Review Sheet Solutions for Midterm Test 2 (Math 156) Fall 2015

Question 1 Solution

a) True. use integration by parts : \( u = x, dv = xe^{-x^2} \Rightarrow du = dx, v = -\frac{1}{2}e^{-x^2} \)
\[
\int_{0}^{\infty} x^2 e^{-x^2} \, dx = x(-\frac{1}{2}e^{-x^2})|_{0}^{\infty} = -\frac{1}{2}e^{-x^2} \, dx = \frac{1}{2} \int_{0}^{\infty} e^{-x^2} \, dx
\]
b) False. \( \lim_{n \to \infty} (1 - \frac{1}{n})^n = L \Rightarrow \ln L = \ln \lim_{n \to \infty} (1 - \frac{1}{n})^n = \ln \lim_{n \to \infty} n(1 - \frac{1}{n}) = \infty \cdot 0 \)
set \( u = \frac{1}{n} \Rightarrow \ln L = \lim_{u \to 0} \frac{\ln(1-u)}{u} = 0 = \lim_{u \to 0} \frac{\frac{1}{2}u^2}{\frac{1}{4}u} = -1 \Rightarrow L = e^{-1} \)
c) True The region is symmetric about the line \( y = x \) and hence by the symmetry principle the CM lies on the line \( y = x \).
d) False. A counterexample was given in class on page 34 of the lecture notes. A triangular plate has vertices in the \( xy \)-plane at \((0,0), (1,0), (0,1)\). Then \( \bar{x} = \frac{1}{3} \). However the area to the left of the line \( x = \bar{x} \) is \( \frac{5}{18} \) and the area to the right is \( \frac{4}{18} \).
e) False. A counterexample is given by the normal distribution \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \). Then \( f(\mu) = \frac{1}{\sqrt{2\pi}} > 1 \) for \( \sigma < \sqrt{2\pi} \).
f) True. 1. \( f(x) \geq 0 \) for all \( x \)
\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi}\sqrt{1-x^2} \, dx \quad \text{set} \quad x = \sin \theta, \, dx = \cos \theta \, d\theta
\]
\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi}\cos \theta \, d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \, d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{\pi}{\pi} \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1
\]
g) False. It is the median \( m \) which satisfies \( \text{prob}(X \leq m) = \text{prob}(X \geq m) \). In general, \( \mu \neq m \) and \( \text{prob}(X \leq m) \neq \text{prob}(X \geq m) \), unless the pdf is symmetric about \( x = \mu \).
h) True. The pdf of a normal distribution is \( f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \) and we know from hw7 that
\[
\int_{-\infty}^{\infty} f(x) \, dx = 1, \int_{-\infty}^{\infty} xf(x) \, dx = \mu. \quad \text{So the given integral is} \quad \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \int_{-\infty}^{\infty} (x-\mu)f(x) \, dx = \int_{-\infty}^{\infty} xf(x) \, dx - \mu \int_{-\infty}^{\infty} f(x) \, dx = \mu - \mu = 0.
\]
i) False. See page 42 of the lecture notes. When a population grows at a rate proportional to itself, the population increases at an exponential rate, not a linear rate. The cell count is \( y(t) = 1000(2.5)^{t/2} \), so \( y(4) = 6250 \) cells.
j) False. \( y(t) = 0 \) is a constant solution, but it is unstable because if \( y_0 \neq 0 \), then \( \lim_{t \to \infty} y(t) = \lim_{t \to \infty} y_0 e^t \neq 0 \).
k) False. As shown in class and on homework, if \( \Delta t \) is reduced by a factor of \( \frac{1}{2} \), then the error in Euler’s method decreases by a factor of approximately \( \frac{1}{4} \), not \( \frac{1}{2} \).
l) True. \( \cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x} > 0 \) for all \( x \)
m) False. \( \sinh^2 x + \cosh^2 x = (\frac{e^x + e^{-x}}{2})^2 + (\frac{e^x - e^{-x}}{2})^2 = \frac{e^{2x} - 2e^{-2x}}{4} + \frac{e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \neq 1 \) for \( x \neq 0 \)
n) False. \( \lim_{x \to \infty} \frac{\sinh x}{\cosh x} = \lim_{x \to \infty} \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \infty = \lim_{x \to \infty} \frac{\frac{1 - e^{-2x}}{2}}{\frac{1 + e^{-2x}}{2}} = 1 \)
o) True. By definition \( \cosh x = \frac{e^x + e^{-x}}{2} \), so \( \cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x \), so \( \cosh x \) is an even function.
p) True. \( \frac{d}{dx} \tanh x = \frac{d}{dx} \sinh x \cosh x = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \)

