

1a. True. We showed in class that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, so $\sum_{i=1}^{12} (2i) = 2 \sum_{i=1}^{12} i = 2 \frac{12 \cdot 13}{2} = 156$. (Of course this can be done by brute force without too much effort.)

1b. True. $\sum_{i=1}^{12} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{12} - \frac{1}{13} \right) = 1 - \frac{1}{13} = \frac{12}{13}$

1c. True. The two sums contain the same terms in the opposite order. (This could be justified with the substitution $j = n - i$.) Alternatively, using our known formula for $\sum_{i=0}^n i$, we have

$$\sum_{i=0}^n (n-i)^2 = \sum_{i=0}^n (n^2 - 2in + i^2) = n^2 \sum_{i=0}^n 1 - 2n \sum_{i=0}^n i + \sum_{i=0}^n i^2 = n^2(n+1) - 2n \frac{n(n+1)}{2} + \sum_{i=0}^n i^2 = \sum_{i=0}^n i^2.$$

1d. True. In class we derived $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$, so $\left(\sum_{i=1}^n i \right)^2 = \left(\frac{n(n+1)}{2} \right)^2 = \sum_{i=1}^n i^3$.

1e. True. $1 + 3 + 5 + 7 + \cdots + (2n-1) = \sum_{i=1}^n (2i-1) = 2 \sum_{i=1}^n i - \sum_{i=1}^n 1 = 2 \cdot \frac{n(n+1)}{2} - n = n^2$

1f. False. Consider the counterexample $f(x) = x, a = -1, b = 2$.

1g. False. As we increase the number of intervals using any Riemann sum the error will decrease, not increase. In the case of the right-hand Riemann sum, if the number of intervals is doubled, the error is approximately cut in half.

1h. False. One needs to use the chain rule, $\frac{d}{dx} \int_0^{x^3} \sqrt{1+t^2} dt = \sqrt{1+x^6} \cdot (x^3)' = 3x^2 \sqrt{1+x^6}$.

1i. True. Apply integration by parts to the first integral, $\int u dv = uv - \int v du$, set $u = e^{-x}, dv = \cos x$, so $du = -e^{-x} dx, v = \sin x$. Then we have

$$\int_0^{\infty} e^{-x} \cos x dx = e^{-x} \sin x \Big|_0^{\infty} + \int_0^{\infty} e^{-x} \sin x dx = 0 - 0 + \int_0^{\infty} e^{-x} \sin x dx = \int_0^{\infty} e^{-x} \sin x dx.$$

1j. False. In general it requires 4 times the work to displace a spring twice as far from equilibrium. In this case, $2 = \int_0^5 kx dx = \frac{25}{2}k$, so $k = .16$. Then the work required to stretch the spring from 10 cm to 20 cm is $\int_0^{10} .16x dx = 50(.16) = 8$ joules.

1k. False. For an integral to be proper, both the integrand and the interval must be bounded.

1l. False. There are many counterexamples; the simplest is $f(x) = 1/x$. Clearly, $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, but $\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln b = \infty$ which shows that $\int_0^{\infty} \frac{1}{x} dx$ diverges.

1m. True. $\int_1^{\infty} \frac{dx}{x^2} = -\frac{1}{x} \Big|_1^{\infty} = 0 - (-1) = 1$; the result also follows from the p -test with $p = 2$

1n. True. Using the comparison test, if $0 \leq f(x) \leq g(x)$ for $x \geq 1$, then $0 \leq \int_1^{\infty} f(x) dx \leq \int_0^{\infty} g(x) dx < \infty$. The last inequality follows from the fact that $\int_1^{\infty} g(x) dx$ converges.

1o. True. Note that $\operatorname{erf}(0) = 0$. Then by the FTC we have $\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$, so $\operatorname{erf}''(x) = \frac{2}{\sqrt{\pi}} (-2x) e^{-x^2}$, and hence $\operatorname{erf}''(0) = 0$.

