} d\theta = \frac{\pi}{4}$

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Question 6 Solution

a) convergent by $p-$test $\quad \int_1^{\infty} \frac{1}{x^2} dx = \int_0^{\infty} \left(-\frac{1}{x}\right)' dx = -\frac{1}{x}\Big|_1^{\infty} = 1$

b) divergent by $p-$test $\quad \int_1^{\infty} \frac{dx}{x} = \ln x|_1^{\infty} = \infty$

c) $\int_1^{\infty} \frac{dx}{x-1} = \int_0^{\infty} \frac{dy}{y}$ divergent by $p-$test

d) divergent by $p$-test $\quad \int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^1 = \infty$

e) convergent by $p$-test $\quad \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}|_0^1 = 2$

f) divergent by $p$-test $\quad$ Since both $\int_{-1}^{0} \frac{dx}{x}$ and $\int_0^1 \frac{dx}{x}$ diverges.

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Question 7 Solution

substitute $u = r^2 - 2rx + a^2$, so $du = -2rdx$

limits $x = -a \Rightarrow u = r^2 - 2r(-a) + a^2 = r^2 + 2ra + a^2 = (r+a)^2$, $x = a \Rightarrow u = r^2 - 2ra + a^2 = (r-a)^2$

$$V(r) = \frac{q}{2a}\int_{-a}^{a} \frac{dx}{\sqrt{r^2-2rx+a^2}} = \frac{q}{2a}\int_{(r+a)^2}^{(r-a)^2} \frac{-\frac{1}{2r}du}{\sqrt{u}} = -\frac{q}{4ar}\cdot 2\sqrt{u}\Big|_{(r+a)^2}^{(r-a)^2} = -\frac{q}{2ar}\left(|r-a| - |r+a|\right) = \begin{cases} \frac{q}{r} & \text{if } r \geq a \\ \frac{q}{a} & \text{if } 0 \leq r \leq a \end{cases}$$

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Question 8a Solution

Assume the aquarium has length $l$, width $w$, and height $h$, let $x$ be the vertical coordinate with $x = 0$ at the bottom of the aquarium (positive $x$ is up), divide the water into layers of width $\Delta x = \frac{h}{n}$, each layer is a rectangular box with volume $lw\Delta x$, the force on a layer is $\rho g l w \Delta x$, the layer at level $x_i = i\Delta x$ is raised a distance $h - x_i$, after letting $n \to \infty, \Delta x \to 0$, the work done is $W = \int_0^h \rho g l w(h-x)dx = \rho g l w(hx - \frac{x^2}{2})\big|_0^h = \frac{1}{2}\rho g l w h^2$, substituting the given values, the work done is $W = \frac{1}{2}\rho g \cdot 2 \cdot 0.5 \cdot 1^2 = \frac{1}{2}\rho g$ Joule. From the formula $W = \frac{1}{2} g l w h^2$, we see that if the width $w$ is doubled, then the work is also doubled, and if the height $h$ is doubled, then the work is multiplied by 4.

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Question 8b Solution TO BE COMPLETED

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Question 9 Solution

a) On the $x$-axis, one ion is held fixed at $x = 0$, so the distance between the ions is $x$, so replace $r$ in $F = -\frac{q^2}{4\pi\epsilon_0 r^2}$ by $x$.

Work $= \int_a^b F(x)dx = \int_3^2 -\frac{q^2}{4\pi\epsilon_0 x^2} dx = \frac{q^2}{4\pi\epsilon_0}\ \frac{1}{x}\Big|_3^2 = \frac{q^2}{4\pi\epsilon_0}(\frac{1}{2} - \frac{1}{3}) = \frac{q^2}{24\pi\epsilon_0}$

b) On the $x$-axis, one ion is held fixed at $x = 1$, so the distance between the ions is $x - 1$, so replace $r$ in $F = -\frac{q^2}{4\pi\epsilon_0 r^2}$ by $x - 1$.

Work $= \int_a^b F(x)dx = \int_3^2 -\frac{q^2}{4\pi\epsilon_0 (x-1)^2} dx = \frac{q^2}{4\pi\epsilon_0}\ \frac{1}{x-1}\Big|_3^2 = \frac{q^2}{4\pi\epsilon_0}(\frac{1}{1} - \frac{1}{2}) = \frac{q^2}{8\pi\epsilon_0}$

c) The work can be calculated separately for each ion then added together. Work $= \frac{q^2}{24\pi\epsilon_0} + \frac{q^2}{8\pi\epsilon_0} = \frac{q^2}{6\pi\epsilon_0}$.

d) Let $w$ be a coordinate along the rod, so $0 \leq w \leq 1$. Divide the rod into small pieces of width $\Delta w$. The charge in each piece is $q\Delta w$, and the force on the piece at coordinate $w$ is $F(x) = -\frac{q \cdot q\Delta w}{4\pi\epsilon_0 (x-w)^2}$. The work contributed by that piece is

Work $= \int_3^2 F(x)dx = \int_3^2 -\frac{q^2\Delta w}{4\pi\epsilon_0(x-w)^2}dx = \left.\frac{q^2\Delta w}{4\pi\epsilon_0(x-w)}\right|_3^2 = \frac{q^2\Delta w}{4\pi\epsilon_0}\left(\frac{1}{2-w}-\frac{1}{3-w}\right)$.

We need a second integral for $w$ from 0 to 1 to sum the work due all the pieces, and in this process we have $\Delta w \to dw$.

Total Work $= \frac{q^2}{4\pi\epsilon_0}\int_0^1(\frac{1}{2-w}-\frac{1}{3-w})dw = \frac{q^2}{4\pi\epsilon_0}\left(-\ln(2-w)+\ln(3-w)\right)|_0^1 = \frac{q^2}{4\pi\epsilon_0}(-\ln 1+\ln 2-(-\ln 2+\ln 3)) = \frac{q^2}{4\pi\epsilon_0}\ln\frac{4}{3}$

Note that Work (a) $= 0.04167\frac{q^2}{\pi\epsilon_0}$, Work (b) $= 0.1667\frac{q^2}{\pi\epsilon_0}$, Work (d) $= 0.07192\frac{q^2}{\pi\epsilon_0}$, so Work (a) $<$ Work (d) $<$ Work (b). This is reasonable because case (a) and case (b) are two extreme cases (all the charge is at one end of the rod or the other), and the total charge is the same in all three cases.

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Question 10 Solution

$f(x) = \cosh x \Rightarrow f'(x) = \sinh x \Rightarrow 1+(f'(x))^2 = 1+\sinh^2 x = \cosh^2 x \Rightarrow \sqrt{1+(f'(x))^2} = \cosh x$

a) arclength $= \int_{-1}^1 \sqrt{1+(f'(x))^2}dx = \int_{-1}^1 \cosh x dx = \sinh x|_{-1}^1 = \sinh 1 - \sinh(-1) = 2\sinh 1$

b) surface area $= \int_{-1}^1 2\pi f(x)\sqrt{1+(f'(x))^2}dx = 2\pi\int_{-1}^1\cosh^2 x dx = 2\pi\int_{-1}^1\left(\frac{e^x+e^{-x}}{2}\right)^2 dx = 2\pi\int_{-1}^1\left(\frac{e^{2x}+2+e^{-2x}}{4}\right)dx$

$= \frac{2\pi}{4}\left(\frac{e^{2x}}{2}+2x+\frac{e^{-2x}}{-2}\right)\Big|_{-1}^1 = \frac{\pi}{2}\left(\frac{e^2}{2}+2+\frac{e^{-2}}{-2}-\left(\frac{e^{-2}}{2}-2+\frac{e^2}{-2}\right)\right) = \frac{\pi}{2}(e^2+4-e^{-2}) = \frac{\pi}{2}(2\sinh 2+4) = \pi(\sinh 2+2)$

You can save some time using the fact that $\cosh x$ is an even function, so $\int_{-1}^1\cdots = 2\int_0^1\cdots$.

