

You may use the integrals: $\int x^n dx = \frac{x^{n+1}}{n+1}$ ($n \neq -1$), $\int \frac{dx}{x} = \ln x$, $\int e^x dx = e^x$, $\int \sin x dx = -\cos x$, $\int \cos x dx = \sin x$, $\int \sec \theta d\theta = \ln(\sec \theta + \tan \theta)$, $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta))$, but all others should be derived.

1. **True or False.** Justify your answer with a reason or counterexample.

a) $\sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$ b) $\sum_{i=1}^n \frac{a_i}{b_i} = \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}$

c) $\lim_{n \rightarrow \infty} (\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2}) = \frac{1}{2}$

d) $\sum_{i=1}^{n^2} i = \frac{n^2(n+1)^2}{4}$ d) $\lim_{n \rightarrow \infty} \sum_{i=1}^n (1 + \frac{i}{n}) \frac{1}{n} = \frac{3}{2}$

e) If the integral $\int_a^b f(x) dx$ is computed using the right-hand Riemann sum and the number of intervals n is doubled, then the error in the approximation is approximately also doubled.

f) In evaluating an integral $\int_a^b f(x) dx$, the right-hand Riemann sum is more accurate than the midpoint rule.

g) If $f(x) = c_1 + c_2x$, then the midpoint rule for $\int_a^b f(x) dx$ is exact.

h) If $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, then $\lim_{n \rightarrow \infty} \sum_{i=1}^n f'(x_i) \Delta x = f(b) - f(a)$.

i) $\frac{d}{dx} \int_0^x \sqrt{1+t} dt = \sqrt{1+x^2}$ j) $\int_0^\infty e^{-x} \cos x dx = \int_0^\infty e^{-x} \sin x dx$

k) $\int_0^1 (1-x^2) dx \geq \int_0^1 \sqrt{1-x^2} dx$

l) A spring has natural length 10 cm. If 2 Joule of work is needed to stretch it from length 10 cm to 15 cm, then 4 Joule of work is needed to stretch it from length 10 cm to 20 cm.

m) If $\lim_{x \rightarrow \infty} f(x) = 0$, then the improper integral $\int_1^\infty f(x) dx$ converges.

n) If $0 \leq f(x) \leq g(x)$ for $x \geq 1$ and $\int_1^\infty g(x) dx$ converges, then $\int_1^\infty f(x) dx$ also converges.

o) The error function defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is concave down for $x > 0$.

appendix E, sections 5.1 - 5.4 (integrals)

2. Derive the formula $\int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3)$ two ways: (a) Riemann sums, (b) FTC.

3. Express the integral as a limit of Riemann sums and evaluate the limit. Check your answer using the FTC. a) $\int_0^2 x dx$ b) $\int_0^1 x^3 dx$ c) $\int_0^1 e^{-x} dx$

4. Evaluate the limit (by any means).

a) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1/n}{1+i/n}$ b) $\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x f(t) dt$ c) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$ d) $\lim_{r \rightarrow 1} \frac{r^{10} - 1}{r - 1}$ e) $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

5. Find the antiderivative.

a) $\int x e^{-x^2} dx$ b) $\int x^2 e^{-x} dx$ c) $\int x \sin x dx$ d) $\int \frac{dx}{4-x^2}$ e) $\int \frac{dx}{\sqrt{4-x^2}}$ f) $\int \sqrt{4-x^2} dx$

6. Prove. a) $\frac{1}{20} \leq \int_0^1 \frac{x^9}{1+x} dx \leq \frac{1}{10}$ b) $\int_0^1 x(1-x)^{11} dx = \frac{1}{156}$

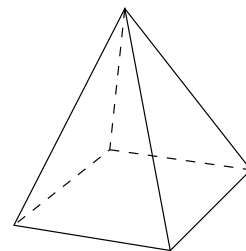
section 6.4 (work)

7. A cable of length L meter is hanging from the top of a building. The cable has cross-sectional area A meter² and density ρ kg/meter³. a) Find the work needed to raise the cable to the top of the building. b) If the length of the cable is doubled, is the work also doubled?

