EXAM 1 REVIEW SOLUTIONS

1a. True.

$$\begin{aligned}\sum_{i=1}^{n}(a_i - b_i) =&(a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n)\\ =&(a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots b_n)\\ =&\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\end{aligned}$$

1b. False. Choose $n = 2, a_1 = 1$, $a_2 = 0$, $b_1 = 1$, and $b_2 = 1$. Then

$$\sum_{i=1}^{2} \frac{a_i}{b_i} = 1, \qquad \frac{\sum_{i=1}^{2} a_i}{\sum_{i=1}^{2} b_i} = \frac{1}{2}$$

So $\sum_{i=1}^{2} \frac{a_i}{b_i} \neq \frac{\sum_{i=1}^{2} a_i}{\sum_{i=1}^{2} b_i}$. False.

1c. True. Two ways:

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2}$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}.$$

Or
$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n^2}\right) = \int_0^1 x dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$.

1d. True. $\lim_{n\to\infty}\sum_{i=1}^{n}(1 + \frac{i}{n})\frac{1}{n} = \int_0^1 (1 + x)\ dx = x + \frac{x^2}{2}\Big|_0^1 = 3/2 - 0 = 3/2$.

1e. False. As we increase the number of intervals using *any* scheme the error should decrease, not increase. The right-hand sum is a linear approximation so if the number of intervals is doubled, the error should half, roughly speaking.

1f. False. Usually the right hand side Riemann sum is less accurate than the midpoint rule. Roughly speaking, when the number of intervals is doubled, the errors for right hand Riemann sum decrease by a half and the errors by midpoint rule decrease by a quarter.

1g. True. To see this is true informally, draw a picture. The function $f(x) = c_1 + c_2 x$ is a straight line. The rectangles in the midpoint rule have error above and below the function which are exactly equal and thus cancel out. A more formal proof would go like this:

Proof. The exact integral is: $\int_a^b c_1 + c_2 x \; dx = c_1(b-a) + c_2 \frac{b^2-a^2}{2}$.

Using the midpoint rule we have: $\Delta x = \frac{b-a}{n}$, midpoints $x_i^* = a + (i - 1/2)\frac{b-a}{n}$
Midpoint approximation (finite n, no limit):

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i^*)\Delta x &= \frac{b-a}{n} \sum_{i=1}^{n} c_1 + c_2(a + (i-1/2)\frac{b-a}{n} \\
&= \frac{b-a}{n}\left[c_1 n + c_2\left[(a - \frac{b-a}{2n})n + (b-a)\frac{n(n+1)}{2n}\right]\right] \\
&= c_1(b-a) + c_2(b-a)\left[a - \frac{1}{2n} + \frac{1}{2} + \frac{1}{2n}\right] \\
&= (b-a)\left[c_1 + c_2\left[a - \frac{b-a}{2n} + \frac{b-a}{2} + \frac{b-a}{2n}\right]\right] \\
&= (b-a)\left[c_1 + c_2\frac{b+a}{2}\right] = c_1(b-a) + c_2\frac{b^2-a^2}{2}
\end{aligned}$$

Which is the same thing we get from the exact integral.

1h. True. $\lim_{n\to\infty} \sum_{i=1}^{n} f'(x_i)\Delta x \overset{def.}{=} \int_a^b f'(x)\, dx \overset{F.T.C.}{=} f(b) - f(a)$. Here we are assuming $f'(x)$ is continuous so we may apply the F.T.C.

1i. False. One needs to use the chain rule: $\frac{d}{dx}\int_0^{x^2} \sqrt{1+t}dt = \sqrt{1+x^2} \cdot \frac{d}{dx}x^2 = 2x\sqrt{1+x^2}$.