1p. True method 1: sketch the graph of each function, argue that the area is the same by symmetry

method 2: use integration by parts, $u = \sin x, dv = \sin x \Rightarrow du = \cos x, v = -\cos x \Rightarrow \int_0^{\pi/2} \sin^2 x dx = \sin x \cdot -\cos x \Big|_{x=0}^{\pi/2} - \int_0^{\pi/2} -\cos^2 x dx = \int_0^{\pi/2} \cos^2 x dx$

method 3: use trigonometric identities, $\sin^2 x = \frac{1}{2}(1 - \cos 2x), \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

2a. FTC: $\int_0^2 x dx = \frac{x^2}{2} \Big|_0^2 = 2 - 0 = 2.$

Riemann Sum: $\lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \frac{2i}{n} = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{4}{n^2} \frac{n(n+1)}{2} = 2.$

2b. FTC: $\int_0^1 x^3 dx = \frac{x^4}{4} \Big|_0^1 = 1/4 - 0 = 1/4.$

Riemann Sum: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n i^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right]^2 = 1/4.$

2c. FTC $\int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1 = -e^{-1} - (-1) = 1 - e^{-1}.$

Riemann Sum: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} e^{-i/n} = \lim_{n \rightarrow \infty} \frac{1}{n} \frac{1-e^{-1}}{1-e^{-1/n}} = 1 - e^{-1}.$ L'Hôpital's Rule is used in the last step. NOTE: I used a left-hand sum to simplify some of the algebra, but you should get the same thing using a right hand sum, the algebra will just be a little uglier.

3a. Note that $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x.$

$$\begin{aligned} \int_a^b x^2 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^2 \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n (a + i\Delta x)^2 \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(a^2 + 2ia \frac{b-a}{n} + i^2 \frac{(b-a)^2}{n^2} \right) \frac{b-a}{n} \\ &= \lim_{n \rightarrow \infty} \left(a^2 \frac{b-a}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} 2a \sum_{i=1}^n i + \frac{(b-a)^3}{n^3} \sum_{i=1}^n i^2 \right) \\ &= \lim_{n \rightarrow \infty} \left(a^2 \frac{b-a}{n} n + \frac{(b-a)^2}{n^2} 2a \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \frac{n(n+1)(2n+1)}{6} \right) \\ &= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} \\ &= \frac{(b-a)(b^2 + ab + a^2)}{3} = \frac{b^3 - a^3}{3} \end{aligned}$$

3b. By FTC

$$\int_a^b x^2 dx = \frac{x^3}{3} \Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

4a. $\lim_{x \rightarrow \infty} x e^{-x} = 0 \cdot \infty = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0;$ we used l'Hôpital's rule

4b. Recognize the summation as a (right) Riemann sum for $f(x) = x^3$ on the interval $1 \leq x \leq 2.$ The limit is $\int_1^2 x^3 dx = [x^4/4]_1^2 = 15/4.$

4c. use l'Hôpital's rule : $\lim_{x \rightarrow 0} \frac{\int_0^x f(t) dt}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{1} = f(0);$ we assume f is continuous at 0

4d. use l'Hôpital's rule : $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^x}{1} = 1$

4e. use l'Hôpital's rule : $\lim_{r \rightarrow 1} \frac{1-r^{11}}{1-r} = \lim_{r \rightarrow 1} \frac{-11r^{10}}{-1} = 11$

or alternatively use the sum of a finite geometric series

$$\frac{1-r^{11}}{1-r} = \sum_{i=0}^{10} r^i \Rightarrow \lim_{r \rightarrow 1} \frac{1-r^{11}}{1-r} = \lim_{r \rightarrow 1} \sum_{i=0}^{10} r^i = \sum_{i=0}^{10} 1 = 11$$

5a. substitute : $u = -x^2 \quad du = -2x dx$ we have $\int x e^{-x^2} dx = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + c = -\frac{1}{2} e^{-x^2} + c.$

5b. Integrate by parts twice to reduce the integral:

$$\begin{aligned} \int x^2 e^{-x} dx & & u = x^2, dv = e^{-x} dx & \Rightarrow du = 2x dx, v = -e^{-x} \\ & = -x^2 e^{-x} - 2 \int x e^{-x} dx & u = x, dv = e^{-x} dx & \Rightarrow du = dx, v = -e^{-x} \\ & = -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C \end{aligned}$$