q) True. \( 2 \sinh x \cosh x = 2(e^{x} - e^{-x})(e^{x} + e^{-x}) = \frac{1}{2} (e^{2x} - e^{-2x}) = \sinh(2x) \)

r) True. \( T_1(x) = f(a) + f'(a)(x - a) \), then \( f(x) = e^x, a = 0 \Rightarrow f(0) = 1, f'(0) = 1 \Rightarrow T_1(x) = 1 + x \)

s) True. This was a theorem stated in class.

t) False. A convergent sequence may converge by damped oscillations, for example \( a_n = (-1)^n \).

u) False. counterexample: \( a_n = \frac{1}{n}, b_n = n \Rightarrow \text{lim} \ a_n = 0, \text{lim} \ b_n = \infty \), but \( \text{lim} \ a_nb_n = 1 \).

v) False. counterexample: \( a_n = \frac{1}{2} + \frac{1}{2^n} \Rightarrow 0 \leq a_n \leq 1 \) and \( a_{n+1} < a_n \), but \( \text{lim} \ a_n = \frac{1}{2} \neq 0 \)

The two given conditions imply the sequence converges, but the limit is not necessary zero.

w) False. The 3rd equal sign is incorrect, i.e. \( \text{lim} \ (n + 1 - n) \neq \text{lim} \ (n + 1) - \text{lim} \ n \), because the left side is equal to 1, but the right side is undefined.

x) False. This is the harmonic series; it is divergent, but it is not a geometric series.

y) False. counterexample: \( a_n = \frac{1}{n} \), then \( \text{lim} \ a_n = \text{lim} \ \frac{1}{n} = 0 \), but \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} \) diverges as shown in class.

z) True. We proved in class that if \( \sum_{n=1}^{\infty} a_n \) converges, then \( \text{lim} \ a_n = 0 \). Hence if \( \text{lim} \ a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

2a.

\[
S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2} \, dx = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} \, dx = \int_0^1 2\pi \sqrt{1 + 9x^4} \, dx
\]

set \( u = 1 + 9x^4 \), so that \( du = 9 \cdot 4x^3 \, dx \), and when \( x = 0, u = 1 \); when \( x = 1, u = 10 \), then we have

\[
S = \int_1^{10} 2\pi \sqrt{u} \cdot \frac{1}{9} \cdot 4 \, du = \frac{1}{18} \pi \int_1^{10} \sqrt{u} \, du = \frac{1}{18} \pi \frac{2}{3} u^{\frac{3}{2}} \bigg|_1^{10} = \frac{\pi}{27} (\sqrt{10^3} - 1)
\]

2b.

\[
S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2} \, dx = \int_0^1 2\pi \sqrt{1 - x} \sqrt{1 + \left( \frac{1}{2\sqrt{1 - x}} \right)^2} \, dx
\]

\[
= \int_0^1 2\pi \sqrt{1 - x} \sqrt{1 + \frac{1}{4(1-x)}} \, dx = \int_0^1 2\pi \sqrt{1 - x} \sqrt{\frac{5 - 4x}{4(1-x)}} \, dx
\]

\[
= \int_0^1 \pi \sqrt{5 - 4x} \, dx = \left[ \frac{\pi}{6} \sqrt{5 - 4x} (-\frac{1}{4}) (5 - 4x) \right]_0^1 = -\frac{\pi}{6} \left( 5 - 4 \right)^{\frac{3}{2}} = -\frac{\pi}{6} ((\sqrt{5})^3 - 1)
\]
2c. \[
S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} \, dx = 2\pi \int_0^1 \left( \frac{e^x + e^{-x}}{2} \right) \sqrt{1 + \left( \frac{e^x - e^{-x}}{2} \right)^2} \, dx
\]