1j. True. Integrating the first term using integration by parts. Set $u = e^{-x}$ and $dv = \cos(x)$, so $du = -e^{-x}dx$ and $v = \sin(x)$. Then using the integration by parts formula $\int udv = uv - \int vdu$ we have

$$\begin{aligned}
\int_0^\infty e^{-x}\cos(x)dx &= e^{-x}\sin(x)\Big|_0^\infty + \int_0^\infty e^{-x}\sin(x)dx \\
&= 0 - 0 + \int_0^\infty e^{-x}\sin(x)dx = \int_0^\infty e^{-x}\sin(x)dx.
\end{aligned}$$

1k. False. 2 J to stretch the spring 20cm beyond its natural length: $\Rightarrow \int_0^{20} kx dx = 2J$ We can solve this for k to get $k = \frac{1J}{100cm^2}$. Now lets use this to find the work used to stretch the spring 10cm beyond its natural length: $W = \int_0^{10} \frac{1}{100} x dx = \frac{x^2}{200}\Big|_0^{10} = \frac{1}{2}J \neq 1J \quad \square$

1l. False. There are many counterexamples one can use, the simplest is $f(x) = 1/x$. Clearly, $\lim_{x\to\infty} \frac{1}{x} = 0$ However, $\int_1^\infty \frac{1}{x} dx = \lim_{b\to\infty} \ln(b) = \infty$ which shows that $\int_0^\infty \frac{1}{x} dx$ diverges.

1m. True. Using the comparison test, if $0 \leq f(x) \leq g(x)$ for $x \geq 1$, then $0 \leq \int_1^\infty f(x)dx \leq \int_0^\infty g(x)dx < \infty$. The last inequality follows from the fact that $\int_1^\infty g(x)dx$ converges.

2a. Note that $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

$$\begin{aligned}
\int_a^b x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^n x_i^2 \Delta x = \lim_{n\to\infty} \sum_{i=1}^n (a + i\Delta x)^2 \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(a^2 + 2ia\frac{b-a}{n} + i^2\frac{(b-a)^2}{n^2} \right) \frac{b-a}{n} \\
&= \lim_{n\to\infty} \left(a^2 \frac{b-a}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} 2a \sum_{i=1}^n i + \frac{(b-a)^3}{n^3} \sum_{i=1}^n i^2 \right) \\
&= \lim_{n\to\infty} \left(a^2 \frac{b-a}{n} n + \frac{(b-a)^2}{n^2} 2a \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \frac{n(n+1)(2n+1)}{6} \right) \\
&= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} \\
&= \frac{(b-a)(b^2+ab+a^2)}{3} = \frac{b^3-a^3}{3}
\end{aligned}$$

2b. By FTC

$$\int_a^b x^2 dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

3a. FTC: $\int_0^2 x dx = \frac{x^2}{2}\Big|_0^2 = 2 - 0 = 2$.
Riemann Sum: $\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^n \frac{2i}{n} = \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^n i = \lim_{n\to\infty} \frac{4}{n^2} \frac{n(n+1)}{2} = 2$.

3b. FTC: $\int_0^1 x^3 dx = \frac{x^4}{4}\Big|_0^1 = 1/4 - 0 = 1/4$.
Riemann Sum: $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (\frac{i}{n})^3 = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^n i^3 = \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right]^2 = 1/4$.

3c. FTC $\int_0^1 e^{-x}dx = -e^{-x}\Big|_0^1 = -e^{-1} - (-1) = 1 - e^{-1}$.
Riemann Sum: $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} e^{-i/n} = \lim_{n\to\infty} \frac{1}{n}\frac{1-e^{-1}}{1-e^{-1/n}} = 1 - e^{-1}$. L'Hôspital's Rule is used in the last step. NOTE: I used a left-hand sum to simplify some of the algebra, but you should get the same thing using a right hand sum, the algebra will just be a little uglier.

4a. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1/n}{1+i/n} = \int_0^1 \frac{dx}{1+x} = \ln(2) - \ln(1)$.

4b. Assuming $f(t)$ is continuous, $\frac{1}{x}\int_0^x f(t)dt \overset{def.}{=}$ Avgerage value of $f(t)$ for $0 \le t \le x$. Thus, for any function continuous at $x = 0$, $\lim_{x\to 0} \frac{1}{x}\int_0^x f(t)dt = f(0)$.

4c. $\lim_{x\to 0} \frac{e^x - 1}{x} \overset{l'hopital}{=} \lim_{x\to 0} \frac{e^x}{1} = 1$

4d. Using L'Hospital's Rule: $\lim_{r\to 1} \frac{r^{10}-1}{r-1} = \lim_{r\to 1} 10r^9 = 10$.