5c. Integrate by parts: $u = x$, $dv = \sin(x)dx$, then $du = dx$ and $v = -\cos(x)$

$$\int x \sin(x) dx = -x \cos(x) + \int \cos(x) dx = -x \cos(x) + \sin(x) + C$$

5d. Easiest method is partial fractions:

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{1/4}{2-x} + \frac{1/4}{2+x}.$$

So we get:

$$\int \frac{dx}{4-x^2} = \frac{1}{4} \int \frac{dx}{2-x} + \frac{1}{4} \int \frac{dx}{2+x} = \frac{1}{4} [\ln(2+x) - \ln(2-x)] = \frac{1}{4} \ln \left(\frac{2+x}{2-x} \right) + C$$

you could also use the trig. substitution $x = 2 \sin(\theta)$, but as you'll see if you try it, the algebra is much more difficult.

5e. Because of the presence of the square root we use a trig. substitution this time. Set $x = 2 \sin(\theta)$. Then $dx = 2 \cos(\theta) d\theta$ and we get:

$$\begin{aligned} \int \frac{dx}{\sqrt{4-x^2}} &= \int \frac{2 \cos(\theta) d\theta}{\sqrt{4-4 \sin^2(\theta)}} \\ &= \int \frac{\cos(\theta)}{\sqrt{1-\sin^2(\theta)}} = \int d\theta = \theta + C \end{aligned}$$

To complete the integration we need to invert the substitution to get back to the variable x . Solving our original substitution $x = 2 \sin(\theta)$ for θ give $\theta = \arcsin(x/2)$ so finally we have:

$$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin(x/2) + C$$

5f. Again we need to use a trig. substitution. Set $x = 2 \sin(t)$ so that $dx = 2 \cos(t) dt$. Plugging this in gives:

$$(1) \quad \int \sqrt{4-x^2} dx = \int \sqrt{4-4 \sin^2(t)} 2 \cos(t) dt = 4 \int \cos^2(t) dt$$

There are several routes to finding this antiderivative. We will use a useful trig. identity from your past:

$$\cos(2t) = \cos^2(t) - \sin^2(t) = 2 \cos^2(t) - 1 \implies \cos^2(t) = \frac{1}{2} + \frac{1}{2} \cos(2t)$$

Inserting this into the above integral (1) we have:

$$2 \int 1 + \cos(2t) dt = 2t + \sin(2t) = 2t + 2 \sin(t) \cos(t) + C$$

where we used the trig identity $\sin(2t) = 2 \sin(t) \cos(t)$ to simplify above. Now we use the original substitution $x = 2 \sin(t)$ to go back to the original variables: $\sin(t) = x/2$ so $\cos(t) = \sqrt{1-(x/2)^2}$ and $t = \arcsin(x/2)$. Plugging these in we get the final answer:

$$\int \sqrt{4-x^2} dx = 2 \arcsin \left(\frac{x}{2} \right) + x \sqrt{1 - \left(\frac{x}{2} \right)^2} + C.$$

6a. On the interval $0 \leq x \leq 1$, we have the following inequality: $1/2 \leq 1/(1+x) \leq 1$. Using this estimate it follows from the comparison test that:

$$\begin{aligned} \int_0^1 \frac{1}{2}x^9 dx &\leq \int_0^1 \frac{x^9}{1+x} dx \leq \int_0^1 x^9 dx \\ \frac{1}{20}x^{10} \Big|_0^1 &\leq \int_0^1 \frac{x^9}{1+x} dx \leq \frac{1}{10}x^{10} \Big|_0^1 \\ \frac{1}{20} &\leq \int_0^1 \frac{x^9}{1+x} dx \leq \frac{1}{10} \end{aligned}$$

6b. We could expand $(1-x)^{11}$, but we seek a simpler method; use a substitution. Set $u = 1-x$, so $x = 1-u$ and $dx = -du$. Our integral becomes upon substitution:

$$\begin{aligned} \int_0^1 x(1-x)^{11} dx &= \int_1^0 (1-u)u^{11}(-du) = - \int_1^0 u^{11} - u^{12} du \\ &= \int_0^1 u^{11} - u^{12} du = \frac{1}{12}u^{12} - \frac{1}{13}u^{13} \Big|_0^1 = \frac{1}{12} - \frac{1}{13} = \frac{1}{156}. \end{aligned}$$

