

1. True or false

a) FALSE. $\sum_{i=1}^{n^2} i = \frac{n^2(n^2+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n^2(n+1)^2}{4}$

b) TRUE. $\lim_{n \rightarrow \infty} \sum_{i=1}^n f'(x_i) \Delta x = \int_a^b f'(x) dx = f(b) - f(a)$ by FTC

c) FALSE. If n is doubled, then Δx decreases by a factor of $\frac{1}{2}$ and the error in the right-hand Riemann sum decreases by a factor of $\frac{1}{2}$, not $\frac{1}{4}$, as we saw in examples in class.

d) TRUE. Since $\int_0^1 f(x)g''(x)dx = \int_0^1 f(x)dg'(x) = f(x)g'(x)|_0^1 - \int_0^1 g'(x)df(x) = f(1)g'(1) - f(0)g'(1) - \int_0^1 f'(x)g'(x)dx = -\int_0^1 f'(x)g'(x)dx$ and $\int_0^1 g(x)f''(x)dx = \int_0^1 g(x)df'(x) = g(x)f'(x)|_0^1 - \int_0^1 f'(x)dg(x) = g(1)f'(1) - g(0)f'(1) - \int_0^1 g'(x)f'(x)dx = -\int_0^1 f'(x)g'(x)dx$, left hand side equals right hand side.

e) FALSE. Since $\int_0^1 \frac{dx}{x^2}$ diverges by p -test $\Rightarrow \int_0^\infty \frac{dx}{x^2}$ diverges.

f) FALSE. The work done for spring $W = \int_{10}^{15} k(x-10)dx = \int_0^5 kx dx = \frac{1}{2}kx^2|_0^5 = 200$ (N·cm) $\Rightarrow k = 16$. $W = \int_{10}^{20} k(x-10)dx = \int_0^{10} kx dx = \frac{1}{2}kx^2|_0^{10} = 800$ (N·cm) = 8 Joule. (The general rule is: both starting from the natural length, if the length stretched is doubled, the work is multiplied by 4.)

g) FALSE. Do not try to find the true CM, it is complicated, instead, draw a graph and think. $\bar{x} = 0$ is correct since the graph is symmetric about $x = 0$. But $\bar{y} = \frac{1}{2}$ is lower than the actual \bar{y} . If the region is a rectangle $-1 \leq x \leq 1$ and $0 \leq y \leq 1$, this CM is true, but the actual graph is well above $y = 1$ (the lowest point of $\cosh x$ is 1 at $x = 0$). Actually, the true $\bar{y} = 0.5985$.

h) FALSE. A counterexample is an exponential distribution, $f(x)$ attains its maximum value at $x = 0$ rather than $\mu = \frac{1}{c}$. (For normal distribution, the statement is true.)

i) TRUE. Since $\int_{-\infty}^{\infty} xf(x)dx = \mu$ and $\int_{-\infty}^{\infty} f(x) = 1$, one has $\int_{-\infty}^{\infty} (x - \mu)f(x)dx = 0$.

j) FALSE. The statement describes a linear decay. But the radioactive material obeys exponential decay ($y(t) = y_0 e^{-kt}$), after 100 year (half-life) only $\frac{1}{2}$ kg left, after 400 year $(\frac{1}{2})^4 = \frac{1}{16}$ kg left.

k) TRUE. Compounded Continuously $y(t) = y_0 e^{rt} = y_0(1 + rt + \frac{(rt)^2}{2} + \text{Remainder})$, where $t = 2$, $r = 0.05 \Rightarrow y(2) = 2000 \cdot (1 + 0.05 \cdot 2 + \frac{1}{2}(0.05 \cdot 2)^2) + \text{Remainder} = 2210 + \text{Remainder} > 2210$ (The Remainder is positive.)

l) TRUE. $y(t) = 1$ is a constant solution of the differential equation.

m) FALSE. This is only true if the constant solution is stable.

n) FALSE. A counterexample is $a_n = \frac{1}{n}$, $b_n = n$, $\lim_{n \rightarrow \infty} a_n b_n = 1$.

o) FALSE. If the step size h decreases, then the error also decreases.

p) FALSE. A counterexample is $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$, so that $\sum_{n=0}^{\infty} a_n$ converges, but $\sum_{n=0}^{\infty} b_n$ diverges.

q) FALSE. In class we showed that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$.

r) FALSE. Since $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$, the ratio test is inconclusive.

s) FALSE. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n$ converges at $x = 1$ by AST. However, it does NOT converge at $x = -1$ by p -test with $p = 1$.

t) TRUE. The radius of convergence is at least $R = 1$, so the interval of convergence is at least $0 < x \leq 2$, which contains $x = \frac{1}{2}$.

u) FALSE. Do not try to find $f^{(3)}(0)$ and $f^{(6)}(0)$ directly, instead, using the Taylor series to derive them, since the general form for Taylor series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \frac{f^{(n)}(0)}{n!} \Rightarrow f^{(n)}(0) = n! \cdot c_n$. Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \text{Remainder}$. Here $c_3 = 0$ since there is not x^3 term, $c_6 = -\frac{1}{6}$, thus $f^{(3)}(0) = 0$, $f^{(6)}(0) = 6! \cdot c_6 = 720 \cdot (-\frac{1}{6}) = -120$. Thus the statement is false.

v) TRUE. Since $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$. Differentiate on both side yields $-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \Rightarrow \frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

w) TRUE. Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, let $x = 1$, $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} \dots$, it is clear than $e > 2$. Consider a series $1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-\frac{1}{2}} = 3$, this series is greater than the series of e , since except the first three terms ($1 = 1, 1 = 1, \frac{1}{2} = \frac{1}{2}$), each term in this series is greater than the corresponding term in the series of e , $\frac{1}{4} > \frac{1}{3!} = \frac{1}{6}$, $\frac{1}{2^3} = \frac{1}{8} > \frac{1}{4!} = \frac{1}{24}$, \dots if $\frac{1}{2^{n-1}} > \frac{1}{n!}$, then $\frac{1}{2^{n-1.2}} = \frac{1}{2^{n+1}} > \frac{1}{n!(n+1)} = \frac{1}{(n+1)!}$, since $n+1 > 2$ for $n \geq 1$, by induction, the series of 3 is greater than the series of e , since each term in the former is greater (or equal) the corresponding term in the latter. Thus $2 < e < 3$.

- x) TRUE. $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \text{Remainder} \Rightarrow e^{-x^2} > 1 - x^2$ since the next term $\frac{1}{2}x^4$ is positive, one can show that the remainder is positive (omit here). Thus $\int_0^1 e^{-x^2} dx > \int_0^1 (1 - x^2) dx = (x - \frac{x^3}{3}) \Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$.
- y) TRUE. Since the series is $\sin \frac{\pi}{2}$ (note that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$) and $\sin \frac{\pi}{2} = 1$.
- z) TRUE. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1$.
- aa) FALSE. $\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{\cosh x} d \cosh x = \ln(\cosh x) \neq \text{sech}^2 h$. Note that $(\tanh x)' = \text{sech}^2 x$.
- bb) TRUE. Note that $T_1(x) = f(a) + f'(a)(x - a)$, thus $T_1'(a) = f'(a)$.
- cc) TRUE. $e^x = 1 + x + \frac{1}{2}x^2 + \dots$, so $e^{-0.1} = 1 + (-0.1) + \frac{1}{2}(-0.1)^2 + \dots = 1 - 0.1 + 0.005 - \dots$, so $|e^{-0.1} - 0.9| \leq 0.005$, so $0.895 \leq e^{-0.1} \leq 0.905$.
- dd) FALSE. Using the binomial series $(1 + x)^k = 1 + kx + \dots$, replace x with $x^2 \Rightarrow (1 + x^2)^k = 1 + kx^2 + \dots$, since $k = \frac{1}{2}$, $\sqrt{1 + x^2} = (1 + x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \dots$.
- ee) TRUE. Since $\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i \sin x + \cos x - i \sin x}{2} = \cos x$
- ff) TRUE. Since $e^{\pi i} = \cos \pi + i \sin \pi = -1 \Rightarrow \pi i = \log(-1)$. (Actually, more rigorous $\log(-1) = (2k + 1)\pi i$, where k is an integer.)
- gg) FALSE. $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$
- hh) TRUE. $\binom{10}{2} = \frac{10!}{2!8!} = \frac{10!}{8!2!} = \binom{10}{8}$.
- ii) TRUE. $\sum_{n=0}^{\infty} \binom{k}{n} (-1)^n = (1 + (-1))^k = 0$.

