1a. True. We showed in class that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, so $\sum_{i=1}^{12}(2i) = 2\sum_{i=1}^{12} i = 2\frac{12\cdot 13}{2} = 156$. (Of course this can be done by brute force without too much effort.)

1b. True. $\sum_{i=1}^{12}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{12} - \frac{1}{13}\right) = 1 - \frac{1}{13} = \frac{12}{13}$

1c. True. The two sums contain the same terms in the opposite order. (This could be justified with the substitution $j = n - i$.) Alternatively, using our known formula for $\sum_{i=0}^{n} i$, we have
$\sum_{i=0}^{n}(n-i)^2 = \sum_{i=0}^{n}(n^2 - 2in + i^2) = n^2\sum_{i=0}^{n} -2n\sum_{i=0}^{n} i + \sum_{i=0}^{n} i^2 = n^2(n+1) - 2n\frac{n(n+1)}{2} + \sum_{i=0}^{n} i^2 = \sum_{i=0}^{n} i^2$.

1d. True. In class we derived $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, so $\left(\sum_{i=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \sum_{i=1}^{n} i^3$.

1e. True. $1 + 3 + 5 + 7 + \cdots + (2n-1) = \sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} = 2\cdot\frac{n(n+1)}{2} - n = n^2$

1f. False. Consider the counterexample $f(x) = x, a = -1, b = 2$.

1g. False. As we increase the number of intervals using any Riemann sum the error will decrease, not increase. In the case of the right-hand Riemann sum, if the number of intervals is doubled, the error is approximately cut in half.

1h. False. One needs to use the chain rule, $\frac{d}{dx}\int_0^{x^3}\sqrt{1+t^2}\,dt = \sqrt{1+x^6}\cdot(x^3)' = 3x^2\sqrt{1+x^6}$.

1i. True. Apply integration by parts to the first integral, $\int udv = uv - \int vdu$, set $u = e^{-x}, dv = \cos x$, so $du = -e^{-x}dx, v = \sin x$. Then we have

$$\int_0^\infty e^{-x}\cos x dx \;=\; e^{-x}\sin x\Big|_0^\infty + \int_0^\infty e^{-x}\sin x dx \;=\; 0 - 0 + \int_0^\infty e^{-x}\sin x dx \;=\; \int_0^\infty e^{-x}\sin x dx.$$

1j. True. The work required to stretch the spring from 10 cm to 15 cm is $\int_0^5 kxdx = \frac{25}{2}k = 2$ J, so $k = \frac{4}{25}$. Then the work required to stretch the spring from 10 cm to 20 cm is $\int_0^{10}\frac{4}{25}xdx = \frac{4}{25}\cdot 50 = 8$ J.

1k. False. Let x be a vertical coordinate with $x = 0$ at the top of the building and $x = L$ at the bottom of the cable. Consider a slice of the cable at position x_i. The volume of the slice is $A\Delta x$; the mass of the slice is $\rho A\Delta x$; the force acting on the slice is $\rho gA\Delta x$; the work done in pulling the slice to the top of the building is $\rho gA\Delta x\cdot x_i$. Hence the total work is $W = \lim_{n\to\infty}\rho gAx_i\Delta x = \int_0^L \rho gAxdx = \frac{1}{2}\rho gAL^2$. If the length of the cable is doubled from L to $2L$, then the work becomes $W = \frac{1}{2}\rho gA(2L)^2 = 2\rho gAL^2$, which is four times the work done in raising a cable of length L, not double.

1l. False. There are many counterexamples; the simplest is $f(x) = 1/x$. Clearly, $\lim_{x\to\infty}\frac{1}{x} = 0$, but $\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\ln b = \infty$ which shows that $\int_0^\infty \frac{1}{x}dx$ diverges.

1m. True. $\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = 0 - (-1) = 1$; the result also follows from the p-test with $p = 2$

1n. True. Using the comparison test, if $0 \le f(x) \le g(x)$ for $x \ge 1$, then $0 \le \int_1^\infty f(x)dx \le \int_1^\infty g(x)dx < \infty$. The last inequality follows from the fact that $\int_1^\infty g(x)dx$ converges.