NOTE: In the integral I had to change the boundaries of the integral when I made the substitution $u = 1-x$. Also, in the third step I used the property that $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

7a) Let x be a vertical coordinate with $x = 0$ at the top of the building and $x = L$ at the bottom of the cable. Consider a slice of the cable at position x_i . The volume of the slice is $A\Delta x$; the mass of the slice is $\rho A\Delta x$; the force acting on the slice is $\rho g A\Delta x$; the work done in pulling the slice to the top of the building is $\rho g A\Delta x \cdot x_i$. Hence the total work is $W = \lim_{n \rightarrow \infty} \rho g A x_i \Delta x = \int_0^L \rho g A x dx = \frac{1}{2} \rho g A L^2$.

7b) No, the work is not also doubled. If the length of the cable is doubled from L to $2L$, then the work becomes $W = \frac{1}{2} \rho g A (2L)^2 = 2 \rho g A L^2$, which is four times the work done in raising a cable of length L , not double.

8. Using Hooke's Law, we have that $30N = k(12cm - 15cm)$ so $k = 10 \frac{N}{cm}$. To calculate the work to stretch from 12cm (the natural length) to 20cm we have $W = \int_0^8 kx dx = \int_0^8 10x dx = 5x^2 \Big|_0^8 = 320 N \cdot cm = 3.2J$.

9. If one ion is held fixed and the second ion is moved from distance r_1 to distance r_2 (relative to the first ion), then the work done is

$$W = \int_{r_1}^{r_2} -\frac{q^2}{4\pi\epsilon_0 r^2} dr = -\frac{q^2}{4\pi\epsilon_0} \int_{r_1}^{r_2} \frac{dr}{r^2} = -\frac{q^2}{4\pi\epsilon_0} \cdot -\frac{1}{r} \Big|_{r_1}^{r_2} = \frac{q^2}{4\pi\epsilon_0} \left(\frac{1}{r_2} - \frac{1}{r_1} \right).$$

The distances are $r_1 = 3$ mm, $r_2 = 2$ mm, so $W = \frac{q^2}{4\pi\epsilon_0} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{q^2}{4\pi\epsilon_0} \cdot \frac{1}{6} = \frac{q^2}{24\pi\epsilon_0}$ mJ.

10a. Draw a vertical x -axis with $x = 0$ at the base of the pyramid and $x = H$ at the top. Define $\Delta x = \frac{H}{n}$, $x_i = i\Delta x$, for $i = 0 : n$, where Δx is the width of a slice of the pyramid and x_i is the height of the i th slice. Each slice has the shape of a thin square box, so if l_i is the side length of the i th slice, then using similar triangles we see that $\frac{l_i}{H-x_i} = \frac{L}{H}$.

volume of i th slice = $l_i^2 \cdot \Delta x = (H-x_i)^2 \frac{L^2}{H^2} \Delta x$

mass of i th slice = $\rho(H-x_i)^2 \frac{L^2}{H^2} \Delta x$

force acting on i th slice = $\rho g(H-x_i)^2 \frac{L^2}{H^2} \Delta x$

work done on i th slice = force \times distance = $\rho g(H - x_i)^2 \frac{L^2}{H^2} \Delta x \cdot x_i$

total work = $W = \int_0^H \rho g(H - x)^2 x \frac{L^2}{H^2} dx = \rho g \frac{L^2}{H^2} \int_0^H (H - x)^2 x dx$

$$\int_0^H (H-x)^2 x dx = \int_0^H (H^2 x - 2Hx^2 + x^3) dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^H = H^4 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{H^4}{12}$$

so the work done in constructing the pyramid is $W = \rho g \frac{L^2}{H^2} \cdot \frac{H^4}{12} = \frac{1}{12} \rho g L^2 H^2$

10b. If L and H are doubled, then W increases by a factor of 16.