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Question 11 Solution

$\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$

case 1: $m = \int_a^b f(x)dx$, $M_x = \frac{1}{2}\int_a^b f^2(x)dx$, $M_y = \int_a^b xf(x)dx$

case 2: $m = \int_a^b(f(x)-g(x))dx$, $M_x = \frac{1}{2}\int_a^b(f^2(x)-g^2(x))dx$, $M_y = \int_a^b x(f(x)-g(x))dx$

answers : $(\bar{x},\bar{y}) =$ (a) $\left(\frac{3}{2},\frac{6}{5}\right)$, (b) $\left(\frac{3}{4},\frac{12}{5}\right)$, (c) $\left(0,\frac{2}{5}\right)$, (d) $\left(\infty,\frac{1}{4}\right)$

(b) $m = \int_0^2(4-x^2)dx = (4x-\frac{x^3}{3})\big|_0^2 = 8-\frac{8}{3} = \frac{16}{3}$

$M_x = \frac{1}{2}\int_0^2(16-x^4)dx = \frac{1}{2}(16x-\frac{x^5}{5})\big|_0^2 = \frac{1}{2}(32-\frac{32}{5}) = \frac{64}{5} \Rightarrow \bar{y} = \frac{M_x}{m} = \frac{64}{5}\cdot\frac{3}{16} = \frac{12}{5}$

$M_y = \int_0^2(4x-x^3)dx = (4\frac{x^2}{2}-\frac{x^4}{4})\big|_0^2 = 8-\frac{16}{4} = 4 \Rightarrow \bar{x} = \frac{M_y}{m} = 4\cdot\frac{3}{16} = \frac{3}{4}$

(d) $m = \int_0^\infty \frac{1}{1+x^2}dx$ $\left(= \arctan x\big|_0^1 = \frac{\pi}{2}\right)$

change variable: $x = \tan\theta$, $dx = (1+\tan^2\theta)d\theta$, $x = 0 \Rightarrow \theta = 0, x = \infty \Rightarrow \theta = \frac{\pi}{2}$, since $\tan 0 = 0, \tan\frac{\pi}{2} = \infty$

$m = \int_0^\infty\frac{1}{1+x^2}dx = \int_0^{\frac{\pi}{2}}\frac{1+\tan^2\theta}{1+\tan^2\theta}d\theta = \int_0^{\frac{\pi}{2}}1d\theta = \arctan x\big|_0^1 = \frac{\pi}{2}$

$\bar{x} = \frac{\int_0^\infty xf(x)dx}{m} = \frac{\int_0^\infty\frac{x}{1+x^2}dx}{m} = \frac{\int_0^\infty\frac{\frac{1}{2}dx^2}{1+x^2}}{\frac{\pi}{2}} = \frac{\frac{1}{2}\ln(1+x^2)\big|_0^\infty}{\frac{\pi}{2}} = \infty$ !!!

The area is finite, but it has an infinitely large $\bar{x}$.

$\bar{y} = \frac{\int_0^\infty\frac{1}{2}f^2(x)dx}{m} = \frac{1}{m}\cdot\int_0^\infty\frac{1}{2}f^2(x)dx = \frac{1}{\frac{\pi}{2}}\int_0^\infty\frac{1}{2}\left(\frac{1}{1+x^2}\right)^2dx = \frac{2}{\pi}\int_0^\infty\frac{\frac{1}{2}}{(1+x^2)^2}dx = \frac{2}{\pi}\int_0^\infty\frac{\frac{1}{2}+\frac{1}{2}x^2-\frac{1}{2}x^2}{(1+x^2)^2}dx$

$= \frac{2}{\pi}\left(\int_0^\infty\frac{\frac{1}{2}(1+x^2)}{(1+x^2)^2}dx - \int_0^\infty\frac{\frac{1}{2}x^2}{(1+x^2)^2}dx\right) = \frac{2}{\pi}\left(\int_0^\infty\frac{\frac{1}{2}}{1+x^2}dx - \int_0^\infty\frac{\frac{1}{2}x\cdot x}{(1+x)^2}dx\right)$

$= \frac{2}{\pi}\left(\frac{1}{2}\arctan x\big|_0^\infty - \int_0^\infty\frac{\frac{1}{2}x\cdot\frac{1}{2}dx^2}{(1+x^2)^2}\right) = \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2}+\int_0^\infty\frac{x}{4}d\left(\frac{1}{1+x^2}\right)\right] = \frac{2}{\pi}\left(\frac{\pi}{4}+\frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\int_0^\infty\frac{1}{1+x^2}dx\right)$

$= \frac{2}{\pi}\left(\frac{\pi}{4}+\frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\int_0^\infty\frac{1}{1+x^2}dx\right) = \frac{2}{\pi}\left(\frac{\pi}{4}+\frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\arctan x\big|_0^\infty\right)$

$= \frac{2}{\pi}\left(\frac{\pi}{4}+0-\frac{1}{4}\cdot\frac{\pi}{2}\right) = \frac{2}{\pi}\cdot\frac{\pi}{8} = \frac{1}{4}$

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Question 12 Solution

$f(t) = ce^{-ct}$, where $c = \frac{1}{1000}$ and $t \geq 0$

a) Prob$(0 \leq t \leq 200) = \int_0^{200}ce^{-ct}dt = -e^{-ct}|_0^{200} = 1-e^{-\frac{1}{5}} \approx 0.18$

b) Prob$(t \geq 800) = \int_{800}^\infty ce^{-ct}dt = -e^{-ct}|_{800}^\infty = e^{-\frac{4}{5}} \approx 0.45$

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Question 13 Solution

First notice that $f(x) \geq 0$ for all $x$. Now we need only show that $\int_0^1 f(x)dx = 1$

$\int_0^1 f(x)dx = \int_0^1\frac{1}{\pi\sqrt{x(1-x)}}dx \xrightarrow{x=u+\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\pi\sqrt{(u+\frac{1}{2})(1-u-\frac{1}{2})}}du = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\pi\sqrt{(\frac{1}{2}+u)(\frac{1}{2}-u)}}du$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\pi\sqrt{\frac{1}{4}-u^2}}du \xrightarrow{u=\frac{1}{2}\sin\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\pi\cdot\frac{1}{2}\cos\theta}\cdot\frac{1}{2}\cos\theta d\theta = \frac{1}{\pi}\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1$

An alternative approach to the integral is to make the substitution $w = \sqrt{x}, x = w^2, dw = dx/(2\sqrt{x})$. This gives $\int_0^1 f(x)dx = \int_0^1\frac{1}{\pi\sqrt{x(1-x)}}dx = \int_0^1\frac{2dw}{\pi\sqrt{1-w^2}} = \frac{2}{\pi}\arcsin w]_0^1 = \frac{2}{\pi}(\pi/2-0) = 1$. The alternative is more efficient but requires some ingenuity to think of.

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**Differential Equations**

Question 14 Solution

a) $y' = -2y$, $y_0 = 1$ (standard exponential decay) $\Rightarrow$ $y = C \cdot e^{-2t}$.

$y_0 = 1 \Rightarrow C = 1 \Rightarrow y(t) = e^{-2t}$ and $\lim_{t\to\infty} y(t) = 0$

b) $y' = 1-2y \Rightarrow \frac{dy}{dt} = 1-2y$ separation of variables $\Rightarrow \frac{dy}{1-2y} = dt$ integrate both sides $\Rightarrow -\frac{1}{2}\ln|1-2y| = t+C$ $\Rightarrow 1-2y = C \cdot e^{-2t} \Rightarrow y = \frac{1-C\cdot e^{-2t}}{2}$

$y_0 = 0 \Rightarrow C = 1 \Rightarrow y = \frac{1-e^{-2t}}{2}$ and $\lim_{t\to\infty} y(t) = \frac{1}{2}$

c) $y' = 1-y^2 \Rightarrow \frac{dy}{dt} = (1+y)(1-y)$ separation of variables $\Rightarrow \frac{dy}{(1+y)(1-y)} = dt$ partial fraction $\Rightarrow \frac{\frac{1}{2}}{1+y}dy + \frac{\frac{1}{2}}{1-y}dy = dt$ integrate both sides $\Rightarrow \frac{1}{2}\ln|1+y| - \frac{1}{2}\ln|1-y| = t+C \Rightarrow \ln\left|\frac{1+y}{1-y}\right| = 2t+C \Rightarrow \frac{1+y}{1-y} = C\cdot e^{2t}$ where $C$ is constant may be positive or negative $\Rightarrow y(t) = \frac{C\cdot e^{2t}-1}{C\cdot e^{2t}+1}$

$y_0 = 0 \Rightarrow C = 1 \Rightarrow y(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Furthermore $y(t) = \frac{(e^{2t}-1)e^{-t}}{(e^{2t}+1)e^{-t}} = \frac{e^t-e^{-t}}{e^t+e^{-t}} = \tanh t$

Check... $y(t) = \tanh t$ is the solution.

$\lim_{t\to\infty} y(t) = 1$

d) $y' = -ty \Rightarrow \frac{dy}{dt} = -ty$ separation of variables $\Rightarrow \frac{dy}{y} = -tdt$ integrate both sides $\Rightarrow \ln|y| = -\frac{1}{2}t^2 + C \Rightarrow$ $y = Ce^{-\frac{1}{2}t^2}$

$y_0 = 1 \Rightarrow C = 1 \Rightarrow y(t) = e^{-\frac{1}{2}t^2}$

$\lim_{t\to\infty} y(t) = 0$

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Question 15 Solution

a) $y = c_1e^t + c_2e^{-t} \Rightarrow y' = c_1e^t - c_2e^{-t} \Rightarrow y'' = c_1e^t + c_2e^{-t} = y$, thus it is a solution of $y'' = y$ for any constants $c_1$, $c_2$.

b) $y(0) = 1, y'(0) = 0 \Rightarrow c_1 + c_2 = 1,\ c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2} \Rightarrow y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh x$

c) $y(0) = 0, y'(0) = 1 \Rightarrow c_1 + c_2 = 0,\ c_1 - c_2 = 1 \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2} \Rightarrow y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh x$

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Question 16 Solution a) $y' = ky$. Solve the equation, we have $y(t) = y_0e^{kt}$. $200 = y(30) = y_0e^{30k}$ and $800 = y(90) = y_0e^{90k}$. Therefore, $y_0 = 10^2 = 100$ cells.

b)$200 = 100e^{30k}$, so $k = \frac{\ln 2}{30}$. Therefore, $y(t) = 100 \cdot 2^{t/30}$. Solve the equation $6400 = 100 \cdot 2^{t/30}$. Then $t = 30\ln 64/\ln 2 = 30 \cdot 6 = 180$ hours.

