

You may use these integrals: $\int x^n dx = \frac{x^{n+1}}{n+1}$ ($n \neq -1$), $\int \frac{dx}{x} = \ln x$, $\int e^x dx = e^x$, $\int \sin x dx = -\cos x$, $\int \cos x dx = \sin x$, $\int \sec \theta d\theta = \ln(\sec \theta + \tan \theta)$, $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta))$, but all others should be derived.

1. True or False? Justify your answer with a reason or counterexample.

a) $\int_0^\infty x^2 e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-x^2} dx$

b) $\bar{x} = M_y/m = \int_a^b x f(x) dx / \int_a^b f(x) dx = \int_a^b x dx / \int_a^b dx = \frac{1}{2}(b^2 - a^2)/(b - a) = \frac{1}{2}(a + b)$

c) If a region R is symmetric about a line l , then the center of mass of R lies on l .

d) If $f(x)$ is the pdf of a random variable X with mean μ , then $\text{prob}(X \leq \mu) = \text{prob}(X \geq \mu)$.

e) If X is a normally distributed random variable with mean μ and standard deviation σ , then $\text{prob}(\mu \leq X \leq \mu + \sigma) = \text{prob}(\mu + \sigma \leq X \leq \mu + 2\sigma)$.

f) If a radioactive material has half-life 100 years and a sample has initial mass 1 kg, then the amount remaining after 50 years is $\frac{1}{2}$ kg.

g) The function defined by $f(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$ is a valid pdf.

h) In solving a differential equation by Euler's method, if the time step h decreases by a factor of $\frac{1}{2}$, then the error in the numerical solution decreases by a factor of approximately $\frac{1}{4}$.

i) $\cosh x > \sinh x$ for all x j) $\tanh x$ is an even function k) $\sinh^2 x + \cosh^2 x = 1$

l) The Taylor polynomial of degree 1 for $f(x) = e^x$ at $x = 0$ is $T_1(x) = 1 + x$.

m) A convergent sequence is bounded.

n) A convergent sequence is either increasing or decreasing.

o) If $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} b_n = \infty$, then $\lim_{n \rightarrow \infty} a_n b_n = 0$.

p) If $0 \leq a_n \leq 1$ and $a_{n+1} < a_n$, then $\lim_{n \rightarrow \infty} a_n = 0$.

q) $1 = \lim_{n \rightarrow \infty} 1 = \lim_{n \rightarrow \infty} (n + 1 - n) = \lim_{n \rightarrow \infty} (n + 1) - \lim_{n \rightarrow \infty} n = \infty - \infty = 0$

r) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is a divergent geometric series

s) If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. t) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

section 9.3 (center of mass)

2. A one-dimensional metal rod on the interval $a \leq x \leq b$ has variable density $\rho(x)$ in units of kg/m. Find an expression for the center of mass of the rod.

3. Sketch the region, find the center of mass, and indicate CM on the sketch.

a) $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$ b) $\{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$

c) $\{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq \sqrt{x(2-x)}\}$ d) $\{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq \cosh x\}$

e) $\{(x, y) : -2 \leq x \leq 2, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$

f) $\{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq x^{-1}\}$ - also, use this result to show that $\ln 2 > \frac{2}{3}$

4. Find the volume of the shape using the theorem of Pappus.

a) a sphere of radius r

b) a cone of height h and base radius r

c) the shape formed by rotating a circle of radius r about the tangent line to the circle at a point on its circumference (“a doughnut with no hole”)

5. Consider a set of n particles located on the x -axis with mass m_i and position x_i , for $i = 1, \dots, n$. Define the function $f(x) = \sum_{i=1}^n m_i(x - x_i)^2$. Show that $f(x)$ is minimized when x is the center of mass of the particle distribution.

section 9.5 (probability)

6. The speed of cars on a certain highway is normally distributed with mean 55 mph and standard deviation 5 mph. What fraction of cars are traveling between 50 mph and 60 mph? ... greater than 70 mph?