5a. Using substitution, $u = -x^2$ $du = -2xdx$ we have $\int xe^{-x^2}dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-x^2} + c$.

5b. Integrate by parts twice to reduce the integral:

$$\begin{array}{ll} \int x^2 e^{-x}dx & u = x^2,\ dv = e^{-x}dx \ \Rightarrow\ du = 2xdx,\ v = -e^{-x} \\ = -x^2e^{-x} - 2\int xe^{-x}dx & u = x,\ dv = e^{-x}dx \ \Rightarrow\ du = dx,\ v = -e^{-x} \\ = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x} = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C & \end{array}$$

5c. Integrate by parts: $u = x,\ dv = \sin(x)dx$, then $du = dx$ and $v = -\cos(x)$

$$\int x\sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + C$$

5d. Easiest method is partial fractions:

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{1/4}{2-x} + \frac{1/4}{2+x}.$$

So we get:

$$\int \frac{dx}{4-x^2} = \frac{1}{4}\int \frac{dx}{2-x} + \frac{1}{4}\int \frac{dx}{2+x} = \frac{1}{4}\left[\ln(2+x) - \ln(2-x)\right] = \frac{1}{4}\ln\left(\frac{2+x}{2-x}\right) + C$$

you could also use the trig. substitution $x = 2\sin(\theta)$, but as you'll see if you try it, the algebra is much more difficult.

5e. Because of the presence of the square root we use a trig. substitution this time. Set $x = 2\sin(\theta)$. Then $dx = 2\cos(t)dt$ and we get:

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos(t)dt}{\sqrt{4-4\sin^2(t)}}$$
$$= \int \frac{\cos(t)}{\sqrt{1-\sin^2(t)}} = \int dt = t + C$$

To complete the integration we need to invert the substitution to get back to the variable x. Solving our original substitution $x = 2\sin(t)$ for t give $t = \arcsin(x/2)$ so finally we have:

$$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin(x/2) + C$$

5f. Again we need to use a trig. substitution. Set $x = 2\sin(t)$ so that $dx = 2\cos(t)dt$. Plugging this in gives:

$$\int \sqrt{4-x^2}dx = \int \sqrt{4-4\sin^2(t)}2\cos(t)dt = 4\int \cos^2(t)dt \tag{1}$$

There are several routes to finding this antiderivative. We will use a useful trig. identity from your past:

$$\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 \Longrightarrow \cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

Inserting this into the above integral (1) we have:

$$2\int 1 + \cos(2t)dt = 2t + \sin(2t) = 2t + 2\sin(t)\cos(t) + C$$

where we used the trig identity $\sin(2t) = 2\sin(t)\cos(t)$ to simplify above. Now we use the original substitution $x = 2\sin(t)$ to go back to the original variables: $\sin(t) = x/2$ so $\cos(t) = \sqrt{1-(x/2)^2}$ and $t = \arcsin(x/2)$. Plugging these in we get the final answer:

$$\int \sqrt{4-x^2}dx = 2\arcsin\left(\frac{x}{2}\right) + x\sqrt{1-\left(\frac{x}{2}\right)^2} + C.$$

6a. On the interval $0 \le x \le 1$, we have the following inequality: $1/2 \le 1/(1+x) \le 1$. Using this estimate it follows from the comparison test that:

$$\int_0^1 \frac{1}{2}x^9 dx \le \int_0^1 \frac{x^9}{1+x}dx \le \int_0^1 x^9 dx$$
$$\frac{1}{20}x^{10}\Big|_0^1 \le \int_0^1 \frac{x^9}{1+x}dx \le \frac{1}{10}x^{10}\Big|_0^1$$
$$\frac{1}{20} \le \int_0^1 \frac{x^9}{1+x}dx \le \frac{1}{10}$$

6b. We could, of course, expand $(1-x)^{11}$ everything out but we are not savages so we seek a slicker path: use a substitution. Set $u = 1-x$, so $x = 1-u$ and $dx = -du$. Our integral becomes upon substitution:

$$\int_0^1 x(1-x)^{11}dx = \int_1^0 (1-u)u^{11}(-du) = -\int_1^0 u^{11} - u^{12}du$$
$$= \int_0^1 u^{11} - u^{12}du = \frac{1}{12}u^{12} - \frac{1}{13}u^{13}\Big|_0^1 = \frac{1}{12} - \frac{1}{13} = \frac{1}{156}.$$

NOTE: In the integral I had to changed the boundaries of the integral when I made the substitution $u = 1-x$. Also, in the third step I used the property that $\int_a^b f(x)dx = -\int_b^a f(x)dx$. This is a useful fact that some of you may have forgot (or never really known), now you do!