1o. True. Note that $\operatorname{erf}(0) = 0$. Then by the FTC we have $\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$, so $\operatorname{erf}''(x) = \frac{2}{\sqrt{\pi}}(-2x)e^{-x^2}$, and hence $\operatorname{erf}''(0) = 0$.

1p. True. method 1: sketch the graph of each function, argue that the area is the same by symmetry

method 2: use integration by parts, $u = \sin x, dv = \sin x \Rightarrow du = \cos x, v = -\cos x \Rightarrow \int_0^{\pi/2}\sin^2 xdx = \sin x\cdot -\cos x\Big|_{x=0}^{\pi/2} - \int_0^{\pi/2} -\cos^2 dx = \int_0^{\pi/2}\cos^2 xdx$

method 3: use trigonometric identities, $\sin^2 x = \frac{1}{2}(1-\cos 2x), \cos^2 x = \frac{1}{2}(1+\cos 2x)$

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2. Express the integral as a limit of Riemann sums, evaluate the limit, and check by the FTC.

2a. $\int_0^2 x dx \Rightarrow a=0, b=2, \Delta x = \frac{b-a}{n} = \frac{2}{n}, x_i = a + i\Delta x = \frac{2i}{n}, f(x) = x, f(x_i) = x_i = \frac{2i}{n}$

Riemann sums: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{2i}{n} = \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{4}{n^2} \frac{n(n+1)}{2} = 2$

FTC: $\int_0^2 x dx = \left.\frac{x^2}{2}\right|_0^2 = 2 - 0 = 2$

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2b. $\int_0^1 x^3 dx \Rightarrow a=0, b=1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x) = x^3, f(x_i) = x_i^3 = (\frac{i}{n})^3$

Riemann sums: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\frac{i}{n})^3 = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right]^2 = 1/4$

FTC: $\int_0^1 x^3 dx = \left.\frac{x^4}{4}\right|_0^1 = 1/4 - 0 = 1/4$

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2c.$\int_a^b x^2 dx \Rightarrow \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x, f(x) = x^2, f(x_i) = x_i^2$

$$\begin{aligned}
\text{Riemann sums : } \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (a + i\Delta x)^2 \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a^2 + 2ia\frac{b-a}{n} + i^2\frac{(b-a)^2}{n^2}\right)\frac{b-a}{n} \\
&= \lim_{n\to\infty} \left(a^2\frac{b-a}{n}\sum_{i=1}^{n} 1 + \frac{(b-a)^2}{n^2}2a\sum_{i=1}^{n} i + \frac{(b-a)^3}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left(a^2\frac{b-a}{n}n + \frac{(b-a)^2}{n^2}2a\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6}\right) \\
&= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} = (b-a)\left(a^2 + a(b-a) + \frac{1}{3}(b-a)^2\right) \\
&= (b-a)\left(ab + \frac{1}{3}(b^2 - 2ab + a^2)\right) = \frac{1}{3}(b-a)(b^2+ab+a^2) = \frac{1}{3}(b^3 - a^3)
\end{aligned}$$

FTC: $\int_a^b x^2 dx = \left.\frac{x^3}{3}\right|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$

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2d. $\int_0^1 e^{-x} dx \Rightarrow a=0, b=1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x) = e^{-x}, f(x_i) = e^{-x_i} = e^{-i/n}$

$$\begin{aligned}
\text{Riemann sums : } \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{-i/n} = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1-\left(e^{-1/n}\right)^{n+1}}{1-e^{-1/n}} - 1\right) = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1-e^{-(n+1)/n}}{1-e^{-1/n}}\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1-e^{-1-1/n}}{1-e^{-1/n}}\right) = \lim_{n\to\infty}(1-e^{-1-1/n}) \cdot \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1-e^{-1/n}}\right) \\
&= (1-e^{-1}) \cdot \lim_{t\to 0} \frac{t}{1-e^{-t}} = (1-e^{-1}) \cdot \frac{0}{0} = (1-e^{-1}) \cdot \lim_{t\to 0} \frac{1}{e^{-t}} = 1 - e^{-1}
\end{aligned}$$

note: l'Hôpital's rule is used in the last step; this derivation uses the right-hand Riemann sum; the left-hand Riemann sum can also be used, but you should get the same answer.