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Question 17 Solution

$y(t) = y_0e^{-kt}$

$y(t) = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4\times10^{-4}}}$

$30 = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4\times10^{-4}}}$

$t = 1.4\times10^{-4}\frac{\ln\frac{3}{4}}{\ln\frac{1}{2}} = 0.581\times10^{-4}\text{s}$

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Question 18 Solution

$y(t)$ : tiger body mass (kg) as a function of time $t$ (day)

$y' = \text{rate in} - \text{rate out} = 2500\frac{\text{cal}}{\text{day}} \cdot \frac{1\,\text{kg}}{10000\,\text{cal}} - 20\,\text{cal} \cdot \frac{y\,\text{kg}}{10000\,\text{cal}} \cdot \frac{1}{\text{day}} = \frac{2500-20y}{10000}\frac{\text{kg}}{\text{day}}$

$y' = -\frac{1}{500}(y-125)$ Newton's heating/cooling $y' = k(y-T)$

$y(t) = T + (y_0 - T)e^{kt} = 125 + (y_0 - 125)e^{-\frac{t}{500}} \Rightarrow \lim_{t\to\infty} y(t) = 125$ kg

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Question 19 Solution

$y(t) = T + (y_0 - T)e^{-kt}$ , note that the patient's temperature is $T$ , $y_0 = 70°\text{F}$

$y(1) = 95 = T + (70-T)e^{-k}$

$y(2) = 100 = T + (70-T)e^{-2k}$

$\Rightarrow (\frac{95-T}{70-T})^2 = \frac{100-T}{70-T} \Rightarrow (95-T)^2 = (100-T)(70-T) \Rightarrow T^2 - 190T + 95^2 = T^2 - 170T + 70000 \Rightarrow 20T = 2025$

$\Rightarrow T = 101.25°$ F

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Question 20 Solution

$y' = ky(M-y)$

$y(t) = \frac{My_0}{y_0+(M-y_0)e^{-kMt}}$

$y_0 = 10,\ M = 4000$

$y(t) = \frac{40000}{10+(4000-10)e^{-4000kt}}$

measure time in days

$20 = \frac{40000}{10+(4000-10)e^{-4000\cdot7k}}$

$e^{-k} = \left(\frac{199}{399}\right)^{\frac{1}{28000}}$
$y(t) = \frac{40000}{10+3990\left(\frac{199}{399}\right)^{\frac{t}{7}}}$
let $y(t) = \frac{1}{2} \cdot 4000 = 2000$, solve $t = \frac{7\ln 399}{\ln 399 - \ln 199} \approx 60$ days.

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Question 21 Solution

This equation, $y' = f(y)$ with $f(y) = 2y$, is particularly simple to study with Euler's Method. The recursion for Euler's Method in this case is $u_{k+1} = u_k + (\Delta t)f(u_k) = (\Delta t)(2u_k)$. After factoring, $u_{k+1} = (1+2\Delta t)(u_k)$. This means that, in this case, we don't actually need to do each iteration of the process; we can foresee that $u_k = (1+2\Delta t)^k u_0 = (1+2\Delta t)^k$. So $u_n = (1+2/n)^n$, where $n = 1/\Delta t$ is the number of steps.

| $\Delta t$ | $u_n$ |
|---|---|
| 1 | $(1+2)^1 = 3$ |
| 1/2 | $(1+1)^2 = 4$ |
| 1/4 | $(1+1/2)^4 = 5.0625$ |

*Remark.* The general solution of this differential equation is $y = Ce^{2t}$, and the solution which fits our initial condition is $y = e^{2t}$. Thus the true answer is $y(1) = e^2 \approx 7.389$. We need to use smaller steps to get a satisfactory approximation. With 1000 steps, we would still only have $u_n \approx 7.374$.

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**Series**

Question 22 Solution

a) divergent $\sum\limits_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum\limits_{n=1}^{\infty} \frac{1}{n}$ by $p$-test of series, $p = 1$.

b) convergent since $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2}\sum\limits_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}} = 1 < \infty$.

c) convergent by $p$-test of series, $p = 2$.

d) convergent by Alternating Series Test, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^2} = 0$, $a_{n+1} < a_n$ and the sign is alternating.

e) divergent by Ratio Test, $L = \lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n} = 2 > 1$, ($L > 1$ divergent)

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Question 23 Solution

a) $0.111111.... = 0.1 + 0.01 + 0.001 + 0.0001 + \cdots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{1000} + \cdots = \sum\limits_{n=1}^{\infty}\frac{1}{10^n} = \frac{1}{10}\sum\limits_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = \frac{1}{10}\cdot\frac{1}{1-\frac{1}{10}} = \frac{1}{9}$

b) $0.1212121212... = \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \cdots = \sum\limits_{n=1}^{\infty}\frac{12}{100^n} = \frac{12}{100}\sum\limits_{n=0}^{\infty}\frac{1}{100^n} = \frac{12}{100}\cdot\frac{1}{1-\frac{1}{100}} = \frac{12}{99}$

c) $0.4999999... = 0.45 + 0.045 + 0.0045 + 0.00045 + \cdots = \frac{45}{100} + \frac{45}{1000} + \frac{45}{10000} + \cdots$
$= \frac{45}{100}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right) = \frac{45}{100}\sum\limits_{n=0}^{\infty}\frac{1}{10^n} = \frac{45}{100}\cdot\frac{1}{1-\frac{1}{10}} = \frac{1}{2}$ (ie, 0.49999999... = 0.5)

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Question 24 Solution

a) recall : $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!} = e^x$ for all $x$, so setting $x = 2$ we obtain $\sum\limits_{n=0}^{\infty}\frac{2^n}{n!} = e^2$

b) recall : $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1 < x < 1$, so setting $x = \frac{1}{3}$ we obtain $\sum\limits_{n=0}^{\infty}\frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \Rightarrow \sum\limits_{n=1}^{\infty}\frac{1}{3^n} = \sum\limits_{n=0}^{\infty}\frac{1}{3^n} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

c) differentiating the first equation in part (b) with respect to $x$ we obtain $\sum\limits_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ for $-1 < x < 1$, so setting $x = \frac{1}{3}$ we obtain $\sum\limits_{n=1}^{\infty} n(\frac{1}{3})^{n-1} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4} \Rightarrow \sum\limits_{n=1}^{\infty} n(\frac{1}{3})^n\cdot(\frac{1}{3})^{-1} = \frac{9}{4} \Rightarrow \sum\limits_{n=1}^{\infty} n(\frac{1}{3})^n = \frac{9}{4}\cdot\frac{1}{3} = \frac{3}{4}$

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Question 25 Solution

Given that $\sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, note that $\sum\limits_{n=0}^{\infty}\frac{1}{(2n+1)^2}$ includes all the odd terms. $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ =odd terms+even terms $\Rightarrow \sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \sum\limits_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \sum\limits_{n=1}^{\infty}\frac{1}{(2n)^2} \Rightarrow \sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \sum\limits_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \frac{1}{4}\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$
$\Rightarrow \frac{\pi^2}{6} = \sum\limits_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \frac{1}{4}\frac{\pi^2}{6} \Rightarrow \sum\limits_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}$.

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Question 26 Solution

a) Use $|s - s_{10}| \le \int_n^{\infty} f(x)dx$ since all terms are positive, where $f(x) = \frac{1}{x^2}$
$|s - s_{10}| \le \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty}\frac{1}{x^2}dx = -\frac{1}{x}\Big|_{10}^{\infty} = 0.1$

b) Use $|s - s_{10}| \le a_{n+1}$ where $a_{n+1} = \frac{1}{(n+1)^2}$ since the series is an alternating series.
$|s - s_{10}| \le a_{n+1} = \frac{1}{11^2} = \frac{1}{121}$

