

For full credit, justify your answer, and give units if appropriate. You may use $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

1. True or False? Justify your answer with a reason or counterexample.

a) $\int_0^{\infty} x^2 e^{-x^2} dx = \frac{1}{2} \int_0^{\infty} e^{-x^2} dx$ b) $\lim_{n \rightarrow \infty} (1 - \frac{1}{n})^n = -e$

c) The center of mass of the region $\{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$ lies on the line $y = x$.

d) If (\bar{x}, \bar{y}) is the center of mass of a region in the xy -plane with density $\rho = 1$, then the area of the region to the left of the line $x = \bar{x}$ is the same as the area to the right of the line.

e) If $f(x)$ is a probability density function, then $0 \leq f(x) \leq 1$ for all x .

f) The function defined by $f(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$ is a valid pdf.

g) If $f(x)$ is the pdf of a random variable X with mean μ , then $\text{prob}(X \leq \mu) = \text{prob}(X \geq \mu)$.

h) $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} (x - \mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 0$

i) The function $y(t) = \tanh t$ is a solution of the differential equation $y' = 1 - y^2$.

j) If a bacteria culture starts with 1000 cells and grows at a rate proportional to its size, and there are 2500 cells after 2 hours, then there are 4000 cells after 4 hours.

k) If a radioactive material has half-life 100 years and a sample has initial mass 1 kg, then the amount remaining after 50 years is $\frac{1}{2}$ kg.

l) If $y' = ky$ and $k > 0$, then $y(t) = 0$ is a stable constant solution of the differential equation.

m) If $y' = 1 - y^2$ and $y(0) = \frac{1}{2}$, then $\lim_{t \rightarrow \infty} y(t) = 1$.

n) In solving a differential equation $y' = f(y)$ by Euler's method, if the time step Δt is reduced by a factor of $\frac{1}{2}$, then the error in the numerical solution is reduced by a factor of approximately $\frac{1}{4}$.

o) $\cosh x > \sinh x$ for all x p) $\sinh^2 x + \cosh^2 x = 1$ q) $\lim_{x \rightarrow \infty} \tanh x = 0$

r) $\cosh x$ is an even function s) $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x}$ t) $\sinh 2x = 2 \sinh x \cosh x$

u) The Taylor polynomial of degree 1 for $f(x) = e^x$ at $x = 0$ is $T_1(x) = 1 + x$.

v) If $0 \leq a_n \leq 1$ and $a_{n+1} < a_n$, then $\lim_{n \rightarrow \infty} a_n = 0$.

w) $1 = \lim_{n \rightarrow \infty} 1 = \lim_{n \rightarrow \infty} (n + 1 - n) = \lim_{n \rightarrow \infty} (n + 1) - \lim_{n \rightarrow \infty} n = \infty - \infty = 0$

x) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is a divergent geometric series y) $9 + 0.9 + 0.09 + 0.009 + \dots = 10$

z) If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. aa) If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

bb) If $0 \leq a_n \leq \frac{1}{n^2}$ for all n , then $\sum_{n=1}^{\infty} a_n$ converges.

cc) If $a_n = f(n)$, where $f(x)$ is positive and decreasing, then $\sum_{n=1}^{\infty} a_n$ converges.

dd) If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = s$, then $\frac{1}{2} < s < \frac{3}{2}$.

surface area

2. Find the surface area obtained by rotating the curve about the x -axis.

a) $y = x^3$, $0 \leq x \leq 1$ b) $y = \sqrt{1-x}$, $0 \leq x \leq 1$ c) $y = \cosh x$, $0 \leq x \leq 1$

3. Let S be the surface area of a zone on a sphere between two parallel planes. Show that $S = 2\pi rd$, where r is the radius of the sphere and d is the distance between the planes. (Hence S depends on the distance between the planes, but not on their location.)

4. An ellipse is defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$. (i) Sketch the ellipse in the xy -plane. (ii) Set up the integral for the surface area S obtained by rotating the upper half of the ellipse about the x -axis. (iii) Show that $S = 2\pi b(b + a(\sin^{-1} c)/c)$, where $c = \sqrt{a^2 - b^2}/a$. Check your answer in the limit $a \rightarrow b$.

center of mass

5. Consider 3 point masses on the x -axis with $m_1 = 2, m_2 = 3, m_3 = 1$, and $x_1 = -10, x_2 = 6$. Where should m_3 be located to ensure the CM is at $\bar{x} = 0$?