integration

Question 2 Solution a) geometric series = $\frac{1}{1 - \frac{2011}{2012}} = \frac{2012}{2012 - 2011} = 2012$

b) It equals $\int_0^1 (1 + x) dx = \left(x + \frac{x^2}{2}\right) \Big|_0^1 = \frac{3}{2}$ (change $\frac{i}{n} \rightarrow x$, $\frac{1}{n} \rightarrow dx$)

c) It equals $\int_0^1 \frac{1}{1+x} dx = \ln(1+x) \Big|_0^1 = \ln 2$

d) By L'Hospital's rule, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{(\sin x)'}{(x)'} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$.

e) By L'Hospital's rule, $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{(1 - \cos x)'}{(x^2)'} = \lim_{x \rightarrow 0} \frac{\sin x}{2x} = \frac{1}{2}$.

f) Note that $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{2n} = \left(\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n\right)^2 = (e^x)^2 = e^{2x}$

g) By L'Hospital's rule $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-1)'}{x'} = \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

h) $\lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} = \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3$ (one can use L'hospital rule, note that 'h' is variable here, regard x as constant)

i) By L'Hospital's rule, $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f'(x+h)}{1} = f'(x)$.

j) Using L'Hospital's rule twice, $\lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} = \lim_{t \rightarrow 0} \frac{f''(x+t) + f''(x-t)}{2} = f''(x)$.

k) $\lim_{h \rightarrow 0} \frac{\int_0^h f(x) dx}{h}$ L'Hospital Rule $\rightarrow \lim_{h \rightarrow 0} \frac{(\int_0^h f(x) dx)'}{(h)'} = \lim_{h \rightarrow 0} \frac{f(h)}{1} = f(0)$

l) $\lim_{h \rightarrow 0} \frac{\int_0^h xf(x) dx}{h^2}$ L'Hospital Rule $\rightarrow \lim_{h \rightarrow 0} \frac{(\int_0^h xf(x) dx)'}{(h^2)'} = \lim_{h \rightarrow 0} \frac{hf(h)}{2h} = \lim_{h \rightarrow 0} \frac{f(h)}{2} = \frac{f(0)}{2}$

Question 3 Solution

a) $\lim_{h \rightarrow 0} (1 + h)^n = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e^1 = e$ b) use L'Hospital's rule twice

Question 4 Solution

a) TO BE COMPLETED

b) TO BE COMPLETED

c) TO BE COMPLETED

d) TO BE COMPLETED

e) TO BE COMPLETED

f) TO BE COMPLETED

g) integration by parts

$$\int x \sin x dx = \int x(-1) d \cos x = - \int x d \cos x = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C = \sin x - x \cos x + C$$

h) using integration by parts twice

Using integration once

$$\int e^{-x} \sin x dx = \int e^{-x}(-1) d \cos x = -e^{-x} \cos x + \int \cos x d e^{-x} = -e^{-x} \cos x - \int e^{-x} \cos x dx$$

Using integration twice

$$\int e^{-x} \cos x dx = \int e^{-x} d \sin x = e^{-x} \sin x - \int \sin x d e^{-x} = e^{-x} \sin x + \int e^{-x} \sin x dx$$

Thus

$$\int e^{-x} \sin x dx = -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x dx \Rightarrow 2 \int e^{-x} \sin x = -e^{-x} \cos x - e^{-x} \sin x \Rightarrow \int e^{-x} \sin x = \frac{1}{2} (-e^{-x} \cos x - e^{-x} \sin x) + C = -\frac{1}{2} e^{-x} (\cos x + \sin x) + C$$

i) $\int \frac{dx}{4x^2} = -\frac{1}{4x} + C$

j) $\int \frac{x}{4+x^2} dx = \int \frac{\frac{1}{2}}{4+x^2} dx^2 = \int \frac{\frac{1}{2}}{4+x^2} d(4+x^2) = \frac{1}{2} \ln(4+x^2) + C$

If you do not like the above way, using variable change $u = 4 + x^2$, $du = 2x dx \Rightarrow dx = \frac{1}{2x} du$

$$\int \frac{x}{4+x^2} dx = \int \frac{x}{u} \cdot \frac{1}{2x} du = \int \frac{1}{2u} du = \frac{1}{2} \ln u + C = \frac{1}{2} \ln(4+x^2) + C$$

k) This type of antiderivative, one needs to use substitution $x = 2 \tan \theta$, $dx = 2(1 + \tan^2 \theta) d\theta$

$$\int \frac{dx}{4+x^2} = \int \frac{2(1+\tan^2 \theta)}{4+4 \tan^2 \theta} d\theta = \int \frac{1}{2} d\theta = \frac{1}{2} \theta + C = \frac{1}{2} \arctan \frac{x}{2} + C$$

Write down this formula $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a}$

l) This type of antiderivative, one needs to use the formula $1 + \sinh^2 \theta = \cosh^2 \theta$, using variable change $x = 2 \sinh \theta$, $dx = 2 \cosh \theta d\theta$,

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2 \cosh \theta}{\sqrt{4+4 \sinh^2 \theta}} d\theta = \int \frac{2 \cosh \theta}{\sqrt{4 \cosh^2 \theta}} d\theta = \int \frac{2 \cosh \theta}{2 \cosh \theta} d\theta = \int d\theta = \theta + C = \operatorname{arcsinh} \frac{x}{2} + C = \sinh^{-1} \frac{x}{2} + C$$

Write down this formula $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1} \frac{x}{a} + C$

m) Using partial fraction

$$\int \frac{dx}{4-x^2} = \int \frac{dx}{(2+x)(2-x)} = \int \left(\frac{1/4}{2+x} + \frac{1/4}{2-x} \right) dx = \int \frac{1/4}{2+x} dx + \int \frac{1/4}{2-x} dx = \frac{1}{4} \ln |2+x| - \frac{1}{4} \ln |2-x| + C$$

n) partial fraction again

$$\int \frac{dx}{4x-x^2} = \int \frac{dx}{x(4-x)} = \int \left(\frac{1/4}{x} + \frac{1/4}{4-x} \right) dx = \int \frac{1/4}{x} dx + \int \frac{1/4}{4-x} dx = \frac{1}{4} \ln |x| - \frac{1}{4} \ln |4-x| + C$$

o) TO BE COMPLETED

p) using integration by parts once and using the equality $\sin^2 x + \cos^2 x = 1$

$$\int \sin^2 x dx = \int \sin x \cdot \sin x dx = \int \sin x (-1) d \cos x = -\sin x \cdot \cos x + \int \cos x d \sin x = -\sin x \cdot \cos x + \int \cos^2 x dx = -\sin x \cdot \cos x + \int (1 - \sin^2 x) dx = -\sin x \cdot \cos x + \int 1 dx - \int \sin^2 x dx \Rightarrow 2 \int \sin^2 x dx = -\sin x \cdot \cos x + x + C \Rightarrow \int \sin^2 x dx = -\frac{1}{2} \sin x \cos x + \frac{x}{2} + C = \frac{x}{2} - \frac{1}{4} \sin 2x + C$$

A simple way, using $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2 \cos x \sin x$, $(\cos 2x)^2 = (1 + \cos 4x)/2$

$$\text{Thus } \int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

q) using the equality $\sin^2 x + \cos^2 x = 1$

$$\int \sin^3 x dx = \int \sin^2 x \cdot \sin x dx = \int (1 - \cos^2 x)(-1) d \cos x = -\int (1 - \cos^2 x) d \cos x = -\int 1 d \cos x + \int \cos^2 x d \cos x = -\cos x + \frac{1}{3} \cos^3 x + C$$

r) using integration by parts will be too complicated, using $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2 \cos x \sin x$, $(\cos 2x)^2 = (1 + \cos 4x)/2$

$$\text{Thus } \int \sin^4 x = \int \left(\frac{1-\cos 2x}{2} \right)^2 dx = \int \frac{1-2 \cos 2x + \cos^2 2x}{4} dx = \int \frac{1-2 \cos 2x + 1 - \sin^2 2x}{4} dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x - \frac{1}{4} \sin^2 2x \right) dx = \frac{x}{2} - \frac{1}{4} \sin 2x - \frac{1}{8} \int \sin^2 2x d(2x)$$

$$\xrightarrow{\text{using 3 i) } x \rightarrow 2x} \frac{x}{2} - \frac{1}{4} \sin 2x - \frac{1}{8} \left(\frac{2x}{2} - \frac{\sin 4x}{4} \right) = \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