\[
= 2\pi \int_0^1 \left( \frac{e^x + e^{-x}}{2} \right) \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}} \, dx = 2\pi \int_0^1 \left( \frac{e^x + e^{-x}}{2} \right) \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}} \, dx
\]

\[
= 2\pi \int_0^1 \left( \frac{e^x + e^{-x}}{2} \right) \sqrt{\left( \frac{e^x + e^{-x}}{2} \right)^2} \, dx = 2\pi \int_0^1 \left( \frac{e^x + e^{-x}}{2} \right)^2 \, dx = 2\pi \int_0^1 \left( \frac{e^{2x} + 2 + e^{-2x}}{4} \right) \, dx
\]

\[
= \pi \left( \frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2} \right) \Big|_0^1 = \pi \left( \frac{e^2}{2} + 2 - \frac{e^{-2}}{2} - \left( \frac{1}{2} - \frac{1}{2} \right) \right) = \pi \left( \frac{e^2}{2} + 2 - \frac{e^{-2}}{2} \right)
\]

Note: this can also be solved using the hyperbolic functions sinh \(x\), cosh \(x\).

3. We assume that the planes are in the right half plane, \(x > 0\), for if they are not we can either perform a similar calculation for the left half plane, or if they are in both, perform two calculations, one for the left half plane and the other for the right half plane. A hemisphere is then generated by rotating the curve \(f(x) = \sqrt{r^2 - x^2}\) around the \(x\) axis. The formula for the surface area is

\[
S = 2\pi \int_z^b f(x)\sqrt{1 + (f'(x))^2} \, dx.
\]

Here

\[
1 + (f'(x))^2 = \frac{r^2}{r^2 - x^2},
\]

and so we find that

\[
S = 2\pi \int_z^b \sqrt{r^2 - x^2} \frac{r}{\sqrt{r^2 - x^2}} \, dx = 2\pi r (b - a).
\]

Letting \(d = b - a\) we find that \(S = 2\pi rd\) which shows that the area depends only on the distance between the planes and not their location.

4. (ii) \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm \sqrt{b^2 - \left( \frac{b}{a} \right)^2 x^2}\). Then, using the upper half of the ellipse

\[
f(x) = b \sqrt{1 - \left( \frac{x}{a} \right)^2} = \frac{b}{a} \sqrt{a^2 - x^2}, \quad f'(x) = -\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}},
\]

and \(\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left( \frac{b}{a} \right)^2 \frac{x^2}{a^2 - x^2}} = \sqrt{\frac{a^2 - x^2 + \left( \frac{b}{a} \right)^2 x^2}{a^2 - x^2}}\).

The surface area (S) for \(-a \leq x \leq a\) is
\[ S = 2\pi \int_{-a}^{a} f(x) \sqrt{1 + [f'(x)]^2} \, dx = 2\pi \int_{-a}^{a} \frac{b}{a} \sqrt{a^2 - x^2} \cdot \frac{\sqrt{a^2 - x^2 + (\frac{b}{a})^2} \cdot x^2 \, dx}{\sqrt{a^2 - x^2}} \]

\[ = 2\pi \frac{b}{a} \int_{-a}^{a} \sqrt{a^2 + \left( \frac{b^2 - a^2}{a^2} \right) x^2} \, dx. \]

(iii) Now we will show that \( S = 2\pi b(b + a \sin^{-1}(c)/c) \) where \( c = \frac{\sqrt{a^2 - b^2}}{a} ; \)

\[ S = 2\pi \frac{b}{a} \int_{-a}^{a} \sqrt{a^2 + \left( \frac{b^2 - a^2}{a^2} \right) x^2} \, dx = 2\pi \frac{b}{a} \int_{-a}^{a} \sqrt{a^2 - \left( \frac{a^2 - b^2}{a} \right) x^2} \, dx = 2\pi \frac{b}{a} \int_{-a}^{a} \sqrt{a^2 - (cx)^2} \, dx \]