FTC: $\int_0^1 e^{-x} dx = \left.-e^{-x}\right|_0^1 = -e^{-1} - (-1) = 1 - e^{-1}$

3a. $\lim_{x\to\infty} xe^{-x} = 0 \cdot \infty = \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$; we used l'Hôpital's rule

3b. Recognize the summation as a (right) Riemann sum for $f(x) = x^3$ on the interval $1 \le x \le 2$. The limit is $\int_1^2 x^3 dx = [x^4/4]_1^2 = 15/4$.

3c. use l'Hôpital's rule : $\lim_{x\to 0} \frac{\int_0^x f(t)\,dt}{x} = \lim_{x\to 0} \frac{f(x)}{1} = f(0)$; we assume f is continuous at 0

3d. use l'Hôpital's rule : $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$

3e. use l'Hôpital's rule : $\lim_{r\to 1} \frac{1-r^{11}}{1-r} = \lim_{r\to 1} \frac{-11r^{10}}{-1} = 11$

or alternatively use the sum of a finite geometric series

$\frac{1-r^{11}}{1-r} = \sum_{i=0}^{10} r^i \Rightarrow \lim\limits_{r\to 1} \frac{1-r^{11}}{1-r} = \lim\limits_{r\to 1} \sum\limits_{i=0}^{10} r^i = \sum\limits_{i=0}^{10} 1 = 11$

4a. substitute : $u = -x^2$ $du = -2xdx$ we have $\int xe^{-x^2}dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-x^2} + c$.

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4b. Integrate by parts twice to reduce the integral:

$$\begin{array}{ll} \int x^2 e^{-x} dx & u = x^2,\ dv = e^{-x}dx \ \Rightarrow\ du = 2xdx,\ v = -e^{-x} \\ = -x^2e^{-x} - 2\int xe^{-x}dx & u = x,\ dv = e^{-x}dx \ \Rightarrow\ du = dx,\ v = -e^{-x} \\ = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x} = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C & \end{array}$$

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4c. Integrate by parts: $u = x,\ dv = \sin(x)dx$, then $du = dx$ and $v = -\cos(x)$

$$\int x\sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + C$$

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4d. Easiest method is partial fractions:

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{1/4}{2-x} + \frac{1/4}{2+x}.$$

So we get:

$$\int \frac{dx}{4-x^2} = \frac{1}{4}\int \frac{dx}{2-x} + \frac{1}{4}\int \frac{dx}{2+x} = \frac{1}{4}\left[\ln(2+x) - \ln(2-x)\right] = \frac{1}{4}\ln\left(\frac{2+x}{2-x}\right) + C$$

you could also use the trig. substitution $x = 2\sin(\theta)$, but as you'll see if you try it, the algebra is much more difficult.

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4e. Because of the presence of the square root we use a trig. substitution this time. Set $x = 2\sin(\theta)$. Then $dx = 2\cos(t)dt$ and we get:

$$\begin{aligned} \int \frac{dx}{\sqrt{4-x^2}} &= \int \frac{2\cos(t)dt}{\sqrt{4-4\sin^2(t)}} \\ &= \int \frac{\cos(t)}{\sqrt{1-\sin^2(t)}} = \int dt = t + C \end{aligned}$$

To complete the integration we need to invert the substitution to get back to the variable x. Solving our original substitution $x = 2\sin(t)$ for t give $t = \arcsin(x/2)$ so finally we have:

$$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin(x/2) + C$$

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4f. Again we need to use a trig. substitution. Set $x = 2\sin(t)$ so that $dx = 2\cos(t)dt$. Plugging this in gives:

$$\int \sqrt{4-x^2}dx = \int \sqrt{4-4\sin^2(t)}2\cos(t)dt = 4\int \cos^2(t)dt \tag{1}$$