Question 5 Solution

Using variable substitution $u = \frac{\pi}{2} - x$, $du = -dx$

$$\int_{\pi/2}^0 \frac{\sin(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)} (-1) du = -\int_{\pi/2}^0 \frac{\cos u}{\cos u + \sin u} du = \int_0^{\pi/2} \frac{\cos u}{\cos u + \sin u} du$$

both u and x are integral variables changing from 0 to $\frac{\pi}{2}$, one may replace them with θ , thus $\int_0^{\pi/2} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta =$

$$\int_0^{\pi/2} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta \text{ on one hand, on the other hand } \int_0^{\pi/2} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta + \int_0^{\pi/2} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta = \int_0^{\pi/2} \left(\frac{\sin \theta}{\sin \theta + \cos \theta} + \frac{\cos \theta}{\sin \theta + \cos \theta} \right) d\theta =$$

$$\int_0^{\pi/2} 1 d\theta = \frac{\pi}{2}.$$

$$\text{Thus } \int_0^{\pi/2} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta = \int_0^{\pi/2} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta = \frac{\pi}{4}$$

Question 6 Solution

a) convergent by p -test $\int_1^\infty \frac{1}{x^2} dx = \int_0^\infty \left(-\frac{1}{x}\right)' dx = -\frac{1}{x} \Big|_1^\infty = 1$

b) divergent by p -test $\int_1^\infty \frac{dx}{x} = \ln x \Big|_1^\infty = \infty$

c) $\int_1^\infty \frac{dx}{x-1} = \int_0^\infty \frac{dy}{y}$ divergent by p -test

d) divergent by p -test $\int_0^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_0^1 = \infty$

e) convergent by p -test $\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_0^1 = 2$

f) divergent by p -test Since both $\int_{-1}^0 \frac{dx}{x}$ and $\int_0^1 \frac{dx}{x}$ diverges.

Question 7 Solution

Using substitution $r^2 - 2rx + a^2 = y$, $-2r dx = dy$, since x changes from $-a$ to a , y changes from $r^2 - 2r(-a) + a^2 = r^2 + 2ra + a^2 = (r+a)^2$ to $r^2 - 2ra + a^2 = (r-a)^2$

$$V(r) = \frac{q}{2a} \int_{-a}^a \frac{dx}{\sqrt{r^2 - 2rx + a^2}} \xrightarrow{x \rightarrow y} \frac{q}{2a} \int_{(r+a)^2}^{(r-a)^2} \frac{-\frac{1}{2r} dy}{\sqrt{y}} = -\frac{q}{4ar} \cdot 2\sqrt{y} \Big|_{(r+a)^2}^{(r-a)^2}$$

$$= -\frac{q}{2ar} (|r-a| - |r+a|) = \begin{cases} \frac{q}{r} & \text{if } r \geq a \\ \frac{q}{a} & \text{if } 0 \leq r \leq a \end{cases}$$

Question 8 Solution

Divide the water into many layers, each is a rectangle with length l , width w , and height dz , put the origin ($z = 0$) on the top layer (downward), the work done is $\int_0^h g \cdot l \cdot w \cdot z dz = g \cdot l \cdot w \cdot \frac{1}{2} z^2 \Big|_0^h = \frac{1}{2} glwh^2$ (where g is the acceleration due to gravity). Actually the center of the mass is on the level $\frac{h}{2}$, the total mass is lwh , assume $\rho = 1$, pumping the water out the top is equivalent to move the center of the mass from $z = \frac{h}{2}$ to the top $z = 0$, thus $\frac{1}{2} glwh^2$.

Substitute values, the work done is $\frac{1}{2} g \cdot 2 \cdot 1 \cdot 0.5^2 = \frac{1}{4} g$

From the formula $\frac{1}{2} glwh^2$, it is clear that if the width is double, the work is also doubled. If the height is doubled, the work is multiplied by 4.

Question 9 Solution

a) On x -axis, since one ion is held fixed at $x = 0$, the distance is x , replace r in $F = -\frac{q^2}{r^2}$ with x , Work = $\int F(x) dx = \int_3^2 -\frac{q^2}{x^2} dx = \frac{q^2}{x} \Big|_3^2 = \frac{q^2}{2} - \frac{q^2}{3} = \frac{q^2}{6}$

b) On x -axis, since one ion is held fixed at $x = 1$, the distance becomes $x - 1$, replace r in $F = -\frac{q^2}{r^2}$ with $x - 1$, Work = $\int F(x) dx = \int_3^2 -\frac{q^2}{(x-1)^2} dx = \frac{q^2}{x-1} \Big|_3^2 = \frac{q^2}{2-1} - \frac{q^2}{3-1} = \frac{q^2}{2}$

c) Add the results in a) and b) together Work = $\frac{q^2}{6} + \frac{q^2}{2} = \frac{2}{3} q^2$. ie, the work can be calculated with respect to A ($x = 0$) and B ($x = 1$), respectively, then put together.

d) Divide the rod into many small pieces, each has width Δw , here we use w to denote the position of small pieces (w changes from 0 to 1, it overlaps $x = 0$ to $x = 1$), for each piece the charge is $q\Delta w$, and the force $F(x) = -\frac{q \cdot q\Delta w}{(x-w)^2}$, work contributed by each piece = $\int_3^2 F(x) dx = \int_3^2 -\frac{q^2 \Delta w}{(x-w)^2} dx = \frac{q^2 \Delta w}{x-w} \Big|_3^2 = \left(\frac{1}{2-w} - \frac{1}{3-w} \right) q^2 \Delta w$. Then we need a second integral for w from 0 to 1 to sum all the pieces, Total Work = $\int_0^1 \left(\frac{1}{2-w} - \frac{1}{3-w} \right) q^2 \Delta w \xrightarrow{\Delta w \rightarrow dw} \int_0^1 \left(\frac{1}{2-w} - \frac{1}{3-w} \right) q^2 dw = q^2 [-\ln(2-w) + \ln(3-w)] \Big|_0^1 = q^2 \ln \frac{4}{3} \approx 0.28 q^2$ (this result is reasonable since it is larger than $\frac{q^2}{6} \approx 0.16 q^2$ (case a) and smaller than $\frac{q^2}{2} \approx 0.5 q^2$ (case b), cases a and b are two extreme cases (if we put all charge to one end of the rod), given that in the three cases (a,b,d) the total charge is the same.

Question 10 Solution

$$f(x) = \cosh x$$

a) arclength = $\int_{-1}^1 \sqrt{1 + [f'(x)]^2} dx = \int_{-1}^1 \sqrt{1 + [\cosh(x)]^2} dx = \int_{-1}^1 \sqrt{1 + \sinh^2 x} dx = \int_{-1}^1 \sqrt{\cosh^2 x} dx = \int_{-1}^1 \cosh x dx = \sinh x \Big|_{-1}^1 = \sinh 1 - \sinh(-1) = 2 \sinh 1$

b) surface = $\int_{-1}^1 2\pi f(x) \sqrt{1 + [\cosh(x)]^2} dx = 2\pi \int_{-1}^1 \cosh^2 x dx = 2\pi \int_{-1}^1 \left(\frac{e^x + e^{-x}}{2} \right)^2 dx = \frac{2\pi}{4} (e^2 - e^{-2} + 4)$
 ($\int \cosh^2 x dx = \frac{1}{2} \cosh x \sinh x + \frac{1}{2} x$ see review problem for 2nd midterm exam.)