Question 27 Solution

Assume the dog starts with student A and runs to student B; if the time interval has duration $t_0$, then we have $10t_0 + 2t_0 = 20$ (the distance the dog runs plus the distance student B walks is equal to the distance between them when they start).
This implies that $12t_0 = 20$ or $t_0 = \frac{20}{12} = \frac{5}{3}$.
Note that the at the end of the first interval, the distance between the students is $20 - 4t_0 = 20 - 4 \cdot \frac{5}{3} = 20 - \frac{20}{3} = \frac{40}{3} = 20 \cdot \frac{2}{3}$; in other words the distance between the students decreased by a factor of $\frac{2}{3}$.
In the next time interval of duration $t_1$, the dog runs back to A; then we have $10t_1 + 2t_1 = \frac{40}{3}$ (the distance the dog runs plus the distance A walks is equal to the distance between the students when they start).
This implies that $12t_1 = \frac{40}{3}$ or $t_1 = \frac{40}{12 \cdot 3} = \frac{10}{9} = \frac{5}{3} \cdot \frac{2}{3}$.
Note that at the end of the second interval, the distance between the students is $\frac{40}{3} - 4 \cdot t_1 = \frac{40}{3} - 4 \cdot \frac{10}{9} = \frac{80}{9} = \frac{40}{3} \cdot \frac{2}{3}$.
In the next time interval of duration $t_2$, the dog runs back to B; then we have $10t_2 + 2t_2 = \frac{80}{9}$ (the distance the dog runs plus the distance B walks is equal to the distance between them when they start).
This implies that $12t_2 = \frac{80}{9}$ or $t_2 = \frac{80}{12 \cdot 9} = \frac{20}{27} = \frac{5}{3} \cdot (\frac{2}{3})^2$.
The pattern repeats.
The total time is $T = t_0 + t_1 + t_2 + \cdots = \frac{5}{3} + \frac{5}{3} \cdot \frac{2}{3} + \frac{5}{3} \cdot \left(\frac{2}{3}\right)^2 + \cdots = \frac{5}{3} \cdot \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = \frac{5}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 5$.
Hence the dog travels a distance $D = 10$ mph $\times$ 5 hours $= 50$ miles.
The question asks us to express D as an infinite series, but we could also find the distance directly as follows. The students meet in the middle after walking a distance of 10 miles; since they walk at 2 mph, this must have taken 5 hours. This is also how long the dog runs, so the dog must have run a total distance of 10 mph $\times$ 5 hours $= 50$ miles.

Question 28 Solution

Consider the remaining points as happening in pairs. In any pair of points, there are 3 possible outcomes; (a) you win both, with probability $p^2$, (b) you lose both, with probability $(1-p)^2$, or (c) you win one and lose one, with probability $2p(1-p)$. In the first case, the game ends and you win. In the second case, the game ends and you lose. In the third case, the game continues. So any scenario in which you win has the following form; you split pairs of points with your opponent a certain number of times and then you win a pair. Thus the probability of winning is $p^2 + (2p(1-p))p^2 + (2p(1-p))^2 p^2 + \cdots = \sum_{n=0}^{\infty} [2p(1-p)]^n p^2 = p^2 \sum_{n=0}^{\infty} [2p(1-p)]^n = p^2 \cdot \frac{1}{1-2p(1-p)} = \frac{p^2}{1-2p+2p^2}$.

$p = \frac{1}{2} \Rightarrow \frac{p^2}{1-2p+2p^2} = \frac{1}{2}$ (no surprise)

$p = \frac{1}{4} \Rightarrow \frac{p^2}{1-2p+2p^2} = \frac{1}{10}$

$p = \frac{3}{4} \Rightarrow \frac{p^2}{1-2p+2p^2} = \frac{9}{10}$ (no surprise that this answer and the previous would sum to 1)

Question 29 Solution

a) The total length removed $= \frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} + \cdots = \frac{1}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right] = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1$

b) Notice that any point which is the endpoint of a remaining interval will never be removed. Every stage doubles the number of remaining intervals, so this accounts for infinitely many points.

**Power Series, Taylor Series**

Question 30 Solution

a) $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \Rightarrow$ the radius of convergence is 1; since at two end points $x = \pm 1$, the series diverges, the interval of convergence is $-1 < x < 1$. The sum is $\frac{1}{1-x}$ for $-1 < x < 1$.

b) $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^n x^{n+1}}{2^{n+1} x^n}\right| = \left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2 \Rightarrow$ the radius of convergence is 2; since at two end points $x = \pm 2$, the series diverges, the interval of convergence is $-2 < x < 2$. The sum is $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$ for $-2 < x < 2$.

c) $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{(x-1)^n}\right| = |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2 \Rightarrow$ the radius of convergence is 1 (the length of the interval divided by 2); since at two end points $x = 0$ and 2, the series diverges, the interval of convergence is $0 < x < 2$. The sum is $\frac{1}{1-(x-1)} = \frac{1}{2-x}$ for $0 < x < 2$.

d) $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{nx^{n+1}}{(n+1)x^n}\right| = |x| < 1 \Rightarrow -1 < x < 1 \Rightarrow$ the radius of convergence is 1 ; since at $x = 1$, the series is harmonic series thus diverges, while at $x = -1$, the series converges by AST (alternating series test), the interval of convergence is $-1 \le x < 1$. Note that $x^n = \int nx^{n-1}dx \Rightarrow \frac{x^n}{n} = \int x^{n-1}dx$ the sum is $\sum_{n=1}^{\infty} \int x^{n-1}dx = \int \sum_{n=1}^{\infty} x^{n-1}dx = \int \sum_{n=0}^{\infty} x^n dx = \int \frac{1}{1-x}dx = -\ln(1-x) = \ln \frac{1}{1-x}$ for $-1 \le x < 1$.

$\ln \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}$

e) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right| = |x| < 1 \Rightarrow -1 < x < 1 \Rightarrow$ the radius of convergence is 1; since at $x = \pm 1$, the series diverges, the interval of convergence is $-1 < x < 1$.

Note that $(x^n)' = nx^{n-1}$, $\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\sum_{n=1}^{\infty}(x^n)' = x\cdot\left(\sum_{n=0}^{\infty} x^n\right)' = x\cdot\left(\frac{1}{1-x}\right)' = x\cdot\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$

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Question 31 Solution

Namely find $c_n$, such that $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} c_n(x-\frac{1}{2})^n$.

$\frac{1}{1-x} = \frac{1}{1-(x-\frac{1}{2}+\frac{1}{2})} = \frac{1}{1-(x-\frac{1}{2})-\frac{1}{2}} = \frac{1}{\frac{1}{2}-(x-\frac{1}{2})} = \frac{1}{2}\cdot\frac{1}{1-2(x-\frac{1}{2})} = \frac{1}{2}\cdot\sum_{n=0}^{\infty}(2(x-\frac{1}{2}))^n = \sum_{n=0}^{\infty} 2^{n-1}(x-\frac{1}{2})^n$

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Question 32 Solution

$f(x) = \sinh x \Rightarrow f(0) = \sinh 0 = 0$
$f'(x) = \cosh x \Rightarrow f'(0) = \cosh 0 = 1$
$f''(x) = \sinh x \Rightarrow f''(0) = \sinh 0 = 0$
$f'''(x) = \cosh x \Rightarrow f'''(0) = \cosh 0 = 1$
$\vdots$

$\sinh x = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$

$\cosh x = (\sinh x)' = \left(\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\right)' = \sum_{n=0}^{\infty}\left(\frac{x^{2n+1}}{(2n+1)!}\right)' = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

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Question 33 Solution

a) $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ , $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sin^2 x + \cos^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2$

$= \left(x^2 - 2\frac{x^4}{3!} + \frac{x^6}{36} + 2\frac{x^6}{5!} + \cdots\right) + \left(1 - 2\frac{x^2}{2} + \frac{x^4}{4} + 2\frac{x^4}{4!} - 2\frac{x^6}{2\cdot 4!} - 2\frac{x^6}{6!} + \cdots\right) = \cdots = 1 + O(x^8)$

b) In fact we know that all terms in the power series for $\sin^2 x + \cos^2 x$ vanish after the first term 1, though proving it is a nice fairly involved exercise.

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Question 34 Solution

Because $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, $e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots$

$T_1(x) = T_0(x) = 1$ and $T_2(x) = 1 - x^2$. See Figure below for sketch.

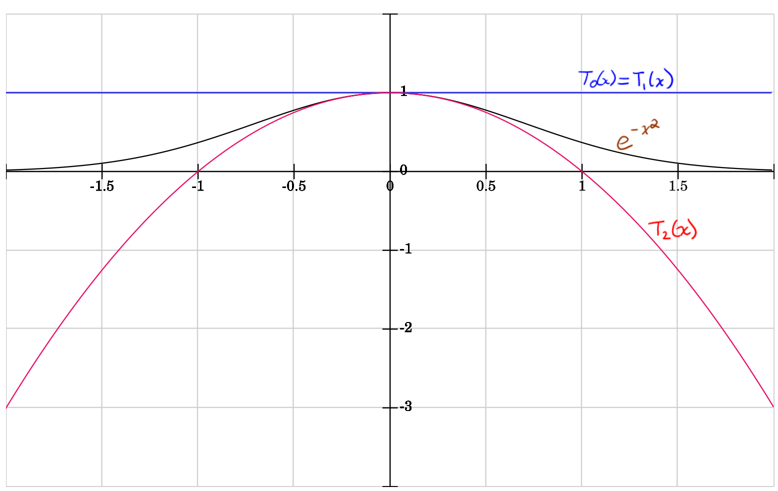


Figure 1: Graph for Problem 34.

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Question 35 Solution

Show that $0 \le f(x) < 1$; $\lim_{x\to\infty} f(x) = 1$; $\lim_{x\to 0^+} f^{(n)}(x) = \lim_{x\to 0^+} P(\frac{1}{x})e^{-1/x}$, where $P(\frac{1}{x})$ is a polynomial of $\frac{1}{x}$; when $x \to 0^+$ $e^{-1/x} \to 0$ exponentially (faster than any polynomial) thus $f^{(n)}(x) \to 0$ regardless of the form of $P(\frac{1}{x})$.