6. Sketch the region, find the center of mass, and indicate CM on the sketch.

a) $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$ b) $\{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq \sqrt{x(2-x)}\}$

c) $\{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq \cosh x\}$ d) $\{(x, y) : -2 \leq x \leq 2, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$

e) $\{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq x^{-1}\}$

7. Find the volume of the shape using the theorem of Pappus.

a) a sphere of radius r b) a cone of height h and base radius r

c) the shape formed by rotating a circle of radius r about the tangent line to the circle at a point on its circumference (“a doughnut with no hole”)

8. Consider n particles located on the x -axis with mass m_i and position x_i , for $i = 1, \dots, n$. Define the function $f(x) = \sum_{i=1}^n m_i(x - x_i)^2$. Show that $f(x)$ is minimized when $x = \bar{x}$, where \bar{x} is the center of mass of the particle distribution.

probability

9. The speed of cars on a highway is normally distributed with mean 55 mph and standard deviation 5 mph. Find the fraction of cars traveling between 50 mph and 60 mph.

10. The lifetime of a car battery is a random variable with pdf $f(t) = \frac{6}{625}t^2(5-t)^2$ for $0 \leq t \leq 5$ and zero otherwise, where the time t is measured in years.

a) Sketch $f(t)$ and show that it defines a valid pdf. Find the mean battery lifetime.

b) Among 1000 batteries chosen at random, about how many will last at least 3 years?

11. The time waiting in line to vote is modeled by an exponential pdf with mean 20 minutes.

a) Find the probability that a voter waits in line less than 10 minutes. (take $\sqrt{e} = 1.6$)

b) Find the probability that a voter waits in line more than 30 minutes.

c) Find the median waiting time. (take $\ln 2 = 0.7$)

12. For a normal pdf, find the probability that the random variable lies within two standard deviations of the mean. Express the answer in terms of the error function $\operatorname{erf}(x)$.

13. Find the standard deviation of an exponential pdf with mean μ .

differential equations

14. Which of the following functions satisfy the differential equation $y'' + 2y' + y = 0$?

a) $y = e^{-t}$ b) $y = -e^{-t}$ c) $y = 2e^{-t}$ d) $y = e^{-2t}$ e) $y = te^{-t}$ f) $y = t^2e^{-t}$

15. Find the constant solutions, sketch the phase plane, and determine whether the constant solutions are stable or unstable.

a) $y' = y - 1$ b) $y' = y^2 - 1$ c) $y' = y^2 - 2y + 1$ d) $y' = y^2 - 3y + 2$ e) $y' = \sin y$

16. A particle of mass m and position $x(t)$ is moving under the influence of a force $f(x)$. Newton's 2nd law states that $x(t)$ satisfies the differential equation $mx'' = f(x)$. Let $f(x) = -V'(x)$, where $V(x)$ is the potential energy function. The total energy of the particle (kinetic + potential) is

$E(t) = \frac{1}{2}mv^2 + V(x)$, where $x = x(t)$ is the particle position and $v = x'(t)$ is the particle velocity. Show that the total energy is constant in time.

17. Solve for $y(t)$ subject to initial condition $y(0) = 1$. Sketch the solution for $t \geq 0$.

a) $y' = y$ b) $y' = ty$ c) $y' = y^2$ d) $y' = y(1 - y)$

18. A tank initially contains 1000 liters of pure water. Sea water containing 0.05 kg of salt per liter enters the tank at a rate of 5 liter/min. The solution is kept well mixed and drains from the tank at the same rate it enters. a) Find the salt concentration in the tank after one hour. b) How much salt is present in the tank in the limit $t \rightarrow \infty$?

19. A country has \$10B in paper currency in circulation at any time. There are two types of bills, depicting the old president and the new president. Due to business transactions, \$50M flows into the banks each day; the old-type bills are removed from circulation and \$50M of new-type bills is released into circulation. Let $x(t)$ denote the value of all new-type bills in circulation at time t . Assume $x(0) = 0$. Use \$1B as the unit of currency and 1 day as the unit of time. a) Write down the differential equation for $x(t)$. b) Solve for $x(t)$. c) How long will it take for the new-type bills to reach 90% of the total currency in circulation? You may use the approximation $\ln 10 \approx 2.3$.

20. In a chemical reactor, one molecule of type A and one molecule of type B combine to form one molecule of type C, $A + B \rightarrow C$. Let a_0, b_0 be the initial concentrations of reactants A, B, and let $c(t)$ be the concentration of product C at time t . The law of mass action says that the reaction rate is proportional to the product of the reactant concentrations, $c' = k(a_0 - c)(b_0 - c)$, where $k > 0$. Take $a_0 = 1$ mole/L, $b_0 = 2$ mole/L, and sketch the phase plane of the system. Find the product concentration $c(t)$ assuming that no product is initially present in the reactor. Find the value of the product concentration in the limit $t \rightarrow \infty$. Would this value change if the initial product concentration was 1.5 mole/L instead of zero?

21. When an object is heated to a high temperature T_0 and then removed from the heat source, the object's temperature $T(t)$ decays according to the radiation equation, $mc_P T' = -e\sigma T^4$, where m is the object's mass, c_P is the specific heat capacity, e is the emissivity, and σ is Stefan's constant. Assume that m, c_P, e, σ are all positive. Find the temperature $T(t)$ and sketch the graph for $t \geq 0$. In the limit $t \rightarrow \infty$, the temperature $T(t)$ decays like $t^{-\alpha}$; find α .