With \( u = cx, \, du = cdx, \, u(-a) = -ac \) and \( u(a) = ac, \) then

\[ S = 2\pi \frac{b}{ac} \int_{-a}^{a} \sqrt{a^2 - (u)^2} \, du = 2\pi \frac{b}{ac} \left[ \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{u}{a} \right) \right]_{-ac}^{ac} \]

\[ = 2\pi \frac{b}{ac} (ac\sqrt{a^2 - a^2c^2} + a^2 \sin^{-1}(c)) = 2\pi b(\sqrt{a^2 - a^2c^2} + a \sin^{-1}(c)). \]

Notice that \( \sqrt{a^2 - a^2c^2} = \sqrt{a^2 - a^2 \left( \frac{a^2 - b^2}{a^2} \right)} = \sqrt{b^2} = b. \) Then \( S = 2\pi b(b + a \sin^{-1}(c)/c). \)

5. \( \bar{x} = \frac{M}{m} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{2 \cdot (-10) + 3 \cdot 6 + 1 \cdot 3}{2 + 3 + 1} = \frac{x_3 - 2}{6} = 0 \Rightarrow x_3 = 2 \)

6. To show that \( f(x) \) is minimized when \( x = \bar{x}, \) we need to show that \( f'(\bar{x}) = 0, f''(\bar{x}) > 0. \)

\[ f(x) = \sum_{i=1}^{n} m_i(x - x_i)^2 \Rightarrow f'(x) = \sum_{i=1}^{n} 2m_i(x - x_i) = \sum_{i=1}^{n} 2m_ix - \sum_{i=1}^{n} 2m_ix_i = 2x \sum_{i=1}^{n} m_i - 2 \sum_{i=1}^{n} m_ix_i \]

\[ = 2xm - 2M, \]

where \( m = \sum_{i=1}^{n} m_i, \, M = \sum_{i=1}^{n} m_i x_i \)

then \( f'(x) = 0 \Rightarrow 2xm - 2M = 0 \Rightarrow x = \frac{M}{m} = \bar{x} \) and \( f''(x) = 2m > 0 \) for all \( x \)

7. \( \bar{x} = \frac{1}{a} \int_{a}^{b} x[f(x) - g(x)] \, dx \)

\[ \bar{x} = \frac{1}{b} \int_{a}^{b} (f(x) - g(x)) \, dx, \]

\[ \bar{y} = \frac{1}{2} \int_{a}^{b} (f(x) - g(x)) \, dx, \]

in some cases \( g(x) = 0 \)

a) \( a = 0, \, b = 1, \, g(x) = 0, \, f(x) = x \)

\[ \bar{x} = \frac{1}{3} \int_{0}^{1} x^2 \, dx = \frac{2}{3} \text{ and } \bar{y} = \frac{1}{2} \int_{0}^{1} x^2 \, dx = \frac{1}{3}. \]

Note that the region is symmetric about line \( y = 1 - x, \) thus \( (\bar{x}, \bar{y}) \) lies on \( y = 1 - x. \) If you notice this, once \( \bar{x} \) is known, then \( \bar{y} \) can be found without extra calculation.

b) \( a = 0, \, b = 2, \, g(x) = 0, \, f(x) = \sqrt{x(2-x)} \)

First consider the curve \( y = \sqrt{x(2-x)}. \)

Then \( y^2 = x(2-x) = 2x - x^2 = -(x^2 - 2x) = -(x^2 - 2(x+1)-1) = -(x-1)^2 + 1 \Rightarrow (x-1)^2 + y^2 = 1, \)

hence the curve is a circle centered at \((x, y) = (1, 0)\) with radius 1. By symmetry we have \( \bar{x} = 1 \)

and then we compute \( \bar{y} = M_{m} = \frac{1}{2} \int_{0}^{1} \sqrt{x(2-x)} \, dx = \frac{1}{2} \int_{0}^{1} x(2-x) \, dx = \frac{2}{3} \int_{0}^{1} (x^2 - \frac{1}{3}x^3) \, dx = \frac{1}{3}(4 - \frac{8}{3}) = \frac{4}{3\pi}. \)

(The denominator is the area of a half circle.)