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7. Their are easier and harder ways to tackle this problem. One way to attack this is to slice the cable into pieces and calculate the work needed to lift each piece to its final location. This works fine when the whole rope will be lifted to the same final destination (i.e. the top of the building) but it gets messy when the different parts of the rope end up in different places (such as when only half the rope is lifted to the top of the building). A better (easier) approach is to think of it as a problem with a variable force $F(x)$ that changes as we lift the rope. The natural length of the rope is L when we have lifted x meters to the top, the remaining hanging weight is $(L-x) \times \rho$. This is the force we are pulling against –we don't have to pull against the weight already at the top–so $F(x) = \rho(L-x)$. It follows that the work done to lift any length ℓ to the top is:

$$W(\ell) = \int_0^\ell F(x)dx = \int_0^\ell \rho(L-x)dx = -\frac{\rho}{2}(L-x)^2\Big|_0^\ell = \frac{\rho}{2}(L^2 - (L-\ell)^2.$$

If we have to lift the whole cable to the top then $\ell = L$ in which case we have:

$$W = \frac{1}{2}\rho L^2.$$

Now if we double the length from L to $2L$ the work becomes $W = \frac{1}{2}\rho(2L)^2 = 2\rho L^2$ which is $4\times$more than the work to lift a cable of length L, not double.

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8. Using Hooke's Law, we have that $30N = k(12cm - 15cm)$ so $k = 10\frac{N}{cm}$. To calculate the work to stretch from 12cm (the natural length) to 20cm we have $W = \int_0^8 kxdx = \int_0^8 10xdx = 5x^2\Big|_0^8 = 320\, N \cdot cm = 3.2J$.

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9. Where the ions are held fixed is immaterial. What matters is the distance between them. If one ion is held fixed and the second is moved from a distance r_1 to a distance r_2

(relative to the position of the first ion) the work done is:

$$W = \int_{r_1}^{r_2} -\frac{q}{r^2} dr = \frac{q}{r}\bigg|_{r_1}^{r_2} = q\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$9a$) The relative distances are $r_1 = 3$ and $r_2 = 2$, so $W = q\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{q}{6}$.
$9b$) The relative distances are $r_1 = 2$ and $r_2 = 1$, so $W = q\left(\frac{1}{1} - \frac{1}{2}\right) = \frac{q}{2}$.

10a. Define the origin of x axis at the top of pyramid and direction of x axis pointing down.
i) the force f_i in order to lift a slice of square to position x_i is : $\rho g(\frac{x_i L}{H})^2 \Delta x$
ii) The moving distance d_i for slice i is :$(H - x_i)$
iii) So the total work W = $\int_0^H \rho g(\frac{xL}{H})^2 (H - x)dx = \frac{1}{12}\rho g L^2 H^2$

10b. If L and H are doubled, then the work increases by a factor of 16.

11a. Converges.

$$\int_0^\infty x^2 e^{-x} dx = \int_0^\infty x^2(-de^{-x}) = -\left[x^2 e^{-x}\Big|_0^\infty - \int_0^\infty e^{-x} 2x dx\right]$$
$$= 2\int_0^\infty e^{-x} x dx = -2\int_0^\infty x de^{-x} = -2\left[xe^{-x}\Big|_0^\infty - \int_0^\infty e^{-x} dx\right]$$
$$= 2\int_0^\infty e^{-x} dx = -2e^{-x}\Big|_0^\infty = 2.$$

11a. Diverges by p-test.

11b. Converges by comparison to e^{-x}. Using integration by parts: $\int_0^\infty xe^{-x} dx = 1$.