Question 36 Solution

method 1 It can be shown using Taylor series for $f(x)=\sqrt{x}$ about $x=a$, that $\sqrt{x}=\sqrt{a}+\frac{1}{2\sqrt{a}}(x-a)-\frac{1}{8}a^{-\frac{3}{2}}(x-a)^2+\cdots$.
Setting $a=9$ yields $\sqrt{x}=3+\frac{1}{6}(x-9)-\frac{1}{216}(x-9)^2+\cdots$.
This is a convergent alternating series (why?), so $|s-s_n|<a_{n+1}$, i.e. $\left|\sqrt{x}-\left(3+\frac{1}{6}(x-9)\right)\right|<\frac{1}{216}(x-9)^2$.
Setting $x=10$ yields $\left|\sqrt{10}-3.16666\right|<\frac{1}{216}<\frac{1}{200}=0.005$.
The approximate value is $\sqrt{10}\approx 3.16666$.

method 2 Use the binomial series, $(1+x)^k=1+kx+\frac{k(k-1)}{2}x^2+\cdots$ for $-1<x<1$.
$\sqrt{10}=\sqrt{9+1}=\sqrt{9(1+\frac{1}{9})}=\sqrt{9}\sqrt{1+\frac{1}{9}}=3(1+\frac{1}{9})^{1/2}$; we have $x=\frac{1}{9}$ and $k=\frac{1}{2}$
$\sqrt{10}=3(1+\frac{1}{2}\cdot\frac{1}{9}-\frac{1}{8}\cdot\frac{1}{9^2}+\cdots)=3+\frac{1}{6}-\frac{1}{216}+\cdots\Rightarrow\left|\sqrt{10}-3.16666\right|<\frac{1}{216}<\frac{1}{200}=0.005$

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Question 37 Solution

Since $f(x)=\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ is an alternating series, we can use the error bound for alternating series: $|s-s_n|\le a_{n+1}$ where $a_{n+1}=\frac{x^{n+1}}{n+1}$. We are interested in estimating $\ln\frac{3}{2}=\ln(1+\frac{1}{2})$; hence, we take $x=\frac{1}{2}$. Let us find the smallest $n$ such that $a_{n+1}=\frac{1}{(n+1)2^{n+1}}\le 0.001$. We find that $n=6$ and $s_6\approx 0.4047$; the exact value is $s=\ln\frac{3}{2}\approx 0.4055$, so the error is less than $10^{-3}$, as desired.

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Question 38 Solution

Find the first two nonzero terms in the Taylor series for $f(x)$ about $x=0$.

a) $e^{-x}\sin x=x-x^2+\cdots$
recall that $e^x=1+x+\frac{1}{2}x^2+\cdots$, so we have $e^{-x}=1-x+\frac{1}{2}x^2+\cdots$, also recall that $\sin x=x-\frac{1}{6}x^3+\cdots$
then we have $e^{-x}\sin x=(1-x+\frac{1}{2}x^2+\cdots)(x-\frac{1}{6}x^3+\cdots)=x-x^2+\cdots$ ok

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b) $\frac{1-\cos x}{x}=\frac{1}{2}x-\frac{1}{24}x^3+\cdots$
recall that $\cos x=1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots$, then $\frac{1-\cos x}{x}=\frac{1-(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots)}{x}=\frac{1}{2}x-\frac{1}{24}x^3+\cdots$ ok

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c) $\tan x=x+\frac{x^3}{3}+\cdots$
recall that $\sin x=x-\frac{1}{6}x^3+\cdots$, $\cos x=1-\frac{1}{2}x^2+\cdots$
then $\tan x=\frac{\sin x}{\cos x}=\frac{x-\frac{1}{6}x^3+\cdots}{1-\frac{1}{2}x^2+\cdots}=(x-\frac{1}{6}x^3+\cdots)\cdot\frac{1}{1-(\frac{1}{2}x^2+\cdots)}$ : write the 2nd factor as a geometric series
$\quad=(x-\frac{1}{6}x^3+\cdots)\cdot(1+(\frac{1}{2}x^2+\cdots)+\cdots)=x+x^3(\frac{1}{2}-\frac{1}{6})+\cdots=x-\frac{1}{3}x^3+\cdots$ ok

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d) $\tan^{-1}x=x-\frac{1}{3}x^3+\cdots$
$\tan^{-1}x=c_0+c_1x+c_2x^2+c_3x^3+\cdots$, note that $\tan^{-1}0=0$, so we must have $c_0=0$
$\Rightarrow\tan^{-1}x=c_1x+c_2x^2+c_3x^3+\cdots=y$
$\Rightarrow x=\tan y=y+\frac{1}{3}y^3+\cdots$ : this uses part (c)
$\quad=(c_1x+c_2x^2+\cdots)+\frac{1}{3}(c_1x+c_2x^2+\cdots)^3+\cdots=c_1x+c_2x^2+c_3x^3+\cdots+\frac{1}{3}(c_1^3x^3+\cdots)+\cdots$
we've shown that $x=c_1x+c_2x^2+c_3x^3+\cdots+\frac{1}{3}(c_1^3x^3+\cdots)+\cdots$
the coefficients of each power of $x$ must match on the left and right
$x^1\Rightarrow 1=c_1$
$x^2\Rightarrow 0=c_2$
$x^3\Rightarrow 0=c_3+\frac{1}{3}c_1^3\Rightarrow c_3=-\frac{1}{3}c_1^3=-1$ ok

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Question 39 Solution

$B_0=f(0)=1$; $B_1=f'(0)=-\frac{1}{2}$; $B_2=f''(0)=\frac{1}{6}$ (using L'Hospital rule).

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Question 40 Solution

a) $f(x)=x, f(0)=0, f'(0)=1$
b) $f(x)=\sin x, f(0)=0, f'(x)=\cos x, f'(0)=1$
c) $f(x)=\ln(1+x), f(0)=0, f'(x)=\frac{1}{1+x}, f'(0)=1$
b) $f(x)=e^x-1, f(0)=0, f'(x)=e^x, f'(0)=1$

If the functions are sketched in a neighborhood of $x=0$, the order they appear (from top to bottom), consider their Taylor approximations

$x=x$, $\sin x = x-\frac{1}{6}x^3+\cdots$, $\ln(1+x)=x-\frac{1}{2}x^2+\cdots$ , $e^x-1=x+\frac{1}{2}x^2$

Thus on right hand side of 0, from top to bottom, $e^x-1$, $x$, $\sin x$ and $\ln(1+x)$; on left hand side of 0, from top to bottom, $e^x-1$, $\sin x$, $x$, and $\ln(1+x)$.

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Question 41 Solution

a) $J_0(x)=1-\frac{x^2}{4}+\frac{x^4}{64}-\cdots$ it is alternating series.

$\int_0^1\left(1-\frac{x^2}{4}\right)dx=\frac{11}{12}$

error bound $\int_0^1\frac{x^4}{64}dx=\frac{x^5}{5\cdot 64}=\frac{1}{320}$

b) $J_0(x)'=\sum\limits_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n-1}}{2^{2n}(n!)^2}=\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)x^{2n+1}}{2^{2n+2}((n+1)!)^2}$

$J_0(x)''=\sum\limits_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$

$xJ_0(x)''=\sum\limits_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n-1}}{2^{2n}(n!)^2}=\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+1}}{2^{2n+2}((n+1)!)^2}$

$xJ_0(x)=\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n}(n!)^2}$

$xJ_0(x)''+J_0(x)'+xJ_0(x)=\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{2^{2n}(n!)^2}\left[\frac{2n+2}{2^2(n+1)^2}+\frac{(2n+1)(2n+2)}{2^2(n+1)^2}-1\right]=0$

Thus $J_0(x)$ satisfies $xy''+y'+xy=0$

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Question 42 Solution

$f(t)=\sum\limits_{n=0}^{\infty}t^n=1+t+t^2+t^3+t^4+\cdots$

a) step 1 : differentiate the series, $f'(t)=1+2t+3t^2+4t^3+\cdots$

step 2 : square the series, $f^2(t)=(1+t+t^2+t^3+t^4+\cdots)\cdot(1+t+t^2+t^3+t^4+\cdots)=1+2t+3t^2+4t^3+\cdots$

Thus $f'(t)=f^2(t)$, so $f(t)$ satisfies the differential equation $y'=y^2$ and we can check that the initial condition is $f(0)=1$.

b) $y'=y^2\Rightarrow\frac{dy}{dt}=y^2\Rightarrow\frac{dy}{y^2}=dt\Rightarrow\int\frac{dy}{y^2}=\int dt\Rightarrow-\frac{1}{y}=t+C\Rightarrow y=\frac{1}{-C-t}$, now apply initial condition $t=0,y=1\Rightarrow C=-1\Rightarrow y=\frac{1}{1-t}$

Since we showed in part (a) that $f(t)$ satisfes the differential equation and initial condition, we obtain $f(t)=\frac{1}{1-t}$. This is a round-about way of deriving the sum of a geomertic series, $f(t)=\sum\limits_{n=0}^{\infty}t^n=1+t+t^2+t^3++t^4+\cdots=\frac{1}{1-t}$.