10c. Which requires more work, building the lower half or the upper half of the pyramid?

$$W_{lower} : \int_0^{H/2} (H-x)^2 x dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^{H/2} = H^4 \left(\frac{1}{2} \cdot \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{16} \right) = \frac{11}{192} H^4$$

$$W_{upper} : \int_{H/2}^H (H-x)^2 x dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_{H/2}^H = H^4 \left(\frac{1}{2} \cdot \frac{3}{4} - \frac{2}{3} \cdot \frac{7}{8} + \frac{1}{4} \cdot \frac{15}{16} \right) = \frac{5}{192} H^4$$

So more work is done building the lower half of the pyramid.

11a. $\int_1^\infty \frac{dx}{x^4}$: converges, p -test, $p = 4$; also $\int_1^\infty \frac{dx}{x^4} = \frac{1}{-3x^3} \Big|_1^\infty = 0 - \left(\frac{1}{-3} \right) = \frac{1}{3}$

11b. $\int_0^\infty x^2 e^{-x} dx$: converges

$$\begin{aligned} \int_0^\infty x^2 e^{-x} dx &= \int_0^\infty x^2 (-de^{-x}) = - \left[x^2 e^{-x} \Big|_0^\infty - \int_0^\infty e^{-x} 2x dx \right] \\ &= 2 \int_0^\infty e^{-x} x dx = -2 \int_0^\infty x de^{-x} = -2 \left[x e^{-x} \Big|_0^\infty - \int_0^\infty e^{-x} dx \right] \\ &= 2 \int_0^\infty e^{-x} dx = -2e^{-x} \Big|_0^\infty = 2 \end{aligned}$$

11c. $\int_0^\infty e^{-x} \sin x dx$ Converges by comparison to e^{-x} . Note that

$$\int_0^\infty e^{-x} \sin x dx = - \int_0^\infty e^{-x} d \cos x = -e^{-x} \cos x \Big|_0^\infty + \int_0^\infty -e^{-x} \cos x dx = 1 - \int_0^\infty e^{-x} \cos x dx$$

Using the result of problem (1i) above:

$$\int_0^\infty e^{-x} \sin(x) dx = \int_0^\infty e^{-x} \cos(x) dx$$

we have

$$\int_0^\infty e^{-x} \sin x dx = 1/2.$$

11d. $\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$ Converges. Don't fall into the trap of saying that each term diverges so the difference diverges; $\infty - \infty$ is inconclusive so you have to work harder. Integrating directly:

$$\begin{aligned} \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx &= [\ln x - \ln(1+x)] \Big|_1^\infty = \lim_{x \rightarrow \infty} \ln \left(\frac{x}{1+x} \right) + \ln 2 \\ &= \ln 1 + \ln 2 = \ln 2. \end{aligned}$$

11e. $\int_{-r}^r \sqrt{r^2 - x^2} dx$ Converges; it's a proper integral. To evaluate, use a trig substitution.

$$\sin \theta = \frac{x}{r}, \cos \theta d\theta = \frac{dx}{r}, \sqrt{r^2 - x^2} = \sqrt{r^2 - r^2 \sin^2 \theta} = \sqrt{r^2(1 - \sin^2 \theta)} = r \cos \theta$$

$$\begin{aligned} \int_{-r}^r \sqrt{r^2 - x^2} &= \int_{-\pi/2}^{\pi/2} \cos \theta \cdot r \cos \theta d\theta = r \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = r \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \frac{r}{2} \cdot \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_{-\pi/2}^{\pi/2} = \frac{r}{2} \cdot \left(\frac{\pi}{2} - \frac{1}{2} \sin \pi - \left(-\frac{\pi}{2} - \frac{1}{2} \sin(-\pi) \right) \right) = \frac{r}{2} \cdot \pi = \frac{\pi r}{2} \end{aligned}$$

11f. $\int_{-r}^r \frac{dx}{\sqrt{r^2 - x^2}}$ Converges; it's an improper integral. To evaluate use a trig substitution.

$$\sin \theta = \frac{x}{r}, \cos \theta d\theta = \frac{dx}{r}, \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{x^2}{r^2}} = \sqrt{\frac{r^2 - x^2}{r^2}} = \frac{1}{r} \sqrt{r^2 - x^2}$$

$$\int_{-r}^r \frac{dx}{\sqrt{r^2 - x^2}} = \int_{-\pi/2}^{\pi/2} \frac{r \cos \theta d\theta}{r \cos \theta} = \pi$$