There are several routes to finding this antiderivative. We will use a useful trig. identity from your past:

$$\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 \Longrightarrow \cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

Inserting this into the above integral (1) we have:

$$2\int 1 + \cos(2t)dt = 2t + \sin(2t) = 2t + 2\sin(t)\cos(t) + C$$

where we used the trig identity $\sin(2t) = 2\sin(t)\cos(t)$ to simplify above. Now we use the original substitution $x = 2\sin(t)$ to go back to the original variables: $\sin(t) = x/2$ so $\cos(t) = \sqrt{1-(x/2)^2}$ and $t = \arcsin(x/2)$. Plugging these in we get the final answer:

$$\int \sqrt{4-x^2}dx = 2\arcsin\left(\frac{x}{2}\right) + x\sqrt{1-\left(\frac{x}{2}\right)^2} + C.$$

5a. For $0 \leq x \leq 1$, we have $\frac{1}{2} \leq \frac{1}{1+x} \leq 1$. Then it follows from the comparison test that

$$\int_0^1 \frac{1}{2}x^9 dx \leq \int_0^1 \frac{x^9}{1+x}dx \leq \int_0^1 x^9 dx \;\Rightarrow\; \frac{1}{2}\frac{x^{10}}{10}\bigg|_0^1 \leq \int_0^1 \frac{x^9}{1+x}dx \leq \frac{x^{10}}{10}\bigg|_0^1 \;\Rightarrow\; \frac{1}{20} \leq \int_0^1 \frac{x^9}{1+x}dx \leq \frac{1}{10}.$$

5b. We could expand $(1-x)^{11}$, but we seek we seek a simpler method; use a substitution. Set $u = 1-x$, so $x = 1-u$ and $dx = -du$. Our integral becomes upon substitution:

$$\begin{aligned}\int_0^1 x(1-x)^{11}dx &= \int_1^0 (1-u)u^{11}(-du) = -\int_1^0 u^{11} - u^{12}du \\ &= \int_0^1 u^{11} - u^{12}du = \frac{1}{12}u^{12} - \frac{1}{13}u^{13}\Big|_0^1 = \frac{1}{12} - \frac{1}{13} = \frac{1}{156}.\end{aligned}$$

NOTE: In the integral I had to changed the boundaries of the integral when I made the substitution $u = 1-x$. Also, in the third step I used the property that $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

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6. Using Hooke's Law, we have $30N = k(12cm - 15cm)$ so $k = 10\frac{N}{cm}$. To calculate the work to stretch from 12cm (the natural length) to 20cm we have $W = \int_0^8 kxdx = \int_0^8 10xdx = 5x^2\big|_0^8 = 320\,N\cdot cm = 3.2J.$

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7. If one ion is held fixed and the second ion is moved from distance r_1 to distance r_2 (relative to the first ion), then the work done is

$$W = \int_{r_1}^{r_2} -\frac{q^2}{4\pi\varepsilon_0 r^2}dr = -\frac{q^2}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\frac{q^2}{4\pi\varepsilon_0}\cdot -\frac{1}{r}\bigg|_{r_1}^{r_2} = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

The distances are $r_1 = 3\,\text{mm}$, $r_2 = 2\,\text{mm}$, so $W = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{q^2}{4\pi\varepsilon_0}\cdot\frac{1}{6} = \frac{q^2}{24\pi\varepsilon_0}$ mJ.