11c. Converges by comparison to e^{-x}. Note that

$$\int_0^\infty e^{-x} \sin x dx = -\int_0^\infty e^{-x} d\cos x = -e^{-x}\cos x\Big|_0^\infty + \int_0^\infty -e^{-x}\cos x dx = 1 - \int_0^\infty e^{-x}\cos x dx$$

Using the result of problem (1j) above:

$$\int_0^\infty e^{-x}\sin(x) dx = \int_0^\infty e^{-x}\cos(x) dx$$

we have

$$\int_0^\infty e^{-x}\sin x dx = 1/2.$$

11d. Converges. Don't fall into the trap of saying that each term diverges so the difference diverges; $\infty - \infty$ is inconclusive so you have to work harder. Integrating directly:

$$\int_1^\infty \frac{1}{x} - \frac{1}{x+1}dx = [\ln x - \ln(1+x)]\Big|_1^\infty = \lim_{x\to\infty} \ln\left(\frac{x}{1+x}\right) + \ln 2$$
$$= \ln 1 + \ln 2 = \ln 2.$$

11e. Converges. Using the substitution $x = \tan t$ (so $dx = \sec^2 t dt$) we have

$$\int_1^\infty \frac{dx}{1+x^2} = \int_{\pi/4}^{\pi/2} \frac{\sec^2 t}{1+\tan^2 t}dt = \int_{\pi/4}^{\pi/2} 1dt = \pi/4.$$

11f. Diverges. Morally, we know this because $\frac{1}{\sqrt{1+t^2}} \sim \frac{1}{t}$ when t is large and later integral diverges. To prove this we use comparison. We need to find a function $f(x) < \frac{1}{\sqrt{1+t^2}}$ for $x > 1$ such that $\int_1^\infty f(x)dx$ diverges. To do this we observe the following inequality:

$$\frac{1}{\sqrt{1+t^2}} = \frac{1}{t\sqrt{t^{-2}+1}} \geq \frac{1}{\sqrt{2}t} \quad \text{(for } t > 1)$$

So we have

$$\int_1^\infty \frac{1}{\sqrt{1+t^2}}dt \geq \int_1^\infty \frac{1}{\sqrt{2}t}dt = \infty$$

which establishes divergence.

11g. Diverges. The problem is at $x = 1$ not at $x = \infty$. Observe the following inequality: For $x \in [1, 2]$

$$0 \leq \frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} \leq \frac{1}{3(1-x)}$$

Now use this to write

$$\int_1^\infty \frac{1}{1-x^2}dx = \int_1^2 \frac{1}{1-x^2}dx + \int_2^\infty \frac{1}{1-x^2}dx$$
$$\leq \int_1^2 \frac{1}{3(1-x)}dx + \int_2^\infty \frac{1}{1-x^2}dx \leq \int_1^2 \frac{1}{3(1-x)} = -\infty,$$

which establishes divergence.

11h. Diverges. Use the substitution $u = 1 - x$. The apply the p-test.

11i. Diverges. The proof is nearly identical to 11g.

11j. Converges. The behavior near $x = 1$ of $\frac{1}{\sqrt{1-x^2}} \sim (1-x)^{-1/2}$ so we should have convergence. To get the exact value we use the trigonometric substitution $x = \sin t$, $dx = \cos t$:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \int_0^{\pi/2} \frac{\cos t}{\sqrt{1-\sin^2 t}} = \int_0^{\pi/2} dt = \pi/2.$$

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12. Use substitution $u = \frac{1}{x}$, then $du = -\frac{1}{x^2}dx = -u^2 dx$. Thus $dx = -\frac{1}{u^2}du$. Therefore,

$$\int_0^1 \frac{\ln x}{x^2+1}dx = \int_\infty^1 \frac{\ln(1/u)}{u^{-2}+1}\frac{-du}{u^2} = -\int_\infty^1 \frac{\ln(1/u)}{1+u^2}du$$
$$= \int_1^\infty \frac{\ln(1/u)}{1+u^2}du = -\int_1^\infty \frac{\ln u}{1+u^2}du$$