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Question 43 Solution

a1) $\int_0^\infty\frac{\sin x}{x}dx$ converges

pf: $\int_0^\infty\frac{\sin x}{x}dx=\sum\limits_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin x}{x}dx=\int_0^\pi\frac{\sin x}{x}dx+\sum\limits_{n=1}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin x}{x}dx\le\int_0^\pi\frac{\sin x}{x}dx+\sum\limits_{n=1}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin x}{n\pi}dx$; the last step follows because $x\ge n\pi$ in each interval; then we have $\int_0^\infty\frac{\sin x}{x}dx\le\int_0^\pi\frac{\sin x}{x}dx+\frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n}\int_{n\pi}^{(n+1)\pi}\sin x dx=\int_0^\pi\frac{\sin x}{x}dx+\frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{-\cos x}{n}\Big|_{n\pi}^{(n+1)\pi}=\int_0^\pi\frac{\sin x}{x}dx+\frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{\cos n\pi-\cos(n+1)\pi}{n}=\int_0^\pi\frac{\sin x}{x}dx+\frac{1}{\pi}\sum\limits_{n=1}^{\infty}(-1)^n\frac{2}{n}=\int_0^\pi\frac{\sin x}{x}dx+\frac{2}{\pi}\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges since the integral is proper (note that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$) and the series converges by the AST <u>ok</u>

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a2) $\int_0^\infty\left|\frac{\sin x}{x}\right|dx$ diverges

pf: $\int_0^\infty\left|\frac{\sin x}{x}\right|dx=\sum\limits_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{|\sin x|}{x}dx=\int_0^\pi\frac{|\sin x|}{x}dx+\sum\limits_{n=1}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{|\sin x|}{x}dx\ge\int_0^\pi\frac{\sin x}{x}dx+\sum\limits_{n=1}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{|\sin x|}{(n+1)\pi}dx$; the last step follows because $x\le n(+1)\pi$ in each interval; then we have $\int_0^\infty\frac{\sin x}{x}dx\ge\int_0^\pi\frac{\sin x}{x}dx+\frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n+1}\int_{n\pi}^{(n+1)\pi}|\sin x|\,dx=\int_0^\pi\frac{\sin x}{x}dx+\frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{2}{n+1}=\int_0^\pi\frac{\sin x}{x}dx+\frac{2}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n+1}$, which diverges because the series is the harmonic series <u>ok</u>

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b) $\int_0^\infty\frac{\sin x}{x}dx=\frac{\pi}{2}$

pf: set $f(a)=\int_0^\infty\frac{\sin x}{x}e^{-ax}dx$; then note that $f(0)=\int_0^\infty\frac{\sin x}{x}dx$; so we need to evaluate $f(0)$; to do that we will evaluate $f(a)$ and then set $a=0$; to solve for $f(a)$, we will find an expression for $f'(a)$ and then integrate; hence we have $f'(a)=\int_0^\infty\frac{\sin x}{x}\frac{d}{da}[e^{-ax}]dx=\int_0^\infty\frac{\sin x}{x}\cdot-xe^{-ax}dx=-\int_0^\infty\sin xe^{-ax}dx$

substitute: $u=\sin x, dv=e^{-ax}dx\Rightarrow du=\cos xdx, v=\frac{e^{-ax}}{-a}$

$\Rightarrow f'(a) = -\left[\sin x \cdot \frac{e^{-ax}}{-a}\Big|_0^\infty - \int_0^\infty \cos x \frac{e^{-ax}}{-a} dx\right] = -\frac{1}{a}\int_0^\infty \cos x e^{-ax} dx$

substitute: $u = \cos x, dv = e^{-ax}dx \Rightarrow du = -\sin x dx, v = \frac{e^{-ax}}{-a}$

$\Rightarrow f'(a) = -\frac{1}{a}\left[\cos x \cdot \frac{e^{-ax}}{-a}\Big|_0^\infty - \int_0^\infty \sin x \frac{e^{-ax}}{a} dx\right] = -\frac{1}{a}\left[\frac{1}{a} + \frac{1}{a}f'(a)\right] \Rightarrow f'(a) = -\frac{1}{a}\left[\frac{1}{a} + \frac{1}{a}f'(a)\right] \Rightarrow f'(a)\left[1 + \frac{1}{a^2}\right] = -\frac{1}{a^2} \Rightarrow f'(a)\left[a^2+1\right] = -1 \Rightarrow f'(a) = -\frac{1}{1+a^2} \Rightarrow f(a) = -\tan^{-1}(a) + C$; to evaluate the constant note that $\lim_{a\to\infty} f(a) = \lim_{a\to\infty}\int_0^\infty \frac{\sin x}{x} e^{-ax} dx = \int_0^\infty \frac{\sin x}{x} \lim_{a\to\infty}[e^{-ax}]dx = \int_0^\infty \frac{\sin x}{x} \cdot 0 dx = \int_0^\infty 0 dx = 0$; hence we have $\lim_{a\to\infty} f(a) = \lim_{a\to\infty} -\tan^{-1}(a) + C = -\tan^{-1}(\infty) + C = -\frac{\pi}{2} + C = 0 \Rightarrow C = \frac{\pi}{2} \Rightarrow f(a) = -\tan^{-1}(a) + \frac{\pi}{2} \Rightarrow f(0) = -\tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$ ok

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Question 44 Solution

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots$; it is a convergent alternating series, so $|\cos x - 1| \le \frac{1}{2}x^2$, and $\left|\cos x - (1 - \frac{1}{2}x^2)\right| \le \frac{1}{4!}x^4$; setting $x = \frac{\pi}{5}$ gives the result

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Question 45 Solution

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^x (1 - t^2 + \frac{t^4}{2} + ...)dt = \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3} + \frac{x^5}{10} + ...)$

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Question 46 Solution

a) $\frac{a}{a+b} = \frac{a}{b} \cdot \frac{1}{1+\frac{a}{b}} = \frac{a}{b} \cdot \frac{1}{1-\left(-\frac{a}{b}\right)} = \frac{a}{b} \cdot \sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n$

$\frac{a}{a+b} = \frac{a}{b}\left(1 - \frac{a}{b} + \frac{a^2}{b^2} + \cdots\right) = \frac{a}{b} - \frac{a^2}{b^2} + \frac{a^3}{b^3} + \cdots$

b) Using the Binomial Series Theorem

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots$ for $-1 < x < 1$

$\sqrt{R^2 - r^2} = R\sqrt{1 - \frac{r^2}{R^2}} = R\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} = R\left[1 - \frac{1}{2}\cdot\frac{r^2}{R^2} + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2}\left(-\frac{r^2}{R^2}\right)^2 + \cdots\right] = R - \frac{r}{2}\cdot\frac{r}{R} - \frac{r}{8}\cdot\frac{r^3}{R^3} + \cdots$

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Question 47 Solution

Starting from the Taylor series of $f(x)$ about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots,$$

replace $x \to x + h, a \to x$. It follows that $x - a \to (x+h) - x = h$, and the Taylor series becomes

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \cdots$$

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Question 48 Solution

a) let $y = 0 \;\Rightarrow\; x = \pm(1+\epsilon)$, let $x = 0 \;\Rightarrow\; y = \pm 1$

b) Solve for $y$: $y = f(x) = \pm\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}$

$A(\epsilon) = 2\int_{-(1+\epsilon)}^{1+\epsilon}\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}dx = 4\int_0^{1+\epsilon}\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}dx$

c)

$A(\epsilon) = 4\int_0^{1+\epsilon}\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}dx = 4(1+\epsilon)\int_0^{1+\epsilon}\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}d\frac{x}{1+\epsilon} = 4(1+\epsilon)\int_0^1\sqrt{1-u^2}du = 4(1+\epsilon)\frac{\pi}{4} = (1+\epsilon)\pi$

The first two nonzero terms are $\pi + \pi\epsilon$.