22. A radioactive sample has mass 128 kg after two hours and mass 2 kg after five hours. (a) What was the initial mass of the sample? (b) When will the sample mass decay to 1 kg? Express the answer in simplest form.

23. A \$500 investment has a 12% annual interest rate. Find the investment value after 10 years if it is a) compounded annually, b) compounded continuously. c) Find the equivalent annual interest rate.

24. A thermometer is taken outside from an air-conditioned room where the temperature is 21°C . It reads 27°C after one minute and 30°C after two minutes. Find the outdoor temperature.

25. The university endowment receives donations at a rate of r dollars/year, and a certain amount is spent at a rate proportional to the endowment size with rate constant k . Assume that $r > 0, k > 0$. Let $m(t)$ denote the value of the endowment at time t in years.

a) Which differential equation expresses the rate of change of the endowment in time?

i) $m' = rm + k$ ii) $m' = r + km$ iii) $m' = r - km$ iv) $m' = km - r$

b) Let M be the constant solution of the equation. Find M in terms of r and k . Sketch the phase plane of the differential equation. Is the constant solution stable or unstable?

c) The endowment drops to $\frac{1}{2}M$ in a recession. How long will it take to recover to $\frac{3}{4}M$? Assume that the annual spending rate is $k = 0.05$ and take $\ln 2 \approx 0.7$.

26. Bob and Ray each order 8 oz cups of coffee which are served steaming hot at temperature T_h . They use different strategies to cool the coffee. Bob immediately adds 1 oz of cold milk at temperature $T_c < T_h$ and then waits 2 minutes before drinking, while Ray waits 2 minutes and then adds 1 oz of cold milk. Assume the ambient temperature T_a in the cafe satisfies $T_c < T_a < T_h$. Who ends up drinking cooler coffee, Bob or Ray? You may explain your answer intuitively, but for full credit you should justify your answer by finding $T_{Bob}(t), T_{Ray}(t)$ in terms of T_c, T_a, T_h and k , where k is the temperature rate constant for cooling a cup of coffee.

27. Assume the rate at which a rumor spreads is proportional to the product of two terms, the fraction of the population who have already heard the rumor and the fraction of the population who have not yet heard the rumor. Consider a town with 1000 inhabitants. Suppose 10 people have heard a certain rumor at 8am and 20 people have heard the rumor at 9am. At what time will half the population have heard the rumor?

series

28. Find the sum of the series.

a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$ c) $1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots$

29. Consider the series $\sum_{n=0}^{\infty} (2x - 1)^n$. (a) For what values of x does the series converge? (b) For those values, find a formula for the sum of the series in terms of x .

30. Determine whether the series converges or diverges. Justify your answer.

a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ c) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ d) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ e) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ f) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n^2+1}$

31. For each series, if the sum s is approximated by the n th partial sum s_n , how large must n be to ensure the error $|s - s_n|$ is less than 0.02? a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

hyperbolic functions

32. Find the antiderivative. a) $\int \cosh 2x \, dx$ b) $\int \tanh 2x \, dx$ c) $\int \cosh^2 x \, dx$

33. Find the surface area obtained by rotating $y = \cosh x, -1 \leq x \leq 1$ about the x -axis.

34. Derive the following addition formulas for sinh and cosh. (hint: first derive (a), then derive (b), (c), (d) from (a) with relatively little extra work)

(a) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ (b) $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

(c) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$ (d) $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

35. a) Find $\lim_{x \rightarrow \pm\infty} \operatorname{sech} x$. b) Find $\frac{d}{dx} \operatorname{sech} x$. c) Sketch $y = \operatorname{sech} x$ for $-\infty < x < \infty$.

Taylor approximation

36. Consider $f(x)$ and $x = a$ given below. In each case find $T_1(x)$, the linear Taylor approximation to $f(x)$ at $x = a$. Sketch $f(x), T_1(x)$ on the same graph.

a) $f(x) = x^{-1}, a = 1$ b) $f(x) = \sin x, a = \frac{\pi}{2}$ c) $f(x) = \sqrt{x}, a = 4$

In (c), setting $x = 5$ yields $f(5) = \sqrt{5} = 2.2361$; compute the approximation $T_1(5)$.

miscellaneous

37. On homework you showed that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$. Now find $\lim_{n \rightarrow \infty} n \left(\left(1 + \frac{1}{n}\right)^n - e \right)$.

This confirms a result about Euler's method on page 49 of the lecture notes.

38. The alternating series test says that if $\{a_n\}$ satisfies (1) $a_n > 0$, (2) $a_{n+1} < a_n$, (3) $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges. Show by example that if conditions (1) and (3) hold, but (2) doesn't, then the series may converge or diverge.