13a. The total dose is

$$D = \int_0^\infty 2te^{-2t}dt = \int_0^\infty ye^{-y}\frac{1}{2}dy = 1/2.$$

13b. Note that

$$D_5 = \int_0^5 2td^{-2t}dt = \int_0^{10} \frac{1}{2}ye^{-y}dy = -\frac{1}{2}\int_0^{10} yde^{-y}$$
$$= -\frac{1}{2}\left[ye^{-y}\Big|_0^{10} - \int_0^{10} e^{-y}dy\right] = -\frac{1}{2}\left[-10e^{-10} + e^{-y}\Big|_0^{10}\right]$$
$$= \frac{1}{2}[1 - 11e^{-10}]$$

Thus

$$\frac{D_5}{D} = \frac{\frac{1}{2}[1-11e^{-10}]}{1/2} = 0.9995.$$

14a. $y = \sqrt{1-x^2}$, $x \in [0,1]$. So

$$s = \int_0^1 \sqrt{1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2}dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \pi/2.$$

14b. $y = \int_0^x \sqrt{1-t^2}dt$ so $\frac{dy}{dx} = \frac{d}{dx}\int_0^x \sqrt{1-t^2}dt = \sqrt{1-x^2}$. Plugging this in we get

$$s = \int_0^1 \sqrt{1+(1-x^2)}dx = \int_0^1 \sqrt{2-x^2}dx = \frac{x}{\sqrt{2}}\sqrt{1-(x/\sqrt{2})^2} + \arcsin(x/\sqrt{2})\Big|_0^1 = 1/2 + \pi/4.$$

14c.

$$s = \int_0^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}dx = \int_0^1 \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}}dx$$
$$= \int_0^1 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}dx = \int_0^1 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}dx$$
$$= \int_0^1 \frac{e^x + e^{-x}}{2}dx = \frac{e^x - e^{-x}}{2}\Big|_0^1 = \frac{e - e^{-1}}{2}.$$

14d. Note that $f'(x) = \frac{3}{2}x^{1/2}$, thus

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx = \int_0^1 \sqrt{1 + \frac{9}{4}x}dx = \frac{4}{9}\int_1^{13/4} u^{1/2}du$$
$$= \frac{4}{9}\frac{u^{3/2}}{3/2}\Big|_1^{13/4} = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right).$$

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15. The curve $y = \sqrt{2x - x^2}$ is, by completing the square under the radical, $y = \sqrt{1 - (x-1)^2}$. This is the equation of the semi-circular arc of a circle of radius 1 centered at $(1, 0)$. You can verify this by graphing. Therefore, without doing any calculus, the arclength is $\frac{1}{2}2\pi r = \pi$.

16a.

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx = \int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}dx = \int_0^1 2\pi\sqrt{1 + 9x^4}x^3dx$$

set $u = 1 + 9x^4$, so that $du = 9 \cdot 4x^3dx$, and when $x = 0$, $u = 1$; when $x = 1$, $u = 10$, then we have

$$S = \int_1^{10} 2\pi\sqrt{u}\frac{1}{9 \cdot 4}du = \frac{1}{18}\pi\int_1^{10} \sqrt{u}du = \frac{1}{18}\pi\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{10} = \frac{\pi}{27}(\sqrt{10}^3 - 1)$$

16b

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx = \int_0^1 2\pi\sqrt{1-x}\sqrt{1 + (\frac{-1}{2\sqrt{1-x}})^2}dx$$
$$= \int_0^1 2\pi\sqrt{1-x}\sqrt{1 + \frac{1}{4(1-x)}}dx = \int_0^1 2\pi\sqrt{1-x}\sqrt{\frac{5-4x}{4(1-x)}}dx$$
$$= \int_0^1 \pi\sqrt{5-4x}dx = \int_0^1 \pi\sqrt{5-4x}(-\frac{1}{4})d(5-4x) = -\frac{\pi}{4}\frac{2}{3}(5-4x)^{\frac{3}{2}}\Big|_0^1$$
$$= -\frac{\pi}{6} + \frac{\pi}{6}(\sqrt{5})^3 = \frac{\pi}{6}((\sqrt{5})^3 - 1)$$