8a. Draw a vertical x-axis with $x = 0$ at the base of the pyramid and $x = H$ at the top. Define $\Delta x = \frac{H}{n}, x_i = i\Delta x$, for $i = 0 : n$, where Δx is the width of a slice of the pyramid and x_i is the height of the ith slice. Each slice has the shape of a thin square box, so if l_i is the side length of the ith slice, then using similar triangles we see that $\frac{l_i}{H-x_i} = \frac{L}{H}$.

volume of ith slice $= l_i^2 \cdot \Delta x = (H-x_i)^2\frac{L^2}{H^2}\Delta x$

mass of ith slice $= \rho(H-x_i)^2\frac{L^2}{H^2}\Delta x$

force acting on ith slice $= \rho g(H-x_i)^2\dfrac{L^2}{H^2}\Delta x$

work done on ith slice = force × distance = $= \rho g(H-x_i)^2\dfrac{L^2}{H^2}\Delta x \cdot x_i$

total work $= W = \displaystyle\int_0^H \rho g(H-x)^2 x\frac{L^2}{H^2}dx = \rho g\frac{L^2}{H^2}\int_0^H (H-x)^2 x dx$

$$\int_0^H (H-x)^2 x dx = \int_0^H (H^2x - 2Hx^2 + x^3)dx = \left(H^2\frac{x^2}{2} - 2H\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_0^H = H^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{H^4}{12}$$

so the work done in constructing the pyramid is $W = \rho g\dfrac{L^2}{H^2}\cdot\dfrac{H^4}{12} = \dfrac{1}{12}\rho g L^2 H^2$

8b. If L and H are doubled, then W increases by a factor of 16.

8c. Which requires more work, building the lower half or the upper half of the pyramid?

$$W_{lower} : \int_0^{H/2} (H-x)^2 x dx = \left(H^2\frac{x^2}{2} - 2H\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_0^{H/2} = H^4\left(\frac{1}{2}\cdot\frac{1}{4} - \frac{2}{3}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{16}\right) = \frac{11}{192}H^4$$

$$W_{upper} : \int_{H/2}^{H} (H-x)^2 x dx = \left(H^2\frac{x^2}{2} - 2H\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_{H/2}^{H} = H^4\left(\frac{1}{2}\cdot\frac{3}{4} - \frac{2}{3}\cdot\frac{7}{8} + \frac{1}{4}\cdot\frac{15}{16}\right) = \frac{5}{192}H^4$$

So more work is done building the lower half of the pyramid.

9a. $\displaystyle\int_1^\infty \frac{dx}{x^4}$: converges, p-test, $p=4$; also $\displaystyle\int_1^\infty \frac{dx}{x^4} = \frac{1}{-3x^3}\Big|_1^\infty = 0 - \left(\frac{1}{-3}\right) = \frac{1}{3}$

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9b. $\displaystyle\int_0^\infty x^2 e^{-x}dx$: converges

$$\begin{aligned}\int_0^\infty x^2e^{-x}dx &= \int_0^\infty x^2(-de^{-x}) = -\left[x^2e^{-x}\Big|_0^\infty - \int_0^\infty e^{-x}2xdx\right]\\ &= 2\int_0^\infty e^{-x}xdx = -2\int_0^\infty xde^{-x} = -2\left[xe^{-x}\Big|_0^\infty - \int_0^\infty e^{-x}dx\right]\\ &= 2\int_0^\infty e^{-x}dx = -2e^{-x}\Big|_0^\infty = 2\end{aligned}$$

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9c. $\displaystyle\int_0^\infty e^{-x}\sin x dx$: converges by comparison to e^{-x}. Note that

$$\int_0^\infty e^{-x}\sin x dx = -\int_0^\infty e^{-x}d(\cos x) = -e^{-x}\cos x\Big|_0^\infty + \int_0^\infty -e^{-x}\cos x dx = 1 - \int_0^\infty e^{-x}\cos x dx$$

Using the result of problem (1i), $\displaystyle\int_0^\infty e^{-x}\sin(x)dx = \int_0^\infty e^{-x}\cos(x)dx \Rightarrow \int_0^\infty e^{-x}\sin x dx = 1/2.$

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9d. $\displaystyle\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$: converges. Don't fall into the trap of saying that each term diverges so the difference diverges; $\infty - \infty$ is inconclusive, but the integral can be evaluated directly.

$$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = [\ln x - \ln(1+x)]\Big|_1^\infty = \lim_{x\to\infty}\ln\left(\frac{x}{1+x}\right) + \ln 2 = \ln 1 + \ln 2 = \ln 2$$

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9e. $\displaystyle\int_{-r}^{r}\sqrt{r^2-x^2}$: converges; it's a proper integral. To evaluate, use a trig substitution.