Question 11 Solution

$m = \int_a^b f(x) dx$, $\bar{x} = \frac{1}{m} \int x f(x) dx$, $\bar{y} = \frac{1}{m} \int \frac{1}{2} f^2(x) dx$ in (a,c,d)

or $m = \int_a^b [f(x) - g(x)] dx$, $\bar{x} = \frac{1}{m} \int x [f(x) - g(x)] dx$, $\bar{y} = \frac{1}{m} \int \frac{1}{2} [f^2(x) - g^2(x)] dx$ in (b)

$(\bar{x}, \bar{y}) =$ (a) $(\frac{3}{2}, \frac{6}{5})$ (b) $(\frac{3}{4}, \frac{12}{5})$ (c) $(0, \frac{2}{5})$ (d) $(\infty, \frac{1}{4})$

(d) $m = \int_0^\infty \frac{1}{1+x^2} dx = \arctan x \Big|_0^\infty = \frac{\pi}{2}$

variable change $x = \tan \theta$, $dx = (1 + \tan^2 \theta) d\theta$, x changes from 0 to ∞ , corresponding to θ changes from 0 to $\frac{\pi}{2}$ since $x = \tan \theta$, $\tan 0 = 0$, $\tan \frac{\pi}{2} = \infty$

$$m = \int_0^\infty \frac{1}{1+x^2} dx = \int_0^{\frac{\pi}{2}} \frac{1 + \tan^2 \theta}{1 + \tan^2 \theta} d\theta = \int_0^{\frac{\pi}{2}} 1 d\theta = \arctan x \Big|_0^\infty = \frac{\pi}{2}$$

$$\bar{x} = \frac{\int_0^\infty x f(x) dx}{m} = \frac{\int_0^\infty \frac{x}{1+x^2} dx}{\frac{\pi}{2}} = \frac{\int_0^\infty \frac{\frac{1}{2} dx^2}{1+x^2}}{\frac{\pi}{2}} = \frac{\frac{1}{2} \ln(1+x^2) \Big|_0^\infty}{\frac{\pi}{2}} = \infty !!!$$

The area is finite, it has an infinitely large \bar{x}

$$\bar{y} = \frac{\int_0^\infty \frac{1}{2} f^2(x) dx}{m} = \frac{1}{m} \cdot \int_0^\infty \frac{1}{2} f^2(x) dx = \frac{1}{\frac{\pi}{2}} \int_0^\infty \frac{1}{2} \left(\frac{1}{1+x^2} \right)^2 dx = \frac{2}{\pi} \int_0^\infty \frac{\frac{1}{2}}{(1+x^2)^2} dx = \frac{2}{\pi} \int_0^\infty \frac{\frac{1}{2} + \frac{1}{2} x^2 - \frac{1}{2} x^2}{(1+x^2)^2} dx$$

$$= \frac{2}{\pi} \left(\int_0^\infty \frac{\frac{1}{2}(1+x^2)}{(1+x^2)^2} dx - \int_0^\infty \frac{\frac{1}{2} x^2}{(1+x^2)^2} dx \right) = \frac{2}{\pi} \left(\int_0^\infty \frac{\frac{1}{2}}{1+x^2} dx - \int_0^\infty \frac{\frac{1}{2} x \cdot x}{(1+x^2)^2} dx \right)$$

$$\begin{aligned}
&= \frac{2}{\pi} \left(\frac{1}{2} \arctan x \Big|_0^\infty - \int_0^\infty \frac{\frac{1}{2}x \cdot \frac{1}{2} dx^2}{(1+x^2)^2} \right) = \frac{2}{\pi} \left[\frac{1}{2} \cdot \frac{\pi}{2} + \int_0^\infty \frac{x}{4} d\left(\frac{1}{1+x^2}\right) \right] = \frac{2}{\pi} \left(\frac{\pi}{4} + \frac{x}{4} \cdot \frac{1}{1+x^2} \Big|_0^\infty - \frac{1}{4} \int_0^\infty \frac{1}{1+x^2} dx \right) \\
&= \frac{2}{\pi} \left(\frac{\pi}{4} + \frac{x}{4} \cdot \frac{1}{1+x^2} \Big|_0^\infty - \frac{1}{4} \int_0^\infty \frac{1}{1+x^2} dx \right) = \frac{2}{\pi} \left(\frac{\pi}{4} + \frac{x}{4} \cdot \frac{1}{1+x^2} \Big|_0^\infty - \frac{1}{4} \arctan x \Big|_0^\infty \right) \\
&= \frac{2}{\pi} \left(\frac{\pi}{4} + 0 - \frac{1}{4} \cdot \frac{\pi}{2} \right) = \frac{2}{\pi} \cdot \frac{\pi}{8} = \frac{1}{4}
\end{aligned}$$

Question 12 Solution

$f(t) = ce^{-ct}$, where $c = \frac{1}{1000}$ and $t \geq 0$

a) $\text{Prob}(0 \leq t \leq 200) = \int_0^{200} ce^{-ct} dt = -e^{-ct} \Big|_0^{200} = 1 - e^{-\frac{1}{5}} \approx 0.18$

b) $\text{Prob}(t \geq 800) = \int_{800}^\infty ce^{-ct} dt = -e^{-ct} \Big|_{800}^\infty = e^{-\frac{4}{5}} \approx 0.45$

Question 13 Solution

We need to show that $\int_0^1 f(x) dx = 1$

$$\begin{aligned}
\int_0^1 f(x) dx &= \int_0^1 \frac{1}{\pi \sqrt{x(1-x)}} dx \xrightarrow{x=u+\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi \sqrt{(u+\frac{1}{2})(1-u-\frac{1}{2})}} du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi \sqrt{(\frac{1}{2}+u)(\frac{1}{2}-u)}} du \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi \sqrt{\frac{1}{4}-u^2}} du \xrightarrow{u=\frac{1}{2} \sin \theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi \cdot \frac{1}{2} \cos \theta} \cdot \frac{1}{2} \cos \theta d\theta = \frac{1}{\pi} \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1
\end{aligned}$$

Differential Equations

Question 14 Solution

a) $y' = -2y, y_0 = 1$ (standard exponential decay) $\Rightarrow y = C \cdot e^{-2t}$.

$y_0 = 1 \Rightarrow C = 1 \Rightarrow y(t) = e^{-2t}$ and $\lim_{t \rightarrow \infty} y(t) = 0$

b) $y' = 1 - 2y \Rightarrow \frac{dy}{dt} = 1 - 2y$ separation of variables $\Rightarrow \frac{dy}{1-2y} = dt$ integrate both sides $\Rightarrow -\frac{1}{2} \ln |1 - 2y| = t + C$
 $\Rightarrow 1 - 2y = C \cdot e^{-2t} \Rightarrow y = \frac{1 - C \cdot e^{-2t}}{2}$

$y_0 = 0 \Rightarrow C = 1 \Rightarrow y = \frac{1 - e^{-2t}}{2}$ and $\lim_{t \rightarrow \infty} y(t) = \frac{1}{2}$

c) $y' = 1 - y^2 \Rightarrow \frac{dy}{dt} = (1+y)(1-y)$ separation of variables $\Rightarrow \frac{dy}{(1+y)(1-y)} = dt$ partial fraction $\Rightarrow \frac{\frac{1}{2}}{1+y} dy + \frac{\frac{1}{2}}{1-y} dy = dt$ integrate both sides $\Rightarrow \frac{1}{2} \ln |1 + y| - \frac{1}{2} \ln |1 - y| = t + C \Rightarrow \ln \left| \frac{1+y}{1-y} \right| = 2t + C \Rightarrow \frac{1+y}{1-y} = C \cdot e^{2t}$ where C is constant may be positive or negative $\Rightarrow y(t) = \frac{C \cdot e^{2t} - 1}{C \cdot e^{2t} + 1}$

$y_0 = 0 \Rightarrow C = 1 \Rightarrow y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$. Furthermore $y(t) = \frac{(e^{2t} - 1)e^{-t}}{(e^{2t} + 1)e^{-t}} = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \tanh t$

Check... $y(t) = \tanh t$ is the solution.

$\lim_{t \rightarrow \infty} y(t) = 1$

d) $y' = -ty \Rightarrow \frac{dy}{dt} = -ty$ separation of variables $\Rightarrow \frac{dy}{y} = -tdt$ integrate both sides $\Rightarrow \ln |y| = -\frac{1}{2}t^2 + C \Rightarrow y = C e^{-\frac{1}{2}t^2}$

$y_0 = 1 \Rightarrow C = 1 \Rightarrow y(t) = e^{-\frac{1}{2}t^2}$

$\lim_{t \rightarrow \infty} y(t) = 0$

Question 15 Solution

a) $y = c_1 e^t + c_2 e^{-t} \Rightarrow y' = c_1 e^t - c_2 e^{-t} \Rightarrow y'' = c_1 e^t + c_2 e^{-t} = y$, thus it is a solution of $y'' = y$ for any constants c_1, c_2 .

b) $y(0) = 1, y'(0) = 0 \Rightarrow c_1 + c_2 = 1, c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2} \Rightarrow y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh x$

c) $y(0) = 0, y'(0) = 1 \Rightarrow c_1 + c_2 = 0, c_1 - c_2 = 1 \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2} \Rightarrow y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh x$

Question 16 Solution

a) $y' = ky$. Solve the equation, we have $y(t) = y_0 e^{kt}$. $200 = y(30) = y_0 e^{30k}$ and $800 = y(90) = y_0 e^{90k}$. Therefore, $y_0 = 10^2 = 100$ cells.

b) $200 = 100e^{30k}$, so $k = \frac{\ln 2}{30}$. Therefore, $y(t) = 100 \cdot 2^{t/30}$. Solve the equation $6400 = 100 \cdot 2^{t/30}$. Then $t = 30 \ln 64 / \ln 2 = 30 \cdot 6 = 180$ hours.

