Math 156  Review Sheet for 1st Midterm Exam  Fall 2019

For full credit, justify your answer, and give the units if appropriate. You may use the following integrals, but all others should be derived.

\[\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1), \quad \int \frac{dx}{x} = \ln x, \quad \int e^x dx = e^x, \quad \int \sin x dx = -\cos x, \quad \int \cos x dx = \sin x\]

\[\int \sec x dx = \ln(\sec x + \tan x), \quad \int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln(\sec x + \tan x))\]

1. True or False. Justify your answer with a reason or counterexample.
   a) \[\sum_{i=1}^{12} 2i = 156\]
   b) \[\sum_{i=1}^{12} \left( \frac{1}{i} - \frac{1}{i+1} \right) = \frac{12}{13}\]
   c) \[\sum_{i=0}^{n} (n-i)^2 = \sum_{i=0}^{n} i^2\]
   d) \[\left( \sum_{i=1}^{n} i \right)^2 = \sum_{i=1}^{n} i^3\]
   e) \[1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2\]
   f) If \[\int_{a}^{b} f(x) dx > 0, \text{ then } f(x) > 0\] for \(a \leq x \leq b\).
   g) If an integral \[\int_{a}^{b} f(x) dx\] is computed using the right-hand Riemann sum and the number of intervals \(n\) is doubled, then the error is approximately also doubled.
   
2. Express the integral as a limit of Riemann sums, evaluate the limit, and check by the FTC.
   a) \[\int_{0}^{x} x^2 dx\]
   b) \[\int_{0}^{1} x^3 dx\]
   c) \[\int_{1}^{a} x^2 dx\]
   d) \[\int_{0}^{1} e^{-x} dx\]

3. Evaluate the limit by any means.
   a) \[\lim_{x \to \infty} xe^{-x}\]
   b) \[\lim_{n \to \infty} \sum_{i=1}^{n} \left( 1 + \frac{i}{n} \right)^3 \frac{1}{n}\]
   c) \[\lim_{x \to 0} \int_{0}^{x} f(t) dt\]
   d) \[\lim_{x \to 0} \frac{e^x - 1}{x}\]
   e) \[\lim_{r \to 1} \frac{1 - r^{11}}{1 - r}\]

4. Find the antiderivative.
   a) \[\int xe^{-x^2} dx\]
   b) \[\int x^2 e^{-x} dx\]
   c) \[\int x \sin x dx\]
   d) \[\int \frac{dx}{4 - x^2}\]
   e) \[\int \frac{dx}{\sqrt{4 - x^2}}\]
   f) \[\int \sqrt{4 - x^2} dx\]

5. Prove.
   a) \[\frac{1}{20} \leq \int_{0}^{1} \frac{x^9}{1 + x} dx \leq \frac{1}{10}\]
   b) \[\int_{1}^{0} x (1 - x)^{11} dx = \frac{1}{156}\]
section 1.5 work
6. A force of 30 N is needed to stretch a spring from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

7. Two identical ions of charge \( q \) repel each other with force \( f(r) = -\frac{q^2}{4\pi\epsilon_0 r^2} \) N, where \( \epsilon_0 \) is the vacuum permittivity and \( r \) is the distance between the ions in meters. If one ion is held fixed at \( x = 0 \) mm, find the work done in moving the second ion from \( x = 3 \) mm to \( x = 2 \) mm.

8. A pyramid is built of stone with density \( \rho \) kg/m\(^3\). The base of the pyramid is a square, and the vertex is directly above the center of the base. The length of a side of the base is \( L \) m and the height of the vertex above the base is \( H \) m. a) Derive a formula for the work done in building the pyramid (i.e. raising the stone from ground level to its level in the pyramid). b) If the length \( L \) and height \( H \) are doubled, by what factor does the work increase? c) Which requires more work, building the lower half or the upper half of the pyramid?

section 1.6 improper integrals
9. Determine whether the integral converges or diverges. If it converges, find the value. If it diverges, give a reason.

   a) \( \int_1^\infty \frac{dx}{x^4} \)    b) \( \int_0^\infty x^2 e^{-x} dx \)    c) \( \int_0^\infty e^{-x} \sin x dx \)    d) \( \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \)    e) \( \int_r^\infty \sqrt{r^2 - x^2} dx \)

   f) \( \int_{-r}^r \frac{dx}{\sqrt{r^2 - x^2}} \)    g) \( \int_1^\infty \frac{dx}{1 + x^2} \)    h) \( \int_1^\infty \frac{dx}{\sqrt{1 + x^2}} \)    i) \( \int_1^\infty \frac{x}{\sqrt{1 + x^2}} dx \)    j) \( \int_1^\infty \frac{dx}{x^2 - 1} \)

   k) \( \int_0^1 \frac{dx}{x} \)    l) \( \int_0^1 \frac{dx}{x^{3/2}} \)    m) \( \int_0^1 \frac{dx}{1 - x} \)    n) \( \int_0^\infty \frac{\ln x}{1 + x^2} dx \) (hint: substitute \( u = x^{-1} \))

10. A patient receives an intravenous drug at the rate \( r(t) = 2te^{-2t} \) ml/sec, where \( t \) is the time in seconds since the treatment started. (a) Find the total dose the patient receives in the limit \( t \to \infty \). (b) What fraction of the total dose is received in the first 5 seconds?

section 1.7 arclength
11. Find the arclength of the curve on the interval \( 0 \leq x \leq 1 \).

   a) \( y = \sqrt{1 - x^2} \)    b) \( y = \int_0^x \sqrt{1 - t^2} dt \)    c) \( y = \frac{e^x + e^{-x}}{2} \)    d) \( y = \sqrt{x^3} \)    e) \( y = 2x^2 \)

12. Sketch the curve \( y = \sqrt{2x - x^2} \) for \( 0 \leq x \leq 2 \) and find its arclength.

13. Sketch the curve \( y = \sqrt{x} \) for \( 0 \leq x \leq 1 \) and find its arclength. (hint: substitute \( y = \sqrt{x} \) in the arclength integral)

miscellaneous
14. Sketch the graph of each function for \( 0 \leq x \leq 2\pi \). What do you notice about (d) and (e)?

   a) \( \cos x \)    b) \( \cos 2x \)    c) \( \frac{1}{2} \cos 2x \)    d) \( \frac{1}{2} + \frac{1}{2} \cos 2x \)    e) \( \cos^2 x \)

15. The average value of a function \( f(x) \) on the interval \( a \leq x \leq b \) is \( f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx \), and the root mean square value is \( f_{rms} = \sqrt{\frac{1}{b-a} \int_a^b (f(x))^2 dx} \). Find \( f_{avg} \) and \( f_{rms} \) for \( f(x) = \cos x \) on the interval \( 0 \leq x \leq 2\pi \). (hint: do problem 14 first)