$\sin\theta = \frac{x}{r}, \cos\theta d\theta = \frac{dx}{r}, \sqrt{r^2-x^2} = \sqrt{r^2 - r^2\sin^2\theta} = \sqrt{r^2(1-\sin^2\theta)} = r\cos\theta$

$$\int_{-r}^{r}\sqrt{r^2-x^2} = \int_{-\pi/2}^{\pi/2} r\cos\theta\cdot r\cos\theta d\theta = r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = r^2\int_{-\pi/2}^{\pi/2}\frac{1}{2}\left(1+\cos 2\theta\right)d\theta$$

$$= \frac{r^2}{2}\cdot\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{r^2}{2}\cdot\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi - \left(-\frac{\pi}{2}+\frac{1}{2}\sin(-\pi)\right)\right) = \frac{r^2}{2}\cdot\pi = \frac{1}{2}\pi r^2$$

In fact, the graph of $f(x) = \sqrt{r^2-x^2}$ is a semi-circle, so the integral is the area of a semi-circle, and so the value we computed makes sense.

9f. $\displaystyle\int_{-r}^{r}\frac{dx}{\sqrt{r^2-x^2}}$: converges; it's an improper integral. To evaluate se a trig substitution.

$\sin\theta = \frac{x}{r}, \cos\theta d\theta = \frac{dx}{r}, \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{x^2}{r^2}} = \sqrt{\frac{r^2-x^2}{r^2}} = \frac{1}{r}\sqrt{r^2-x^2}$

$$\int_{-r}^{r}\frac{dx}{\sqrt{r^2-x^2}} = \int_{-\pi/2}^{\pi/2}\frac{r\cos\theta d\theta}{r\cos\theta} = \pi$$

9g. $\displaystyle\int_{1}^{\infty}\frac{dx}{1+x^2}$: converges. Using the substitution $x = \tan t$ (so $dx = \sec^2 t dt$) we have

$$\int_{1}^{\infty}\frac{dx}{1+x^2} = \int_{\pi/4}^{\pi/2}\frac{\sec^2 t}{1+\tan^2 t}dt = \int_{\pi/4}^{\pi/2} 1 dt = \pi/4.$$

9h. $\displaystyle\int_{1}^{\infty}\frac{dx}{\sqrt{1+x^2}}$: diverges. We know this because $\frac{1}{\sqrt{1+x^2}} \sim \frac{1}{x}$ when x is large and the integral of the latter diverges. To prove this we use comparison. We need to find a function $f(x) < \frac{1}{\sqrt{1+x^2}}$ for $x > 1$ such that $\int_1^\infty f(x)dx$ diverges. To do this we observe,

$$\frac{1}{\sqrt{1+x^2}} = \frac{1}{x\sqrt{x^{-2}+1}} \geq \frac{1}{\sqrt{2}x} \text{ for } x > 1 \ \Rightarrow \int_1^\infty \frac{1}{\sqrt{1+x^2}}dx \geq \int_1^\infty \frac{1}{\sqrt{2}x}dx = \infty$$

which establishes divergence.

9i. $\displaystyle\int_{0}^{\infty}\frac{x}{\sqrt{x^2+1}}dx$: diverges

substituting $u = x^2+1, du = 2xdx$ yields $\displaystyle\int_0^\infty \frac{x}{\sqrt{x^2+1}}dx = \int_1^\infty \frac{du}{2\sqrt{u}} = \sqrt{u}\Big|_1^\infty = \infty - 1 = \infty$

9j. $\displaystyle\int_{1}^{\infty}\frac{dx}{x^2-1}$: diverges

This can be shown using the FTC (the antiderivative can be derived by partial fractions), but here we'll show how to use the comparison test. The problem is at $x = 1$ not $x = \infty$. If $1 \leq x \leq 2$, then we have