Question 17 Solution

$y(t) = y_0 e^{-kt}$

$y(t) = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4 \times 10^{-4}}}$

$30 = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4 \times 10^{-4}}}$

$t = 1.4 \times 10^{-4} \frac{\ln \frac{3}{4}}{\ln \frac{1}{2}} = 0.581 \times 10^{-4} s$

Question 18 Solution

$y' = \frac{2500 - 20y}{10000}$

$y' = \frac{1}{500}(125 - y)$ Newton's heating/cooling $y' = k(T - y)$

$y(t) = T + (y_0 - T)e^{-kt} = 125 + (y_0 - 125)e^{-\frac{t}{500}}$
 approach to 125 kg.

Question 19 Solution

$y(t) = T + (y_0 - T)e^{-kt}$ Note that the patient's temperature is T , $y_0 = 70^\circ\text{F}$

$$\begin{cases} 95 = T + (70 - T)e^{-k} \\ 100 = T + (70 - T)e^{-2k} \end{cases}$$

$$\left(\frac{95-T}{70-T}\right)^2 = \frac{100-T}{70-T}$$

 $T = 101.25^\circ\text{ F}$

Question 20 Solution

$y' = ky(M - y)$
 $y(t) = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}$
 $y_0 = 10, M = 4000$
 $y(t) = \frac{40000}{10 + (4000 - 10)e^{-4000kt}}$
 measure time in days
 $20 = \frac{40000}{10 + (4000 - 10)e^{-4000 \cdot 7k}}$
 $e^{-k} = \left(\frac{199}{399}\right)^{\frac{1}{28000}}$
 $y(t) = \frac{40000}{10 + 3990\left(\frac{199}{399}\right)^{\frac{t}{7}}}$
 let $y(t) = \frac{1}{2} \cdot 4000 = 2000$, solve $t = \frac{7 \ln 399}{\ln 399 - \ln 199} \approx 60$ days.

Question 21 Solution

TO BE COMPLETED

Series

Question 22 Solution

- a) divergent $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ by p -test of series, $p = 1$.
 b) convergent since $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1 < \infty$.
 c) convergent by p -test of series, $p = 2$.
 d) convergent by Alternating Series Test, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$, $a_{n+1} < a_n$ and the sign is alternating.
 e) divergent by Ratio Test, $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = 2 > 1$, ($L > 1$ divergent)

Question 23 Solution

- a) $0.111111\dots = 0.1 + 0.01 + 0.001 + 0.0001 + \dots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \dots = \sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{9}$
 b) $0.1212121212\dots = \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \dots = \sum_{n=1}^{\infty} \frac{12}{100^n} = \frac{12}{100} \sum_{n=0}^{\infty} \frac{1}{100^n} = \frac{12}{100} \cdot \frac{1}{1-\frac{1}{100}} = \frac{12}{99}$
 c) $0.4999999\dots = 0.45 + 0.045 + 0.0045 + 0.00045 + \dots = \frac{45}{100} + \frac{45}{1000} + \frac{45}{10000} + \dots$
 $= \frac{45}{100} \left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots\right) = \frac{45}{100} \sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{45}{100} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{2}$ (ie, $0.4999999\dots = 0.5$)

Question 24 Solution

- a) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$, where $x = 2$.
 b) $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$ (note that n starts from 1)
 c) $\sum_{n=1}^{\infty} \frac{1}{n3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n} \Big|_{x=\frac{1}{3}} = \sum_{n=1}^{\infty} \int x^{n-1} dx \Big|_{x=\frac{1}{3}} = \int \sum_{n=1}^{\infty} x^{n-1} dx \Big|_{x=\frac{1}{3}} = \int \sum_{n=0}^{\infty} \int x^n dx \Big|_{x=\frac{1}{3}} = \int \frac{1}{1-x} dx \Big|_{x=\frac{1}{3}} = -\ln(1-x) \Big|_{x=\frac{1}{3}} = \ln \frac{1}{1-x} \Big|_{x=\frac{1}{3}} = \ln \frac{3}{2}$
 (Note that $\ln \frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots = \sum_{n=1}^{\infty} \frac{1}{n}x^n$)

Question 25 Solution

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, note that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ includes all the odd terms. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \text{odd terms} + \text{even terms} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

Question 26 Solution

- a) Use $|s - s_{10}| \leq \int_{10}^{\infty} f(x)dx$ since all terms are positive, where $f(x) = \frac{1}{x^2}$
- $$|s - s_{10}| \leq \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty} \frac{1}{x^2}dx = -\frac{1}{x} \Big|_{10}^{\infty} = 0.1$$
- b) Use $|s - s_{10}| \leq a_{n+1}$ where $a_{n+1} = \frac{1}{(n+1)^2}$ since the series is an alternating series.
- $$|s - s_{10}| \leq a_{n+1} = \frac{1}{11^2} = \frac{1}{121}$$

Question 27 Solution

Assume the dog starts with A running towards B, it will take $\frac{20}{10+2}$ hr to meet B, during this time interval A and B traveled $2 \cdot \frac{20}{10+2}$, respectively and the dog traveled $10 \cdot \frac{20}{10+2} = \frac{50}{3} = 25 \cdot \frac{2}{3}$. Then the dog will run from B towards A, the distance between A and B becomes $20 - 2 \cdot \frac{20}{10+2} - 2 \cdot \frac{20}{10+2} = \frac{40}{3}$. Everything is the same except 20 replaced by $\frac{40}{3}$, this time the dog will travel $\frac{10}{10+2} \cdot \frac{40}{3} = \frac{100}{9} = 25 \cdot \left(\frac{2}{3}\right)^2$

The series is $25 \cdot \frac{2}{3} + 25 \cdot \left(\frac{2}{3}\right)^2 + 25 \cdot \left(\frac{2}{3}\right)^3 + 25 \cdot \left(\frac{2}{3}\right)^4 + \dots = 25 \cdot \frac{2}{3} \cdot \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right) = 25 \cdot \frac{2}{3} \cdot \frac{1}{1-\frac{2}{3}} = 50$

The question requires to express D as an infinite series, actually a simple way to find the sum is that using distance = speed \times time, where time = $\frac{20}{2+2} = 5$ hr, total time it will take for the two students to meet, then the distance = $10 \times 5 = 50$ miles.

Question 28 Solution

Consider an arbitrary sequence of a final win :

$$1, -1, -1, -1, 1, -1, 1, -1, \dots, 1, 1$$

where -1 denotes lose, 1 denotes win. The sequence satisfies following properties:

- 1) It has even number of elements, denote the length of the sequence as $2n$ (n round), the sum equals 2, it is required that $a_{2n-1} = 1$ and $a_{2n} = 1$. The sequence must have a length of an even number. This is equivalent to, if you generate a sequence with -1 and 1 , and the sum of the sequence is 2, the length of the sequence has to be an even number.
- 2) In this sequence, $a_{2i-1} = 1$ and $a_{2i} = 1$ ($i < n$) does not exist, otherwise, the game stops at i round rather than n rounds.
- 3) In this sequence, $a_{2i-1} = -1$ and $a_{2i} = -1$ ($i < n$) does not exist, otherwise, the game stops at i round rather than n rounds.
- 4) Thus in each round, there are two cases: either $\{a_{2i-1} = 1, a_{2i} = -1\}$ or $\{a_{2i-1} = -1, a_{2i} = 1\}$. In other words, in each round, a_{2i-1} and a_{2i} have opposite sign, for $i < n$. After each round ($i < n$), the score comes back to zero.
- 5) The possibility in each round $p(1-p) + (1-p)p = 2p(1-p)$.
- 6) The possibility for a sequence with a length $2n$ is $[2p(1-p)]^{n-1}p^2$.