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Question 49 Solution

$V(x) = \frac{Gm_1}{|x-x_1|} + \frac{Gm_2}{|x-x_2|}$ for $x \to \infty$ ie., $x > x_1$ and $x > x_2 \quad\Rightarrow\quad V(x) = \frac{Gm_1}{x-x_1} + \frac{Gm_2}{x-x_2}$

Using the hint set $y = 1/x$, ie., $x = 1/y$ and expand the potential in powers of $y$

$V(1/y) = \frac{Gm_1}{1/y-x_1} + \frac{Gm_2}{1/y-x_2} = \frac{Gm_1y}{1-x_1y} + \frac{Gm_2y}{1-x_2y}$

Using Geometric Series Formula $\frac{1}{1-x_iy} = \sum_{n=0}^{\infty}(x_iy)^n$ where $i = 1, 2$

$V(1/y) = Gm_1y\sum_{n=0}^{\infty}(x_1y)^n + Gm_2y\sum_{n=0}^{\infty}(x_2y)^n = Gm_1\sum_{n=0}^{\infty}x_1^ny^{n+1} + Gm_2\sum_{n=0}^{\infty}x_2^ny^{n+1}$

$= (Gm_1 + Gm_2)y + (Gm_1x_1 + Gm_2x_2)y^2 + (Gm_1x_2^2 + Gm_2x_2^2)y^3 + \cdots$

change $y = \frac{1}{x}$ back

$V(x) = (Gm_1 + Gm_2)\frac{1}{x} + (Gm_1x_1 + Gm_2x_2)\frac{1}{x^2} + (Gm_1x_2^2 + Gm_2x_2^2)\frac{1}{x^3} + \cdots$

Thus $a = (Gm_1 + Gm_2)$, $b = (Gm_1x_1 + Gm_2x_2)$, and $c = (Gm_1x_2^2 + Gm_2x_2^2)$

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Question 50 Solution

a) $\lim\limits_{r\to 0} V(r) = \lim\limits_{r\to 0} V_0((\frac{r_0}{r})^{12} - 2(\frac{r_0}{r})^6) = V_0 \lim\limits_{r\to 0} \frac{r_0^{12} - 2r_0^6 r^6}{r^{12}} = V_0 \frac{r_0^{12}}{0^+} = \infty$

$\lim\limits_{r\to\infty} V(r) = \lim\limits_{r\to\infty} V_0((\frac{r_0}{r})^{12} - 2(\frac{r_0}{r})^6) = V_0 \lim\limits_{r\to\infty} \frac{r_0^{12} - 2r_0^6 r^6}{r^{12}} = V_0 \cdot 0 = 0$

b) $V'(r) = V_0((-12)\frac{r_0^{12}}{r^{13}} - 2(-6)\frac{r_0^6}{r^7}) = V_0(\frac{-12r_0^{12}}{r^{13}} + \frac{12r_0^6}{r^7}) = V_0 \frac{-12r_0^{12} + 12r_0^6 r^6}{r^{13}}$

$V'(r) = 0 \Leftrightarrow -12r_0^{12} + 12r_0^6 r^6 = 0 \Leftrightarrow 12r_0^6 r^6 = 12r_0^{12} \Leftrightarrow r^6 = r_0^6 \Leftrightarrow r = r_0\,.$

d) Find the quadratic Taylor approximation at $\underline{x = x_0}$, ie., $c_0 + c_1(x - x_0) + c_2(x - x_0)^2$ using the Theorem

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots$ for $-1 < x < 1$

$V(x) = V_0\left[\left(\frac{x_0}{x}\right)^{12} - 2\left(\frac{x_0}{x}\right)^6\right] = V_0\left[\left(\frac{x}{x_0}\right)^{-12} - 2\left(\frac{x}{x_0}\right)^{-6}\right] = V_0\left[\left(\frac{x - x_0 + x_0}{x_0}\right)^{-12} - 2\left(\frac{x - x_0 + x_0}{x_0}\right)^{-6}\right]$

$= V_0\left[\left(1 + \frac{x - x_0}{x_0}\right)^{-12} - 2\left(1 + \frac{x - x_0}{x_0}\right)^{-6}\right]$

using the above Theorem

$\left(1 + \frac{x-x_0}{x_0}\right)^{-12} = 1 - 12\frac{x - x_0}{x_0} + \frac{(-12)\cdot(-12-1)}{2}\left(\frac{x-x_0}{x}\right)^2 + \cdots = 1 - \frac{12}{x_0}(x - x_0) + \frac{78}{x_0^2}(x - x_0)^2 + \cdots$

$\left(1 + \frac{x-x_0}{x_0}\right)^{-6} = 1 - 6\frac{x - x_0}{x_0} + \frac{(-6)\cdot(-6-1)}{2}\left(\frac{x-x_0}{x}\right)^2 + \cdots = 1 - \frac{6}{x_0}(x - x_0) + \frac{21}{x_0^2}(x - x_0)^2 + \cdots$

$V(x) = V_0\left[\left(1 - \frac{12}{x_0}(x - x_0) + \frac{78}{x_0^2}(x - x_0)^2 + \cdots\right) - 2\left(1 - \frac{6}{x_0}(x - x_0) + \frac{21}{x_0^2}(x - x_0)^2\right)\right] + \cdots$

$= V_0\left[-1 + \frac{36}{x_0^2}(x - x_0)^2\right] + \cdots = -V_0 + 36\frac{V_0}{x_0^2}(x - x_0)^2 + \cdots$

$T_2(x) = -V_0 + 36\frac{V_0}{x_0^2}(x - x_0)^2$

e) We know that work equals the integral of the force:

$W = \int\limits_{r_0}^{\infty} f(r)dr = \int\limits_{r_0}^{\infty} -V'(r)dr = -V(r)|_{r_0}^{\infty} = -0 - (-V(r_0)) = V(r_0) = V_0(1 - 2) = -V_0$

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**binomial series**

Question 51 Solution

Using the Binomial Series Theorem,

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots$ for $-1 < x < 1$

$(1+x^2)^k = 1 + kx^2 + \frac{k(k-1)}{2}x^4 + \frac{k(k-1)(k-2)}{3!}x^6 + \frac{k(k-1)(k-2)(k-3)}{4!}x^8 + \cdots$

$(1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^4 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^8 + \cdots = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \cdots$

This is an alternating series and the assumptions in the AST apply.

Using the second order Taylor approximation, $T_2(x) = 1 + \frac{1}{2}x^2$, we have $|s - T_2| < \frac{1}{8}x^4$, where $s$ is the exact value.

$\int_0^1 \sqrt{1+x^2}dx \approx \int_0^1 (1 + \frac{1}{2}x^2)dx = x + \frac{1}{6}x^3\Big|_0^1 = \frac{7}{6}$ and the error is less than $\int \frac{1}{8}x^4 dx = \frac{1}{40}x^5\Big|_0^1 = \frac{1}{40}$

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Question 52 Solution

Consider the expansion $\frac{1}{\sqrt{1-2ax+x^2}} = c_0 + c_1 x + c_2 x^2 + \cdots$, where $a$ is a constant. Find $c_0, c_1, c_2$ in terms of $a$.

We can apply the binomial expansion.

$\frac{1}{\sqrt{1-2ax+x^2}} = (1 - 2ax + x^2)^{-1/2} = (1 + (-2ax + x^2))^{-1/2} = 1 + (-\frac{1}{2})(-2ax + x^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-2ax + x^2)^2 + \cdots$

$= 1 + ax - \frac{1}{2}x^2 + \frac{3}{8}(4a^2x^2 - 4ax^3 + x^4) + \cdots = 1 + ax + (-\frac{1}{2} + \frac{3}{2}a^2)x^2 + \cdots$, hence $c_0 = 1, c_1 = a, c_2 = \frac{1}{2}(3a^2 - 1)$

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Question 53 Solution

a) Show that $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$

This is true since the left hand side is number of ways of choosing $n+1$ objects from a set of $k+1$ objects (disregarding the order in which the objects are chosen).

The right hand side means that: assume all $k+1$ objects are white, one may randomly pick one object from the $k+1$ objects, coloring it red, then put it back. Now choose $n+1$ objects from these $k+1$ objects, there are two different situations: one situation is that the red one is chosen, the number of ways is $\binom{k}{n}$ (it is equivalent to choosing $n$ from $k$ objects); the other situation is the red one is not chosen, the number of ways is $\binom{k}{n+1}$ (it is equivalent to choosing $n+1$ objects from $k$ objects).

The left hand side equals the right hand side, since it is the same thing, choosing $n+1$ objects from $k+1$ objects.

$\binom{k+1}{n+1} = \frac{(k+1)!}{(n+1)!(k-n)!} = \frac{k!\cdot(k+1)}{(n+1)!(k-n)!} = \frac{k!\cdot(k-n+n+1)}{(n+1)!(k-n)!} == \frac{k!\cdot(k-n)+k!\cdot(n+1)}{(n+1)!(k-n)!} = \frac{k!\cdot(k-n)}{(n+1)!(k-n)!} + \frac{k!\cdot(n+1)}{(n+1)!(k-n)!} = \frac{k!}{(n+1)!(k-n-1)!} + \frac{k!}{n!(k-n)!} = \binom{k}{n+1} + \binom{k}{n}$

b)

| | |
|---|---|
| $\binom{0}{0}$ | 1 |
| $\binom{1}{0}\ \binom{1}{1}$ | 1  1 |
| $\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$ | 1  2  1 |

$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \qquad\qquad 1 \quad 3 \quad 3 \quad 1$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \qquad\qquad 1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \qquad\qquad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \qquad\qquad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Denote elements in the triangle by $a_{k,n}$, $n$th element on $k$th row. Each row is obtained by adding the two entries diagonally above, $a_{k+1,n+1} = a_{k,n} + a_{k,n+1}$ which is equivalent to $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$.
c) $(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