8. A force of 30 N is needed to maintain a spring at length 15 cm when it is stretched from its natural length of 12 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

9. Two ions repel each other with a force $f(r) = -\frac{q^2}{r^2}$, where q is the ion charge and r is the distance between the ions. a) If one ion is held fixed at $x = 0$, find the work done in moving the second ion from $x = 3$ to $x = 2$. b) If one ion is held fixed at $x = 1$, find the work done in moving the second ion from $x = 3$ to $x = 2$. c) In which case is the work greater?

10. A pyramid (sketched) is built of stone with mass density ρ kg/m³. The base of the pyramid is a square, and the vertex is directly above the center of the base. The length of a side of the base is L meter and the height of the vertex above the base is H meter. a) Derive a formula for the work done in building the pyramid (i.e. raising the stone from ground level). b) If the length L and height H are doubled, by what factor does the work increase?



section 8.8 (improper integrals)

11. Determine whether the integral converges or diverges. If it converges, find the value. If it diverges, give a reason.

a) $\int_1^{\infty} \frac{dx}{x^4}$ b) $\int_0^{\infty} x^2 e^{-x} dx$ c) $\int_0^{\infty} e^{-x} \sin x dx$ d) $\int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$
 e) $\int_1^{\infty} \frac{dx}{1+x^2}$ f) $\int_1^{\infty} \frac{dx}{\sqrt{1+x^2}}$ g) $\int_1^{\infty} \frac{dx}{1-x^2}$ h) $\int_0^1 \frac{dx}{1-x}$ i) $\int_0^1 \frac{dx}{1-x^2}$ j) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

12. Prove. $\int_0^{\infty} \frac{\ln x}{1+x^2} dx = 0$ (hint: substitute $u = \frac{1}{x}$)

13. A patient receives intravenous medicine at the rate $r(t) = 2te^{-2t}$ ml/sec, where t is the time in seconds since the treatment started. a) Find the total dose the patient receives in the limit $t \rightarrow \infty$. b) What fraction of the total dose is received in the first 5 seconds?

section 9.1 (arclength)

14. Find the arclength of the curve on the interval $0 \leq x \leq 1$.

a) $y = \sqrt{1-x^2}$ b) $y = \int_0^x \sqrt{1-t^2} dt$ c) $y = \frac{e^x + e^{-x}}{2}$ d) $y = \sqrt{x^3}$

15. Sketch the curve $y = \sqrt{2x-x^2}$ for $0 \leq x \leq 2$ and find its arclength.

section 9.2 (surface area)

16. Find the surface area obtained by rotating the curve about the x -axis.

a) $y = x^3$, $0 \leq x \leq 1$ b) $y = \sqrt{1-x}$, $0 \leq x \leq 1$ c) $y = \frac{e^x + e^{-x}}{2}$, $0 \leq x \leq 1$

d) $y = \left(\frac{r_2 - r_1}{x_2 - x_1}\right)(x - x_1) + r_1$, $x_1 \leq x \leq x_2$ (this gives the surface area of a truncated cone)

17. Consider the curve $y = e^x$ between $x = 0$ and $x = 1$. Find an expression for the surface area formed by rotating the curve about the (a) x -axis, (b) y -axis. Leave the answers in the form of integrals (do not evaluate). Which surface area is larger?

18. Let S be the surface area of a zone on a sphere between two parallel planes. Show that $S = 2\pi rd$, where r is the radius of the sphere and d is the distance between the planes. (Note that S depends only on the distance between the planes, not on their location.)

19. An ellipse is defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (i) Sketch the ellipse in the xy -plane (assume $a > b > 0$). (ii) Set up integrals giving the area A of the ellipse, the arclength

L of the ellipse, and the surface area S obtained by rotating the ellipse about the x -axis.
(iii) Show that $A = \pi ab$ and $S = 2\pi b(b + a(\sin^{-1} c)/c)$, where $c = \sqrt{a^2 - b^2}/a$. Check your answers in the limit $a \rightarrow b$. (note: L cannot be evaluated in terms of elementary functions; it requires special functions defined in terms of elliptic integrals.)