The total possibility to a final win is

$$\sum_{n=1}^{\infty} [2p(1-p)]^{n-1}p^2 = \sum_{n=0}^{\infty} [2p(1-p)]^n p^2 = p^2 \sum_{n=0}^{\infty} [2p(1-p)]^n = p^2 \cdot \frac{1}{1-2p(1-p)} = \frac{p^2}{1-2p+2p^2}$$

$$p = \frac{1}{2} \Rightarrow \frac{p^2}{1-2p+2p^2} = \frac{1}{2}$$

$$p = \frac{1}{4} \Rightarrow \frac{p^2}{1-2p+2p^2} = \frac{1}{10}$$

$$p = \frac{3}{4} \Rightarrow \frac{p^2}{1-2p+2p^2} = \frac{9}{10}$$

Question 29 Solution

a) The total length removed = $\frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} + \dots = \frac{1}{3} \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right] = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1$

b) The number of intervals is a sequence: $2, 4, 8, \dots, 2^n, \dots$, and $\lim_{n \rightarrow \infty} 2^n = \infty$.

Power Series, Taylor Series

Question 30 Solution

a) $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| = |x| < 1 \Rightarrow$ the radius of convergence is 1; since at two end points $x = \pm 1$, the series diverges, the interval of convergence is $-1 < x < 1$. The sum is $\frac{1}{1-x}$ for $-1 < x < 1$.

b) $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n x^{n+1}}{2^{n+1} x^n} \right| = \left| \frac{x}{2} \right| < 1 \Rightarrow |x| < 2 \Rightarrow$ the radius of convergence is 2; since at two end points $x = \pm 2$, the series diverges, the interval of convergence is $-2 < x < 2$. The sum is $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$ for $-2 < x < 2$.

c) $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \right| = |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2 \Rightarrow$ the radius of convergence is 1 (the length of the interval divided by 2); since at two end points $x = 0$ and 2 , the series diverges, the interval of convergence is $0 < x < 2$. The sum is $\frac{1}{1-(x-1)} = \frac{1}{2-x}$ for $0 < x < 2$.

d) $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{nx^{n+1}}{(n+1)x^n} \right| = |x| < 1 \Rightarrow -1 < x < 1 \Rightarrow$ the radius of convergence is 1; since at $x = 1$, the series is harmonic series thus diverges, while at $x = -1$, the series converges by AST (alternating series test), the interval of convergence is $-1 \leq x < 1$. Note that $x^n = \int nx^{n-1} dx \Rightarrow \frac{x^n}{n} = \int x^{n-1} dx$ the sum is $\sum_{n=1}^{\infty} \int x^{n-1} dx = \int \sum_{n=1}^{\infty} x^{n-1} dx =$

$$\int \sum_{n=0}^{\infty} \int x^n dx = \int \frac{1}{1-x} dx = -\ln(1-x) = \ln \frac{1}{1-x} \text{ for } -1 \leq x < 1$$

$$\ln \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

e) $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = |x| < 1 \Rightarrow -1 < x < 1 \Rightarrow$ the radius of convergence is 1; since at $x = \pm 1$, the series diverges, the interval of convergence is $-1 < x < 1$.

Note that $(x^n)' = nx^{n-1}$, $\sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x \sum_{n=1}^{\infty} (x^n)' = x \cdot \left(\sum_{n=0}^{\infty} x^n \right)' = x \cdot \left(\frac{1}{1-x} \right)' = x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$$

Question 31 Solution

Namely find c_n , such that $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} c_n(x - \frac{1}{2})^n$.

$$\frac{1}{1-x} = \frac{1}{1-(x-\frac{1}{2}+\frac{1}{2})} = \frac{1}{1-(x-\frac{1}{2})-\frac{1}{2}} = \frac{1}{\frac{1}{2}-(x-\frac{1}{2})} = \frac{1}{2} \cdot \frac{1}{1-2(x-\frac{1}{2})} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} (2(x-\frac{1}{2}))^n = \sum_{n=0}^{\infty} 2^{n+1}(x-\frac{1}{2})^n$$

Question 32 Solution

$$f(x) = \sinh x \Rightarrow f(0) = \sinh 0 = 0$$

$$f'(x) = \cosh x \Rightarrow f'(0) = \cosh 0 = 1$$

$$f''(x) = \sinh x \Rightarrow f''(0) = \sinh 0 = 0$$

$$f'''(x) = \cosh x \Rightarrow f'''(0) = \cosh 0 = 1$$

⋮

$$\sinh x = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x = (\sinh x)' = \left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \right)' = \sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{(2n+1)!} \right)' = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Question 33 Solution

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\begin{aligned} \sin^2 x + \cos^2 x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)^2 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)^2 \\ &= \left(x^2 - 2\frac{x^4}{3!} + 2\frac{x^6}{5!} + \dots \right) + \left(1 - 2\frac{x^2}{2} + \frac{x^4}{4} + 2\frac{x^4}{4!} + \dots \right) = 1 \end{aligned}$$

Question 34 Solution

Because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 +$

$$T_1(x) = T_0(x) = 1 \text{ and } T_2(x) = 1 - x^2$$

Question 35 Solution

Show that $0 \leq f(x) < 1$, $\lim_{x \rightarrow \infty} f(x) = 1$, $\lim_{x \rightarrow 0^+} f^{(n)}(x) = P(\frac{1}{x})e^{-1/x}$, where $P(\frac{1}{x})$ is a polynomial of $\frac{1}{x}$, when $x \rightarrow 0^+$ $e^{-1/x} \rightarrow 0$ exponentially (faster than any polynomial) thus $f^{(n)}(x) \rightarrow 0$ regardless of the form of $P(\frac{1}{x})$.

Question 36 Solution

$$\sqrt{x} = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a) - \frac{1}{8}a^{-\frac{3}{2}}(x-a)^2 + \text{Remainder}$$

set $a = 9$

$$\sqrt{x} = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \text{Remainder}$$

This is an alternating series, $|s - s_n| \leq a_{n+1}$, ie $|\sqrt{x} - (3 + \frac{1}{6}(x-9))| \leq \frac{1}{216}(x-9)^2$

set $x = 10$

$$|\sqrt{10} - 3.16666| \leq 0.00463 < 0.005$$

The approximate value is 3.16666

Question 37 Solution

Since $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, an alternating series, thus $|s - s_n| \leq a_{n+1}$ where $a_{n+1} = \frac{x^{n+1}}{n+1}$, let $\frac{x^{n+1}}{n+1} = 10^{-3}$ (need to evaluate $\ln \frac{3}{2} = \ln(1 + \frac{1}{2})$, ie, $x = \frac{1}{2}$), substitute $x = \frac{1}{2}$ yields $\frac{1}{2^{n+1}} = 0.001(n+1)$, test with $n = 1, 2, 3, 4, 5, 6, 7$, find that $n = 6$, roughly satisfies the equality. $n = 6$, $s_6 \approx 0.4047$, exact value $s \approx 0.4055$, error is within 10^{-3}

Question 38 Solution

The first two nonzero terms

- $\tan x = x + \frac{x^3}{3} + \text{Remainder}$
 - $e^{-x} \sin x = x - x^2 + \text{Remainder}$
 - $\frac{1-\cos x}{x} = \frac{1}{2}x - \frac{1}{24}x^3 + \text{Remainder}$
-

Question 39 Solution

$$f(x) = \frac{x}{e^x - 1}, B_0 = f(0) = 1, B_1 = f'(0) = -\frac{1}{2}, B_2 = f''(0) = \frac{1}{6} \text{ (using L'Hospital rule).}$$

Question 40 Solution

- $f(x) = x, f(0) = 0, f'(0) = 1$
- $f(x) = \sin x, f(0) = 0, f'(x) = \cos x, f'(0) = 1$
- $f(x) = \ln(1+x), f(0) = 0, f'(x) = \frac{1}{1+x}, f'(0) = 1$
- $f(x) = e^x - 1, f(0) = 0, f'(x) = e^x, f'(0) = 1$

If the functions are sketched in a neighborhood of $x = 0$, the order they appear (from top to bottom), consider their Taylor approximations

$$x = x, \sin x = x - \frac{1}{6}x^3 + \dots, \ln(1+x) = x - \frac{1}{2}x^2 + \dots, e^x - 1 = x + \frac{1}{2}x^2$$

Thus on right hand side of 0, from top to bottom, $e^x - 1, x, \sin x$ and $\ln(1+x)$; on left hand side of 0, from top to bottom, $e^x - 1, \sin x, x$, and $\ln(1+x)$.