11g. $\int_1^\infty \frac{dx}{1+x^2}$ Converges. Using the substitution $x = \tan t$ (so $dx = \sec^2 t dt$) we have

$$\int_1^\infty \frac{dx}{1+x^2} = \int_{\pi/4}^{\pi/2} \frac{\sec^2 t}{1+\tan^2 t} dt = \int_{\pi/4}^{\pi/2} 1 dt = \pi/4.$$

11h. $\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$ Diverges. We know this because $\frac{1}{\sqrt{1+t^2}} \sim \frac{1}{t}$ when t is large and later integral diverges. To prove this we use comparison. We need to find a function $f(x) < \frac{1}{\sqrt{1+t^2}}$ for $x > 1$ such that $\int_1^\infty f(x) dx$ diverges. To do this we observe the following inequality:

$$\frac{1}{\sqrt{1+t^2}} = \frac{1}{t\sqrt{t^{-2}+1}} \geq \frac{1}{\sqrt{2}t} \quad (\text{for } t > 1)$$

So we have

$$\int_1^\infty \frac{1}{\sqrt{1+t^2}} dt \geq \int_1^\infty \frac{1}{\sqrt{2}t} dt = \infty$$

which establishes divergence.

11i. $\int_0^\infty \frac{x}{\sqrt{x^2+1}} dx$: diverges

substituting $u = x^2 + 1, du = 2x dx$ yields $\int_0^\infty \frac{x}{\sqrt{x^2+1}} dx = \int_1^\infty \frac{du}{2\sqrt{u}} = \sqrt{u} \Big|_1^\infty = \infty - 1 = \infty$

11j. $\int_1^\infty \frac{dx}{x^2-1}$: diverges

This can be shown using the FTC (the antiderivative can be derived by partial fractions), but here we'll show how to use the comparison test. The problem is at $x = 1$ not $x = \infty$. If $1 \leq x \leq 2$, then we have

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} \geq \frac{1}{3(x-1)}.$$

Now use this to write

$$\begin{aligned} \int_1^\infty \frac{1}{x^2-1} dx &= \int_1^2 \frac{1}{x^2-1} dx + \int_2^\infty \frac{1}{x^2-1} dx \geq \int_1^2 \frac{1}{3(x-1)} dx + \int_2^\infty \frac{1}{x^2-1} dx \\ &\geq \int_1^2 \frac{1}{3(x-1)} = \frac{1}{3} \cdot \ln(x-1) \Big|_1^2 = \frac{1}{3} \cdot (\ln 1 - \ln 0) = \infty, \end{aligned}$$

which establishes divergence.

12. substitute $u = \frac{1}{x}$, then $du = -\frac{1}{x^2} dx = -u^2 dx$ and $dx = -\frac{1}{u^2} du$

$$\int_0^\infty \frac{\ln x}{x^2+1} dx = \int_\infty^0 \frac{\ln(1/u)}{u^{-2}+1} \frac{-du}{u^2} = - \int_\infty^0 \frac{\ln(1/u)}{1+u^2} du = \int_0^\infty \frac{\ln(1/u)}{1+u^2} du = - \int_0^\infty \frac{\ln u}{1+u^2} du$$

13a. The total dose is

$$D = \int_0^\infty 2te^{-2t} dt = \int_0^\infty ye^{-y} \frac{1}{2} dy = 1/2.$$

13b. Note that

$$\begin{aligned} D_5 &= \int_0^5 2td^{-2t} dt = \int_0^{10} \frac{1}{2} ye^{-y} dy = -\frac{1}{2} \int_0^{10} yde^{-y} \\ &= -\frac{1}{2} \left[ye^{-y} \Big|_0^{10} - \int_0^{10} e^{-y} dy \right] = -\frac{1}{2} \left[-10e^{-10} + e^{-y} \Big|_0^{10} \right] \\ &= \frac{1}{2} [1 - 11e^{-10}] \end{aligned}$$