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**complex numbers**

Question 54 Solution

a) $1 + i = \sqrt{2}e^{\frac{\pi}{4}i} \quad (x = 1, y = 1$ already in Cartesian form)
b) $(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i = 2e^{\frac{\pi}{2}i} \quad (x = 0, y = 2)$
c) $(1+i)^3 = (1+i)^2(1+i) = 2i(1+i) = -2 + 2i = 2\sqrt{2}e^{\frac{3\pi}{4}i} \quad (x = -2, y = 2)$
d) $\frac{1}{1+i} = \frac{1\cdot(1-i)}{(1+i)\cdot(1-i)} = \frac{1-i}{1-i^2} = \frac{1-i}{1-(-1)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i = \frac{1}{\sqrt{2}}e^{\frac{-\pi}{4}i} \quad (x = \frac{1}{2}, y = -\frac{1}{2})$
e) $\sqrt{1+i} = \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}e^{\frac{\pi}{8}i} = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\,\sin\frac{\pi}{8}\right) = 2^{\frac{1}{4}}\cos\frac{\pi}{8} + i\,2^{\frac{1}{4}}\sin\frac{\pi}{8} \quad (x = 2^{\frac{1}{4}}\cos\frac{\pi}{8}, y = 2^{\frac{1}{4}}\sin\frac{\pi}{8})$

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Question 55 Solution

a) See 49 c) $(1+i)^6 = 1 + 6\cdot i + 15\cdot i^2 + 20\cdot i^3 + 15\cdot i^4 + 6\cdot i^5 + i^6 = 1 + 6i - 15 - 20i + 15 + 6i - 1 = -8i$
b) $1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$, since $x = 1$, $y = 1$, $r = \sqrt{x^2+y^2} = \sqrt{2}$, $\theta = \arctan\frac{y}{x} = \arctan 1 = \frac{\pi}{4}$, $re^{\theta i} = \sqrt{2}e^{\frac{\pi}{4}i}$
$(1+i)^6 = (\sqrt{2}e^{\frac{\pi}{4}i})^6 = 2^{\frac{6}{2}}e^{\frac{6}{4}\pi i} = 2^3e^{\frac{3}{2}\pi i} = 8(\cos\frac{3}{2}\pi + i\,\sin\frac{3}{2}\pi) = -8i$

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Question 56 Solution

a) $z^2 + 2z - 2 = 0 \ \Rightarrow\ a = 1,\ b = 2,\ c = -2,\ z_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4+8}}{2} = -1 \pm \sqrt{3}$ two real roots
b) $z^2 + 2z + 2 = 0 \ \Rightarrow\ a = 1,\ b = 2,\ c = 2,\ z_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2\pm2\sqrt{-1}}{2} = -1\pm\sqrt{-1} = -1\pm i$
c) $z^2 = (re^{\theta i})^2 = r^2e^{2\theta i} = 1 \ \Rightarrow\ r = 1, \theta = \pi k$, where k is any integer $\Rightarrow\ z_{1,2} = \pm1$
d) $z^3 = r^3e^{3\theta i} = 1 \ \Rightarrow\ r = 1, \theta = \frac{2\pi}{3}k$, where k is any integer $\Rightarrow$ three roots: $z_1 = 1$ and $z_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
e) $z^4 = r^4e^{4\theta i} = 1 \ \Rightarrow\ z^2 = \pm1 \ \Rightarrow$ four roots: $z_{1,2} = \pm1$, $z_{3,4} = \pm i$
f) $e^z = 1$ on real axis there is on root $z = 0$, but in complex plane there are infinite roots, let $z = x + yi$, $e^z = e^{x+yi} = e^x(\cos y + i\,\sin y) = 1 \ \Rightarrow\ x = 0$ and $y = 2k\pi$, where $k$ is any integer, roots are $z_k = 2k\pi\,i$.

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Question 57 Solution

$(\cos\theta + i\,\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

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Question 58 Solution

$e^{i(a+b)} = e^{ia}\cdot e^{ib} \Rightarrow \cos(a+b)+i\sin(a+b) = (\cos a+i\sin a)\cdot(\cos b+i\sin b) = \cos a\cos b-\sin a\sin b+i(\sin a\cos b+\cos a\sin b)$
take real and imaginary parts : $\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b$ , $\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b$

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Question 59 Solution

a1) $\int e^{ax}\cos bx\,dx = \int e^{ax}\frac{1}{b}\,d\sin bx = \frac{1}{b}e^{ax}\cdot\sin bx - \int\frac{1}{b}\sin bx\,de^{ax} = \frac{1}{b}e^{ax}\cdot\sin bx - \int\frac{a}{b}e^{ax}\sin bx\,dx = \frac{1}{b}e^{ax}\cdot\sin bx - \int\frac{a}{b}e^{ax}\left(-\frac{1}{b}\right)\,d\cos bx = \frac{1}{b}e^{ax}\cdot\sin bx + \int\frac{a}{b^2}e^{ax}\,d\cos bx = \frac{1}{b}e^{ax}\cdot\sin bx + \frac{a}{b^2}e^{ax}\cdot\cos bx - \frac{a}{b^2}\int\cos bx\,de^{ax} = \frac{1}{b}e^{ax}\cdot\sin bx + \frac{a}{b^2}e^{ax}\cdot\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx \ \Rightarrow\ \underline{\int e^{ax}\cos bx\,dx} = \frac{1}{b}e^{ax}\cdot\sin bx + \frac{a}{b^2}e^{ax}\cdot\cos bx - \frac{a^2}{b^2}\underline{\int e^{ax}\cos bx\,dx} \ \Rightarrow\ \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\cdot\sin bx + \frac{a}{b^2}e^{ax}\cdot\cos bx \ \Rightarrow\ \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(b\cdot\sin bx + a\cdot\cos bx\right)$

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a2) $\int e^{ax}\sin bx\,dx = \int e^{ax}\left(-\frac{1}{b}\right)\,d\cos bx = -\frac{1}{b}e^{ax}\cdot\cos bx + \int\frac{1}{b}\cos bx\,de^{ax} = -\frac{1}{b}e^{ax}\cdot\cos bx + \int\frac{a}{b}e^{ax}\cos bx\,dx = -\frac{1}{b}e^{ax}\cdot\cos bx + \int\frac{a}{b}e^{ax}\cdot\frac{1}{b}\,d\sin bx = -\frac{1}{b}e^{ax}\cdot\cos bx + \int\frac{a}{b^2}e^{ax}\,d\sin bx = -\frac{1}{b}e^{ax}\cdot\cos bx + \frac{a}{b^2}e^{ax}\cdot\sin bx - \frac{a}{b^2}\int\sin bx\,de^{ax} = -\frac{1}{b}e^{ax}\cdot\cos bx + \frac{a}{b^2}e^{ax}\cdot\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx \ \Rightarrow\ \underline{\int e^{ax}\sin bx\,dx} = -\frac{1}{b}e^{ax}\cdot\cos bx + \frac{a}{b^2}e^{ax}\cdot\sin bx - \frac{a^2}{b^2}\underline{\int e^{ax}\sin bx\,dx} \ \Rightarrow\ \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cdot\cos bx + \frac{a}{b^2}e^{ax}\cdot\sin bx \ \Rightarrow\ \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cdot\sin bx - b\cdot\sin bx\right)$

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b) $e^{(a+ib)x} = e^{ax+ibx} = e^{ax}\cdot e^{ibx} = e^{ax}\left(\cos bx + i\,\sin bx\right)$

$\int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x} = \frac{1\cdot(a-ib)}{(a+ib)\cdot(a-ib)}e^{(a+ib)x} = \frac{a-ib}{a^2+b^2}e^{(a+ib)x}$

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c) Since $\int e^{(a+ib)x}dx = \int\left(e^{ax}\cos bx + i\,e^{ax}\sin bx\right)dx = \int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx = \frac{a-ib}{a^2+b^2}e^{(a+ib)x}$, and $\frac{a-ib}{a^2+b^2}e^{(a+ib)x} = \frac{a-ib}{a^2+b^2}\left(e^{ax}\cos bx + i\,e^{ax}\sin bx\right) = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + i\left[\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\sin bx)\right]$, we have

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$ and $\int\ e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\sin bx)$

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Question 60 Solution

a) since $e^{ix} = \cos x + i\ \sin x$ and $e^{-ix} = \cos x - i\ \sin x \quad \Rightarrow \quad e^{ix} + e^{-ix} = 2\cos x \quad \Rightarrow \quad \cos x = \frac{e^{ix}+e^{-ix}}{2}$

b) $e^{ix} - e^{-ix} = 2i\sin x \quad \Rightarrow \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$

c) $\frac{d}{dx}\cos x = \frac{d}{dx}\left(\frac{e^{ix}+e^{-ix}}{2}\right) = \frac{ie^{ix}-ie^{-ix}}{2} = \frac{\left(ie^{ix}-ie^{-ix}\right)i}{2i} = \frac{-e^{ix}+e^{-ix}}{2i} = -\sin x$

d) $\frac{d}{dx}\sin x = \frac{d}{dx}\left(\frac{e^{ix}-e^{-ix}}{2i}\right) = \frac{ie^{ix}+ie^{-ix}}{2i} = \frac{e^{ix}+e^{-ix}}{2} = \cos x$

e) TO BE COMPLETED

f) TO BE COMPLETED

g) TO BE COMPLETED

h) TO BE COMPLETED