16c

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx = \int_0^1 2\pi\frac{e^x + e^{-x}}{2}\sqrt{1 + (\frac{e^x - e^{-x}}{2})^2}dx$$

here, set $u = \sinh x = \frac{e^x - e^{-x}}{2}$, so that $du = \frac{e^x + e^{-x}}{2}dx$, when $x = 0$, $u = 0$; when $x = 1$, $u = \frac{e - e^{-1}}{2} = \sinh 1$, then we have

$$
\begin{aligned}
S &= \int_0^{\sinh 1} 2\pi\sqrt{1+u^2}du \\
&= 2\pi\frac{1}{2}(u\sqrt{u^2+1}+\ln(u+\sqrt{u^2+1}))\Big|_0^{\sinh 1} \\
&= \pi(\sinh 1\sqrt{\sinh 1^2+1}+\ln(\sinh 1+\sqrt{\sinh 1^2+1}))
\end{aligned}
$$

16d

$$
y = (\frac{r_2-r_1}{x_2-x_1})(x-x_1)+r_1
$$

def: $k=\frac{r_2-r_1}{x_2-x_1}$, $b=-\frac{r_2-r_1}{x_2-x_1}x_1+r_1$ then we have $y=kx+b$, so that

$$
\begin{aligned}
S &= \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}dx = \int_{x_1}^{x_2} 2\pi(kx+b)\sqrt{1+k^2}dx \\
&= 2\pi\sqrt{1+k^2}\int_{x_1}^{x_2}(kx+b)dx = 2\pi\sqrt{1+k^2}(\frac{k}{2}x^2+bx)\Big|_{x_1}^{x_2} \\
&= 2\pi\sqrt{1+k^2}\left(\frac{k}{2}(x_2^2-x_1^2)+b(x_2-x_1)\right) \\
&= 2\pi\sqrt{1+k^2}(x_2-x_1)\left(\frac{k}{2}(x_2+x_1)+b\right) \\
&= 2\pi\sqrt{\frac{(x_2-x_1)^2+(r_2-r_1)^2}{(x_2-x_1)^2}}(x_2-x_1)\left(\frac{1}{2}\frac{r_2-r_1}{x_2-x_1}(x_2+x_1)-\frac{r_2-r_1}{x_2-x_1}x_1+r_1\right) \\
&= 2\pi\sqrt{(x_2-x_1)^2+(r_2-r_1)^2}\left(\frac{r_2-r_1}{x_2-x_1}(\frac{x_2+x_1}{2}-x_1)+r_1\right) \\
&= 2\pi\sqrt{(x_2-x_1)^2+(r_2-r_1)^2}\left(\frac{r_2-r_1}{x_2-x_1}\frac{x_2-x_1}{2}+r_1\right) \\
&= \pi\sqrt{(x_2-x_1)^2+(r_2-r_1)^2}(r_1+r_2)
\end{aligned}
$$

17a. The function is $f(x)=e^x$ and so $f'(x)=e^x$. We then find that the integral is given by

$$
\int_0^1 2\pi f(x)\sqrt{1+(f'(x))^2}\mathrm{d}x = \int_0^1 2\pi e^x\sqrt{1+e^{2x}}\mathrm{d}x.
$$

17b. The easiest method is to find the inverse, so $x=g(y)=\ln(y)$ and hence $g'(y)=y^{-1}$. We then find that the integral is given by

$$
\int_1^e 1\pi g(y)\sqrt{1+(g'(y))^2}\mathrm{d}y = \int_1^e 2\pi\ln(y)\sqrt{1+y^{-2}}\mathrm{d}y.
$$

The surface area in part a is larger because the same length of curve is rotated round a larger radius.

---

18. We assume that the planes are in the right half plane, $x > 0$, for if they are not we can either perform a similar calculation for the left half plane, or if they are in both, perform two calculations, one for the left half plane and the other for the right half plane. A hemisphere is then generate by rotation the curve $f(x) = \sqrt{r^2 - x^2}$ around the x axis. The formula for the surface area is

$$S = 2\pi \int_z^b f(x)\sqrt{1 + (f'(x))^2}\mathrm{d}x.$$

Here

$$1 + (f'(x))^2 = \frac{r^2}{r^2 - x^2},$$

and so we find that

$$S = 2\pi \int_z^b \sqrt{r^2 - x^2}\frac{r}{\sqrt{r^2 - x^2}}\mathrm{d}x = 2\pi r(b - a).$$

Letting $d = b - a$ we find that $S = 2\pi rd$ which shows that the area depends only on the distance between the planes and not their location.