Question 41 Solution

- a) $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \dots$ it is alternating series.

$$\int_0^1 \left(1 - \frac{x^2}{4}\right) dx = \frac{11}{12}$$

$$\text{error bound } \int_0^1 \frac{x^4}{64} dx = \frac{x^5}{5 \cdot 64} = \frac{1}{320}$$

$$b) J_0(x)' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n} (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+1}}{2^{2n+2} ((n+1)!)^2}$$

$$J_0(x)'' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n-2}}{2^{2n} (n!)^2}$$

$$x J_0(x)'' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n-1}}{2^{2n} (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n+1}}{2^{2n+2} ((n+1)!)^2}$$

$$x J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n} (n!)^2}$$

$$x J_0(x)'' + J_0(x)' + x J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2^{2n} (n!)^2} \left[\frac{2n+2}{2^2 (n+1)^2} + \frac{(2n+1)(2n+2)}{2^2 (n+1)^2} - 1 \right] = 0$$

Thus $J_0(x)$ satisfies $xy'' + y' + xy = 0$

Question 42 Solution

$$a) f(t)' = \left(\sum_{n=0}^{\infty} t^n \right)' = 1 + t + 2t + 3t^2 + 4t^3 + \dots = \sum_{n=0}^{\infty} (n+1)t^n$$

$$f^2(t) = \left(\sum_{n=0}^{\infty} t^n \right)^2 = 1 + t + 2t + 3t^2 + 4t^3 + \dots = \sum_{n=0}^{\infty} (n+1)t^n$$

Thus $f(t)' = f^2(t)$, $f(t)$ is solution of $y' = y^2$ and $f(0) = 1$

$$b) y' = y^2 \Rightarrow \frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow \int \frac{dy}{y^2} = \int dt \Rightarrow -\frac{1}{y} = t + C \Rightarrow y = \frac{1}{-C-t}$$

$$t = 0, y = 1 \Rightarrow C = -1, \text{ thus } y = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \text{ Geometric Series.}$$

Question 43 Solution

a) $\int_0^\infty \frac{\sin x}{x} dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx = \int_0^\pi \frac{\sin x}{x} dx + \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx \leq \int_0^\pi \frac{\sin x}{x} dx + \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{n\pi} dx$ since in each interval $x \geq n\pi$
 $\int_0^\infty \frac{\sin x}{x} dx \leq \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^\infty \frac{1}{n} \int_{n\pi}^{(n+1)\pi} \sin x dx = \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^\infty \frac{-\cos x}{n} \Big|_{n\pi}^{(n+1)\pi} = \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^\infty \frac{\cos n\pi - \cos(n+1)\pi}{n} =$
 $\int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^\infty (-1)^n \frac{2}{n} = \int_0^\pi \frac{\sin x}{x} dx + \frac{2}{\pi} \sum_{n=1}^\infty \frac{(-1)^n}{n}$ converges since the $n = 0$ term is finite (note that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$) and by AST.

b) $\int_0^\infty \left| \frac{\sin x}{x} \right| dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx = \int_0^\pi \left| \frac{\sin x}{x} \right| dx + \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx \geq \int_0^\pi \frac{\sin x}{x} dx + \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{(n+1)\pi} dx$ since in each interval $x \leq n(+1)\pi$
 $\int_0^\infty \frac{\sin x}{x} dx \geq \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^\infty \frac{1}{n+1} \int_{n\pi}^{(n+1)\pi} |\sin x| dx = \int_0^\pi \frac{\sin x}{x} dx + \frac{1}{\pi} \sum_{n=1}^\infty \frac{2}{n+1} = \int_0^\pi \frac{\sin x}{x} dx + \frac{2}{\pi} \sum_{n=1}^\infty \frac{1}{n+1}$ diverges since the series is Harmonic Series.

Question 44 Solution

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \dots$; it is a convergent alternating series, so $|\cos x - 1| \leq \frac{1}{2}x^2$, and $|\cos x - (1 - \frac{1}{2}x^2)| \leq \frac{1}{4!}x^4$; setting $x = \frac{\pi}{5}$ gives the result

Question 45 Solution

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x (1 - t^2 + \frac{t^4}{2} - \dots) dt = \frac{2}{\sqrt{\pi}} (x - \frac{x^3}{3} + \frac{x^5}{10} - \dots)$$

Question 46 Solution

a) $\frac{a}{a+b} = \frac{a}{b} \cdot \frac{1}{1+\frac{a}{b}} = \frac{a}{b} \cdot \frac{1}{1-\left(-\frac{a}{b}\right)} = \frac{a}{b} \cdot \sum_{n=0}^\infty \left(-\frac{a}{b}\right)^n$
 $\frac{a}{a+b} = \frac{a}{b} \left(1 - \frac{a}{b} + \frac{a^2}{b^2} - \dots\right) = \frac{a}{b} - \frac{a^2}{b^2} + \frac{a^3}{b^3} - \dots$

b) using the Theorem

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \dots \text{ for } -1 < x < 1$$

$$\sqrt{R^2 - r^2} = R\sqrt{1 - \frac{r^2}{R^2}} = R\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} = R\left[1 - \frac{1}{2} \cdot \frac{r^2}{R^2} + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2} \left(-\frac{r^2}{R^2}\right)^2 + \dots\right] = R - \frac{r}{2} \cdot \frac{r}{R} - \frac{r}{8} \cdot \frac{r^3}{R^3} + \dots$$

Question 47 Solution

Starting from the formula derived in class, $f(x) = f(a) + \dots$, replace $x \rightarrow x+h, a \rightarrow x, \dots$

Question 48 Solution

a) let $y = 0 \Rightarrow x = \pm(1+\epsilon)$, let $x = 0 \Rightarrow y = \pm 1$

b) Solve $y \Rightarrow y = f(x) = \pm\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}$

$$A(\epsilon) = 2 \int_{-1-\epsilon}^{1+\epsilon} \sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2} dx = 4 \int_0^{1+\epsilon} \sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2} dx$$

c) $A(\epsilon) = 4 \int_0^{1+\epsilon} \sqrt{1 - \frac{1}{2} \cdot \left(\frac{x}{1+\epsilon}\right)^2} dx = 4(1+\epsilon) \int_0^{1+\epsilon} \sqrt{1 - \frac{1}{2} \cdot \left(\frac{x}{1+\epsilon}\right)^2} d\frac{x}{1+\epsilon} = 4(1+\epsilon) \int_0^1 \sqrt{1 - u^2} du = 4(1+\epsilon) \frac{\pi}{4} = (1+\epsilon)\pi$

The first two nonzero terms are $\pi + \pi\epsilon$.

Question 49 Solution

$$V(x) = \frac{Gm_1}{|x-x_1|} + \frac{Gm_2}{|x-x_2|} \text{ for } x \rightarrow \infty \text{ i.e., } x > x_1 \text{ and } x > x_2 \Rightarrow V(x) = \frac{Gm_1}{x-x_1} + \frac{Gm_2}{x-x_2}$$

Using the hint set $y = 1/x$, i.e., $x = 1/y$ and expand the potential in powers of y

$$V(1/y) = \frac{Gm_1}{1/y-x_1} + \frac{Gm_2}{1/y-x_2} = \frac{Gm_1 y}{1-x_1 y} + \frac{Gm_2 y}{1-x_2 y}$$

Using Geometric Series Formula $\frac{1}{1-x_i y} = \sum_{n=0}^\infty (x_i y)^n$ where $i = 1, 2$

$$V(1/y) = Gm_1 y \sum_{n=0}^\infty (x_1 y)^n + Gm_2 y \sum_{n=0}^\infty (x_2 y)^n = Gm_1 \sum_{n=0}^\infty x_1^n y^{n+1} + Gm_2 \sum_{n=0}^\infty x_2^n y^{n+1} = (Gm_1 + Gm_2)y + (Gm_1 x_1 + Gm_2 x_2)y^2 + (Gm_1 x_1^2 + Gm_2 x_2^2)y^3 + \dots$$

change $y = \frac{1}{x}$ back

$$V(x) = (Gm_1 + Gm_2)\frac{1}{x} + (Gm_1 x_1 + Gm_2 x_2)\frac{1}{x^2} + (Gm_1 x_1^2 + Gm_2 x_2^2)\frac{1}{x^3} + \dots$$