7. The lifetime of a certain car battery is a random variable with pdf $f(t) = \frac{6}{625}t^2(5 - t)^2$ for $0 \leq t \leq 5$ and zero otherwise, where the time t is measured in years.

a) Sketch $f(t)$ and show that it defines a valid pdf. Find the mean battery lifetime.

b) Among 1000 batteries chosen at random, how many will last at least 3 years?

8. The length of time spent waiting in line to vote in a certain district is modeled by an exponential density function with mean 20 minutes.

a) Find the probability that a voter waits in line less than 10 minutes. (take $\sqrt{e} = 1.6$)

b) Find the probability that a voter waits in line more than 30 minutes.

c) Find the median waiting time. (take $\ln 2 = 0.7$)

section 10.1 (differential equations)

9. Which of the following functions satisfies the differential equation $y'' + 2y' + y = 0$?

a) $y = e^{-t}$ b) $y = 2e^{-t}$ c) $y = e^{-2t}$ d) $y = te^{-t}$ e) $y = t^2e^{-t}$

10. Find the constant solutions, sketch the phase plane, and determine whether the constant solutions are stable or unstable.

a) $y' = y - 1$ b) $y' = y^2 - 1$ c) $y' = y^2 - 2y + 1$ d) $y' = y^2 - 3y + 2$ e) $y' = \sin y$

11. A particle of mass m and time-dependent position $x(t)$ is moving under the influence of a force $f(x)$. Newton's 2nd law states that $x(t)$ satisfies the differential equation $mx'' = f(x)$. Let $f(x) = -V'(x)$, where $V(x)$ is the potential energy function. The total energy of the particle (kinetic + potential) is $E(t) = \frac{1}{2}mv^2 + V(x)$, where $x = x(t)$ is the particle position and $v = x'(t)$ is the particle velocity. Show that the total energy is constant in time.

section 10.3 (separation of variables)

12. Solve for $y(t)$ subject to initial condition $y(0) = 1$. Sketch the solution for $t \geq 0$.

a) $y' = y$ b) $y' = ty$ c) $y' = y^2$ d) $y' = y(1 - y)$

13. A tank initially contains 1000 liters of pure water. Sea water containing 0.05 kg of salt per liter enters the tank at a rate of 5 liter/min. The solution is kept well mixed and drains from the tank at the same rate as it enters. Find the salt concentration in the tank after one hour.

14. A certain country has \$10 billion in paper currency in circulation at any time. Due to normal business transactions, each day \$50 million flows into the country's banks and the same amount flows out. The paper currency is becoming worn out and the government decides to replace the old bills with new bills whenever old bills come into a bank. Let $x(t)$ denote the value of new currency in circulation at time t . Assume that $x(0) = 0$.

a) Write down a differential equation for $x(t)$. Use 1 day as the unit of time and \$1 billion as the unit of currency. You may assume that when the new bills are released each day, they are instantaneously mixed with the currency in circulation.

b) Solve for $x(t)$.

c) How long will it take for new currency to reach 90% of the amount in circulation?

15. In a certain chemical reaction, one molecule of type A and one molecule of type B combine to form one molecule of type C, $A + B \rightarrow C$. Let a_0, b_0 denote the initial concentrations of reactants A, B, and let $c(t)$ be the concentration of product C at time t . The law of mass action states that the reaction rate is proportional to the product of the reactant concentrations, $c' = k(a_0 - c)(b_0 - c)$. Take $a_0 = 1$ mole/L, $b_0 = 2$ mole/L, and assume that no product is present at the start of the reaction. Sketch the phase plane of the system. Find the product concentration $c(t)$ and sketch the graph for $t \geq 0$. Find the value of the product concentration in the limit $t \rightarrow \infty$. Would this value change if the initial product concentration was 1.5 mole/L instead of zero?