---

19. (b)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Longrightarrow y = \pm\sqrt{b^2 - \left(\frac{b}{a}\right)^2 x^2}.$$

Then, using the upper half of the ellipse

$$f(x) = b\sqrt{1 - \left(\frac{x}{a}\right)^2} = \frac{b}{a}\sqrt{a^2 - x^2},$$

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}},$$

and

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{b}{a}\right)^2 \frac{x^2}{a^2 - x^2}} = \frac{\sqrt{a^2 - x^2 + \left(\frac{b}{a}\right)^2 x^2}}{\sqrt{a^2 - x^2}}.$$

The area (A) of the ellipse for $-a \leq x \leq a$ is given as:

$$A = 2\int_{-a}^{a} f(x)dx = 2b\int_{-a}^{a} \sqrt{1 - \left(\frac{x}{a}\right)^2}dx.$$

The arc length (L) for $-a \leq x \leq a$ is given as:

$$L = \int_{-a}^{a} \sqrt{1 + [f'(x)]^2}dx = \int_{-a}^{a} \sqrt{1 + \left(\frac{b}{a}\right)^2 \frac{x^2}{a^2 - x^2}}dx.$$

The surface area (S) for $-a \le x \le a$ is given as:

$$\begin{aligned} S = 2\pi \int_{-a}^{a} f(x)\sqrt{1+[f'(x)]^2}dx &= 2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\frac{\sqrt{a^2-x^2+\left(\frac{b}{a}\right)^2 x^2}}{\sqrt{a^2-x^2}}dx \\ &= 2\pi\frac{b}{a}\int_{-a}^{a}\sqrt{a^2+\left(\frac{b^2-a^2}{a^2}\right)x^2dx} \end{aligned}$$

(c) Evaluating the area to show that $A = \pi ab$.

$$\begin{aligned} A = 2\int_{-a}^{a} f(x)dx &= 2b\int_{-a}^{a}\sqrt{1-\left(\frac{x}{a}\right)^2}dx \\ &\text{With } u = \tfrac{x}{a},\ dx = adu,\ u(-a) = -1 \text{ and } u(a) = 1, \text{ then} \\ A &= 2ab\int_{-1}^{1}\sqrt{1-u^2}du \\ &\text{The integral is the area of a semicircle of radius 1 is } \tfrac{\pi}{2}, \text{ thus} \\ A &= 2ab\frac{\pi}{2} = \pi ab. \end{aligned}$$

Now we will show that $S = 2\pi b(b + a\sin^{-1}(c)/c)$ where $c = \frac{\sqrt{a^2-b^2}}{a}$;

$$\begin{aligned} S &= 2\pi\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-+\left(\frac{b^2-a^2}{a^2}\right)x^2dx} \\ &= 2\pi\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-\left(\frac{a^2-b^2}{a^2}\right)x^2dx} \\ &= 2\pi\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-\left(\frac{\sqrt{a^2-b^2}}{a}x\right)^2}dx \\ &= 2\pi\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-(cx)^2}dx \\ &\text{With } u = cx,\ du = cdx,\ u(-a) = -ac \text{ and } u(a) = ac, \text{ then} \\ S &= 2\pi\frac{b}{ac}\int_{-a}^{a}\sqrt{a^2-(u)^2}du \\ &= 2\pi\frac{b}{ac}\left[\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)\right]_{-ac}^{ac} \\ &= 2\pi\frac{b}{ac}(ac\sqrt{a^2-a^2c^2}+a^2\sin^{-1}(c)) \\ &= 2\pi b(\sqrt{a^2-a^2c^2}+\frac{a}{c}\sin^{-1}(c)) \end{aligned}$$

Notice that

$$\sqrt{a^2 - a^2c^2} = \sqrt{a^2 - a^2\left(\frac{a^2 - b^2}{a^2}\right)} = \sqrt{b^2} = b$$

Then

$$S = S = 2\pi b(b + a\sin^{-1}(c)/c)$$