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} \geq \frac{1}{3(x-1)}.$$

Now use this to write

$$\int_1^\infty \frac{1}{x^2-1}dx = \int_1^2 \frac{1}{x^2-1}dx + \int_2^\infty \frac{1}{x^2-1}dx \geq \int_1^2 \frac{1}{3(x-1)}dx + \int_2^\infty \frac{1}{x^2-1}dx$$

$$\geq \int_1^2 \frac{1}{3(x-1)} = \frac{1}{3}\cdot\ln(x-1)\Big|_1^2 = \frac{1}{3}\cdot(\ln 1 - \ln 0) = \infty \Rightarrow \text{the integral diverges.}$$

9k. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$: converges. p-test with $p = \frac{1}{2}$; $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 x^{1/2}dx = \left.\frac{x^{3/2}}{3/2}\right|_0^1 = \frac{2}{3}$

9l. $\displaystyle\int_0^1 \frac{dx}{x^{3/2}}$: diverges. p-test, $p = \frac{3}{2}$, $\displaystyle\int_0^1 \frac{dx}{x^{3/2}} = \int_0^1 x^{-3/2}dx = \left.\frac{x^{-1/2}}{-1/2}\right|_0^1 = -2-(-\infty) = \infty$

9m. $\displaystyle\int_0^1 \frac{dx}{1-x}$: diverges

substitute $u = 1-x, du = -dx$, $\displaystyle\int_0^1 \frac{dx}{1-x} = \int_1^0 \frac{-du}{u} = \int_0^1 \frac{du}{u}$: diverges by p-test, $p = 1$

9n. substitute $u = \frac{1}{x}$, then $du = -\frac{1}{x^2}dx = -u^2dx$ and $dx = -\frac{1}{u^2}du$

$$\int_0^\infty \frac{\ln x}{x^2+1}dx = \int_\infty^0 \frac{\ln(1/u)}{u^{-2}+1}\frac{-du}{u^2} = -\int_\infty^0 \frac{\ln(1/u)}{1+u^2}du = \int_0^\infty \frac{\ln(1/u)}{1+u^2}du = -\int_0^\infty \frac{\ln u}{1+u^2}du$$

10a. The total dose is $D = \displaystyle\int_0^\infty 2te^{-2t}dt = \int_0^\infty ye^{-y}\frac{1}{2}dy = 1/2.$

10b. Note that

$$\begin{aligned} D_5 &= \int_0^5 2td^{-2t}dt = \int_0^{10} \frac{1}{2}ye^{-y}dy = -\frac{1}{2}\int_0^{10} yde^{-y} \\ &= -\frac{1}{2}\left[\left.ye^{-y}\right|_0^{10} - \int_0^{10} e^{-y}dy\right] = -\frac{1}{2}\left[-10e^{-10} + \left.e^{-y}\right|_0^{10}\right] \\ &= \frac{1}{2}[1-11e^{-10}] \end{aligned}$$

Thus $\displaystyle\frac{D_5}{D} = \frac{\frac{1}{2}[1-11e^{-10}]}{1/2} = 0.9995.$

11a. $y = \sqrt{1-x^2}$, $x \in [0,1]$. So

$$s = \int_0^1 \sqrt{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2}dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \pi/2.$$

11b. $y = \int_0^x \sqrt{1-t^2}dt$ so $\frac{dy}{dx} = \frac{d}{dx}\int_0^x \sqrt{1-t^2}dt = \sqrt{1-x^2}$. Plugging this in we get

$$s = \int_0^1 \sqrt{1+(1-x^2)}dx = \int_0^1 \sqrt{2-x^2}dx = \left.\frac{x}{\sqrt{2}}\sqrt{1-(x/\sqrt{2})^2} + \arcsin(x/\sqrt{2})\right|_0^1 = 1/2+\pi/4.$$

11c.

$$\begin{aligned} s &= \int_0^1 \sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}dx = \int_0^1 \sqrt{1+\frac{e^{2x}-2+e^{-2x}}{4}}dx \\ &= \int_0^1 \sqrt{\frac{e^{2x}+2+e^{-2x}}{4}}dx = \int_0^1 \sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2}dx \\ &= \int_0^1 \frac{e^x+e^{-x}}{2}dx = \left.\frac{e^x-e^{-x}}{2}\right|_0^1 = \frac{e-e^{-1}}{2}. \end{aligned}$$