16. When an object is heated to a high temperature T_0 and then removed from the heat source, the object's temperature $T(t)$ decays according to the radiation equation, $mc_P T' = -e\sigma T^4$, where m is the mass of the object, c_P is the specific heat capacity, e is the emissivity, and σ is Stefan's constant. Assume that m, c_P, e, σ are all positive. Find the temperature $T(t)$ and sketch the graph for $t \geq 0$. In the limit $t \rightarrow \infty$, the temperature decays like $t^{-\alpha}$; find the value of α .

section 10.4 (exponential growth and decay)

17. The mass of a radioactive sample is 128 kg after two hours and 2 kg after five hours. What was the initial mass of the sample? How long will it take for the sample to decay from 2 kg to 1 kg?

18. A thermometer is taken outside from an air-conditioned room where the temperature is 21°C. It reads 27°C after one minute and 30°C after two minutes. Find the outdoor temperature.

19. The university endowment receives contributions at a rate of r dollars/year and a certain amount is continually spent at a rate proportional to the endowment size, with proportionality factor α . Assume that $r > 0, \alpha > 0$.

a) Set up a differential equation for $y(t)$, the size of the endowment at time t .

b) Let M be the constant solution of the equation. Find M in terms of r and α . Sketch the phase plane of the differential equation. Is the constant solution stable or unstable?

c) The endowment dropped to $\frac{1}{2}M$ last year. How long will it take to recover to $\frac{3}{4}M$?

20. Bob and Ray order 8 oz cups of coffee which are served steaming hot at temperature T_h . They use different strategies to cool the coffee. Bob immediately adds 1 oz of cold milk with temperature $T_c < T_h$ and waits 2 minutes before drinking, while Ray waits 2 minutes and then adds 1 oz of cold milk. Assume the ambient temperature T_a in the cafe satisfies $T_c < T_a < T_h$. Who ends up drinking cooler coffee, Bob or Ray? You may explain your answer intuitively, but for full credit you should justify your answer by finding $T_{\text{Bob}}(t), T_{\text{Ray}}(t)$ in terms of T_c, T_a, T_h and k (where k is the temperature rate constant for cooling a cup of coffee).

section 10.5 (logistic equation)

21. Assume the rate at which a rumor spreads is proportional to the product of two terms, the fraction of the population who have already heard the rumor and the fraction of the population who have not yet heard the rumor. Consider a town with 1000 inhabitants. Suppose that 10 people have heard a certain rumor at 8am and 20 people have heard the rumor at 9am. At what time will half the population have heard the rumor?

sections 12.1 (sequences), 12.2 (series)

22. Determine whether the sequence $\{a_n\}$ converges or diverges. If it converges, give the limit.

a) $a_n = \frac{n}{n+1}$ b) $a_n = \frac{(-1)^n}{n}$ c) $a_n = \frac{2^n}{n}$ d) $a_n = 1 - \frac{1}{n}$ e) $a_n = \left(1 - \frac{1}{n}\right)^n$

23. Find the sum of the series.

a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$ b) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \dots$ c) $1 + 0.1 + 0.01 + 0.001 + \dots$

hyperbolic functions

24. Find the antiderivative. a) $\int \cosh 2x \, dx$ b) $\int \tanh 2x \, dx$ c) $\int \cosh^2 x \, dx$

25. Consider the curve $y = \cosh x$ on the interval $-1 \leq x \leq 1$. Find (a) the arclength of the curve and (b) the surface area obtained by rotating the curve about the x -axis.

26. Derive the following addition formulas for \sinh and \cosh . (Hint: first derive (a); then (b), (c), (d) can be derived from (a) with little extra work.)

a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ b) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$ d) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

Taylor polynomials

27. Consider $f(x)$ and $x = a$ given below. In each case find $T_1(x)$ and $T_2(x)$, the linear and quadratic Taylor approximations at $x = a$. Sketch $f(x), T_1(x), T_2(x)$ on the same graph.

(i) $f(x) = x^{-1}, a = 1$ (ii) $f(x) = \sin x, a = \frac{\pi}{2}$ (iii) $f(x) = \sqrt{x}, a = 4$

Note that setting $x = 5$ in case (iii) yields $f(5) = \sqrt{5}$, so we can approximate $\sqrt{5}$ by $T_1(5)$ or $T_2(5)$. Find these two values. Which is a more accurate approximation for $\sqrt{5}$? Why?