Thus $a = (Gm_1 + Gm_2)$, $b = (Gm_1 x_1 + Gm_2 x_2)$, and $c = (Gm_1 x_1^2 + Gm_2 x_2^2)$

Question 50 Solution

- a) TO BE COMPLETED
 b) TO BE COMPLETED

- c) $(1+i)^3 = (1+i)^2(1+i) = 2i(1+i) = -2+2i = 2\sqrt{2}e^{\frac{3\pi}{4}i}$ ($x = -2, y = 2$)
d) $\frac{1}{1+i} = \frac{1 \cdot (1-i)}{(1+i)(1-i)} = \frac{1-i}{1-i^2} = \frac{1-i}{1-(-1)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i = \frac{1}{2\sqrt{2}}e^{-\frac{\pi}{4}i}$ ($x = \frac{1}{2}, y = -\frac{1}{2}$)
e) $\sqrt{1+i} = (\sqrt{2}e^{\frac{\pi}{4}i})^{\frac{1}{2}} = 2^{\frac{1}{4}}e^{\frac{\pi}{8}i} = 2^{\frac{1}{4}}(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8}) = 2^{\frac{1}{4}}\cos \frac{\pi}{8} + i 2^{\frac{1}{4}}\sin \frac{\pi}{8}$ ($x = 2^{\frac{1}{4}}\cos \frac{\pi}{8}, y = 2^{\frac{1}{4}}\sin \frac{\pi}{8}$)

Question 54 Solution

- a) See 49 c) $(1+i)^6 = 1 + 6 \cdot i + 15 \cdot i^2 + 20 \cdot i^3 + 15 \cdot i^4 + 6 \cdot i^5 + i^6 = 1 + 6i - 15 - 20i + 15 + 6i - 1 = -8i$
b) $1+i = \sqrt{2}e^{\frac{\pi}{4}i}$, since $x = 1, y = 1, r = \sqrt{x^2+y^2} = \sqrt{2}, \theta = \arctan \frac{y}{x} = \arctan 1 = \frac{\pi}{4}, re^{\theta i} = \sqrt{2}e^{\frac{\pi}{4}i}$
 $(1+i)^6 = (\sqrt{2}e^{\frac{\pi}{4}i})^6 = 2^{\frac{6}{2}}e^{\frac{6\pi}{4}i} = 2^3e^{\frac{3\pi}{2}i} = 8(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) = -8i$

Question 55 Solution

- a) $z^2 + 2z - 2 = 0 \Rightarrow a = 1, b = 2, c = -2, z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$ two real roots
b) $z^2 + 2z + 2 = 0 \Rightarrow a = 1, b = 2, c = 2, z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1} = -1 \pm i$
c) $z^2 = (re^{\theta i})^2 = r^2e^{2\theta i} = 1 \Rightarrow r = 1, \theta = \pi k$, where k is any integer $\Rightarrow z_{1,2} = \pm 1$
d) $z^3 = r^3e^{3\theta i} = 1 \Rightarrow r = 1, \theta = \frac{2\pi}{3}k$, where k is any integer \Rightarrow three roots: $z_1 = 1$ and $z_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
e) $z^4 = r^4e^{4\theta i} = 1 \Rightarrow z^2 = \pm 1 \Rightarrow$ four roots: $z_{1,2} = \pm 1, z_{3,4} = \pm i$
f) $e^z = 1$ on real axis there is on root $z = 0$, but in complex plane there are infinite roots, let $z = x + yi, e^z = e^{x+yi} = e^x(\cos y + i \sin y) = 1 \Rightarrow x = 0$ and $y = 2k\pi$, where k is any integer, roots are $z_k = 2k\pi i$.

Question 56 Solution

$$(\cos \theta + i \sin \theta)^n = (e^{\theta i})^n = e^{n\theta i} = \cos n\theta + i \sin n\theta$$

Question 57 Solution

- $\int e^{ax} \cos bx \, dx = \int e^{ax} \frac{1}{b} d \sin bx = \frac{1}{b} e^{ax} \cdot \sin bx - \int \frac{1}{b} \sin bx de^{ax} = \frac{1}{b} e^{ax} \cdot \sin bx - \int \frac{a}{b} e^{ax} \sin bx \, dx = \frac{1}{b} e^{ax} \cdot \sin bx - \int \frac{a}{b} e^{ax} \left(-\frac{1}{b}\right) d \cos bx = \frac{1}{b} e^{ax} \cdot \sin bx + \int \frac{a}{b^2} e^{ax} d \cos bx = \frac{1}{b} e^{ax} \cdot \sin bx + \frac{a}{b^2} e^{ax} \cdot \cos bx - \frac{a}{b^2} \int \cos bx de^{ax} = \frac{1}{b} e^{ax} \cdot \sin bx + \frac{a}{b^2} e^{ax} \cdot \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx \Rightarrow \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \cdot \sin bx + \frac{a}{b^2} e^{ax} \cdot \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx \Rightarrow \left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \cdot \sin bx + \frac{a}{b^2} e^{ax} \cdot \cos bx \Rightarrow \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (b \cdot \sin bx + a \cdot \cos bx) + C$
 $\int e^{ax} \sin bx \, dx = \int e^{ax} \left(-\frac{1}{b}\right) d \cos bx = -\frac{1}{b} e^{ax} \cdot \cos bx + \int \frac{1}{b} \cos bx de^{ax} = -\frac{1}{b} e^{ax} \cdot \cos bx + \int \frac{a}{b} e^{ax} \cos bx \, dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \int \frac{a}{b} e^{ax} \cdot \frac{1}{b} d \sin bx = -\frac{1}{b} e^{ax} \cdot \cos bx + \int \frac{a}{b^2} e^{ax} d \sin bx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx - \frac{a}{b^2} \int \sin bx de^{ax} = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx \, dx \Rightarrow \left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \sin bx \, dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx \Rightarrow \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cdot \sin bx - b \cdot \cos bx) + C$
b) $e^{(a+ib)x} = e^{ax+ibx} = e^{ax} \cdot e^{ibx} = e^{ax} (\cos bx + i \sin bx)$
 $\int e^{(a+ib)x} \, dx = \frac{1}{a+ib} e^{(a+ib)x} = \frac{1 \cdot (a-ib)}{(a+ib)(a-ib)} e^{(a+ib)x} = \frac{a-ib}{a^2+b^2} e^{(a+ib)x}$
since $\int e^{(a+ib)x} \, dx = \int (e^{ax} \cos bx + i e^{ax} \sin bx) \, dx = \int e^{ax} \cos bx \, dx + i \int e^{ax} \sin bx \, dx = \frac{a-ib}{a^2+b^2} e^{(a+ib)x} = \frac{a-ib}{a^2+b^2} (e^{ax} \cos bx + i e^{ax} \sin bx)$
 $\frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + i \left[\frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right]$
Thus $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$ and $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

Question 58 Solution

- a) since $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x \Rightarrow e^{ix} + e^{-ix} = 2 \cos x \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$
b) $e^{ix} - e^{-ix} = 2i \sin x \Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$
c) $\frac{d}{dx} \cos x = \frac{d}{dx} \left(\frac{e^{ix} + e^{-ix}}{2} \right) = \frac{ie^{ix} - ie^{-ix}}{2} = \frac{(ie^{ix} - ie^{-ix})i}{2i} = \frac{-e^{ix} + e^{-ix}}{2i} = -\sin x$
d) $\frac{d}{dx} \sin x = \frac{d}{dx} \left(\frac{e^{ix} - e^{-ix}}{2i} \right) = \frac{ie^{ix} + ie^{-ix}}{2i} = \frac{e^{ix} + e^{-ix}}{2} = \cos x$
e) $1 = e^{ix} \cdot e^{-ix} (\cos x + i \sin x) \cdot (\cos x - i \sin x) = \cos^2 x - i^2 \sin^2 x = \cos^2 x + \sin^2 x$
f) $e^{2xi} = \cos 2x + i \sin 2x$
 $e^{2xi} = e^{xi} \cdot e^{xi} = (\cos x + i \sin x)(\cos x + i \sin x) = \cos^2 x + i \cdot 2 \cos x \sin x + i^2 \sin^2 x = \cos^2 x - \sin^2 x + i \cdot 2 \cos x \sin x$
Thus $\cos 2x = \cos^2 x - \sin^2 x$ g) and $\sin 2x = 2 \sin x \cos x$