11d. $f'(x) = \dfrac{3}{2}x^{\frac{1}{2}}$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\int_1^{13/4} u^{1/2}du = \frac{4}{9}\frac{u^{3/2}}{3/2}\Big|_1^{13/4} = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right)$$

11e. $f'(x) = 4x \Rightarrow L = \displaystyle\int_a^b \sqrt{1+(f'(x))^2}dx = \int_0^1 \sqrt{1+16x^2}dx.$

Substitute $\tan\theta = 4x; \sec^2\theta\, d\theta = 4\, dx$, leading to

$$\begin{aligned} L &= \int_{x=0}^{x=1} \sqrt{1+\tan^2\theta}\frac{1}{4}\sec^2\theta\, d\theta = \frac{1}{4}\int_{x=0}^{x=1} \sec^3\theta\, d\theta \\ &= \frac{1}{8}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))]_{x=0}^{x=1} = \frac{1}{8}(4x\sqrt{1+16x^2} + \ln(\sqrt{1+16x^2}+4x))]_0^1 \\ &= \frac{1}{8}(4\sqrt{17} + \ln(4+\sqrt{17})) \end{aligned}$$

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12. The curve $y = \sqrt{2x - x^2}$ is, by completing the square, $y = \sqrt{1-(x-1)^2}$. This is the equation of the semi-circular arc of a circle of radius 1 centered at $(1, 0)$. You can verify this by graphing. Therefore, without doing any calculus, the arclength is $\frac{1}{2}2\pi r = \pi$. This gets partial credit; to get full credit it should also be derived using the arclength integral.

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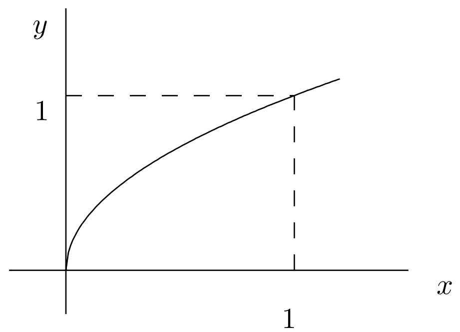
13. The curve $y = \sqrt{x}$ for $0 \le x \le 1$ is the inverse of a parabola.



$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow L = \int_a^b \sqrt{1+(f'(x))^2}dx = \int_0^1 \sqrt{1+\frac{1}{4x}}dx \text{ : improper, converges}$$

$$\text{substitute : } y = \sqrt{x} \Rightarrow dy = \frac{dx}{2\sqrt{x}} = \frac{dx}{2y}$$

$$L = \int_0^1 \sqrt{1+\frac{1}{4y^2}}\cdot 2y\,dy = \int_0^1 \sqrt{4y^2+1}\,dy = \text{arclength of a parabola} = \tfrac{1}{4}(2\sqrt{5} + \ln(2+\sqrt{5}))$$

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14. (TO BE COMPLETED) Sketch the graph of each function on the interval $0 \le x \le 2\pi$.

a) $\cos x$ b) $\cos 2x$ c) $\frac{1}{2}\cos 2x$ d) $\frac{1}{2}+\frac{1}{2}\cos 2x$ e) $\cos^2 x$

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15. $f(x) = \cos x, 0 \le x \le 2\pi$

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2\pi}\int_0^{2\pi}\cos x dx = \frac{1}{2\pi}\sin x\Big|_0^{2\pi} = 0$$

$$\begin{aligned} f_{\text{rms}} &= \sqrt{\frac{1}{b-a}\int_a^b (f(x))^2 dx} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x dx} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx} \\ &= \sqrt{\frac{1}{2\pi}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\Big|_0^{2\pi}} = \sqrt{\frac{1}{2\pi}\cdot\frac{1}{2}2\pi} = \frac{1}{\sqrt{2}} \end{aligned}$$