Thus

$$\frac{D_5}{D} = \frac{\frac{1}{2}[1 - 11e^{-10}]}{1/2} = 0.9995.$$

14a. $y = \sqrt{1-x^2}$, $x \in [0, 1]$. So

$$s = \int_0^1 \sqrt{1 + \left(\frac{x}{\sqrt{1-x^2}} \right)^2} dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \pi/2.$$

14b. $y = \int_0^x \sqrt{1-t^2} dt$ so $\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sqrt{1-t^2} dt = \sqrt{1-x^2}$. Plugging this in we get

$$s = \int_0^1 \sqrt{1 + (1-x^2)} dx = \int_0^1 \sqrt{2-x^2} dx = \frac{x}{\sqrt{2}} \sqrt{1 - (x/\sqrt{2})^2} + \arcsin(x/\sqrt{2}) \Big|_0^1 = 1/2 + \pi/4.$$

14c.

$$\begin{aligned} s &= \int_0^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2} \right)^2} dx = \int_0^1 \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}} dx \\ &= \int_0^1 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}} dx = \int_0^1 \sqrt{\left(\frac{e^x + e^{-x}}{2} \right)^2} dx \\ &= \int_0^1 \frac{e^x + e^{-x}}{2} dx = \frac{e^x - e^{-x}}{2} \Big|_0^1 = \frac{e - e^{-1}}{2}. \end{aligned}$$

14d. Note that $f'(x) = \frac{3}{2}x^{1/2}$, thus

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_0^1 \sqrt{1 + \frac{9}{4}x} dx = \frac{4}{9} \int_1^{13/4} u^{1/2} du \\ &= \frac{4}{9} \frac{u^{3/2}}{3/2} \Big|_1^{13/4} = \frac{8}{27} \left(\left(\frac{13}{4} \right)^{3/2} - 1 \right). \end{aligned}$$

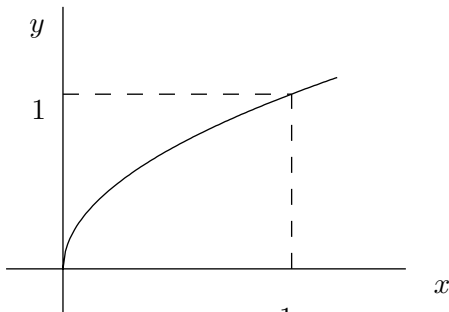
14e. Note that $f'(x) = 4x$, thus $L = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_0^1 \sqrt{1 + 16x^2} dx$.

Substitute $\tan \theta = 4x$; $\sec^2 \theta d\theta = 4 dx$, leading to

$$\begin{aligned} L &= \int_{x=0}^{x=1} \sqrt{1 + \tan^2 \theta} \frac{1}{4} \sec^2 \theta d\theta = \frac{1}{4} \int_{x=0}^{x=1} \sec^3 \theta d\theta \\ &= \frac{1}{8} (\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)) \Big|_{x=0}^{x=1} = \frac{1}{8} (4x\sqrt{1+16x^2} + \ln(\sqrt{1+16x^2} + 4x)) \Big|_0^1 \\ &= \frac{1}{8} (4\sqrt{17} + \ln(4 + \sqrt{17})) \end{aligned}$$

15. The curve $y = \sqrt{2x - x^2}$ is, by completing the square, $y = \sqrt{1 - (x - 1)^2}$. This is the equation of the semi-circular arc of a circle of radius 1 centered at $(1, 0)$. You can verify this by graphing. Therefore, without doing any calculus, the arclength is $\frac{1}{2}2\pi r = \pi$. This gets partial credit; to get full credit it should also be derived using the arclength integral.

16. The curve $y = \sqrt{x}$ for $0 \leq x \leq 1$ is the inverse of a parabola.



$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow L = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_0^1 \sqrt{1 + \frac{1}{4x}} dx : \text{improper, converges}$$

$$\text{substitute : } y = \sqrt{x} \Rightarrow dy = \frac{dx}{2\sqrt{x}} = \frac{dx}{2y}$$

$$L = \int_0^1 \sqrt{1 + \frac{1}{4y^2}} \cdot 2y dy = \int_0^1 \sqrt{4y^2 + 1} dy = \text{arclength of a parabola} = \frac{1}{4}(2\sqrt{5} + \ln(2 + \sqrt{5}))$$