\( \bar{x} = 1 \) and \( \bar{y} = \frac{4}{3\pi}. \)
c) \(a = -1, \ b = 1, \ g(x) = 0, \ f(x) = \cosh x\), since \(\cosh x\) is an even function, thus the region is symmetric about \(x = 0 \Rightarrow \bar{x} = 0\).

\[
\bar{y} = \frac{1}{2} \frac{\int_{-1}^{1} \cosh^2 x \, dx}{\int_{-1}^{1} \cosh x \, dx} = \frac{1}{2} \frac{\frac{1}{2} (\cosh x \sinh x + x) \big|_{-1}^{1}}{\sinh x \big|_{-1}^{1}} = \frac{1}{4} \cosh 1 \sinh 1 + 1 \quad \text{(where } \int \cosh^2 x = \frac{1}{2} \cosh x \sinh x + \frac{1}{2} x)\)
\]

d) \(a = -2, \ b = 2, \ f(x) = \sqrt{4 - x^2}\), but \(g(x) = \begin{cases} 0 & \text{for } -2 \leq x \leq -1 \\ \sqrt{1 - x^2} & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } 1 \leq x \leq 2 \end{cases}\)

The region is symmetric about \(x = 0 \Rightarrow \bar{x} = 0\).

\[
\bar{y} = \frac{M_x}{m} = \frac{1}{2} \frac{\int_{-1}^{1} (\sqrt{4-x^2})^2 \, dx + \int_{-1}^{1} ((\sqrt{4-x^2})^2 - (\sqrt{4-x^2})^2) \, dx + \int_{-1}^{1} (\sqrt{4-x^2})^2 \, dx}{2\pi - \frac{1}{2} \pi} = \frac{1}{2} \frac{\int_{-1}^{1} 4-x^2 \, dx - \int_{-1}^{1} 1-x^2 \, dx}{2\pi - \frac{1}{2} \pi} = \frac{28}{9\pi} \quad \text{(since it is a circular region, the area } m \text{ can be found directly.)}\)
\]
e) \(a = 1, \ b = 2, \ g(x) = 0, \ f(x) = x^{-1}\)

\[
\bar{x} = \frac{\int_{1}^{2} x \cdot x^{-1} \, dx}{\int_{1}^{2} x^{-1} \, dx} = \frac{1}{\ln 2} = \frac{1}{\ln 2} \quad \text{and } \bar{y} = \frac{1}{2} \frac{\int_{1}^{2} (x^{-1})^2 \, dx}{\int_{1}^{2} x^{-1} \, dx} = \frac{1 - x^{-1}}{2} = \frac{1}{4\ln 2}\]

8a) The region is a half circle with radius \(r\), the area is \(\frac{1}{2}\pi r^2\).

The length the center of mass (of the region) traveled is \(2\pi \bar{y}\).

\[
\bar{y} = \frac{M_x}{m} = \frac{1}{2} \frac{\int_{-1}^{1} (\sqrt{\pi^2 - x^2})^2 \, dx}{\frac{4\pi^2}{3\pi}} = \frac{4r}{3\pi}
\]

Thus \(V = A \cdot L = \frac{1}{2}\pi r^2 \cdot 2\pi \frac{4r}{3\pi} = \frac{4}{3}\pi r^3\)

8b) Let the axis (height) of the cone on y axis, the cone can be regards as a triangle (area \(A = \frac{1}{2}hr\)) revolving around y axis. \(V = A \cdot L \) where \(L\) is the length that the center of mass (\(\bar{x}, \bar{y}\)) of the triangle travels (circumference) around y-axis, ie, \(2\pi \bar{x}\) here.

The equation of the hypotenuse is \(y = f(x) = h - \frac{h}{r}x\)

\[
\bar{x} = \frac{\int_{0}^{r} f(x) \, dx}{\int_{0}^{r} f(x) \, dx} = \frac{\int_{0}^{r} (h - \frac{h}{r}x) \, dx}{\int_{0}^{r} (h - \frac{h}{r}x) \, dx} = \frac{\frac{hr^2}{2}}{\frac{hr^2}{2}} = \frac{1}{2}r \quad \text{(Note that the denominator is actually the area)}
\]

\[
V = A \cdot L = \frac{1}{2}hr \cdot 2\pi \frac{r}{3} = \frac{1}{3}\pi hr^2 \quad \text{(known result, 1/3 volume of a cylinder)}
\]

8c) The region is a circle, the center of mass is the center of the circle, from the center of the circle to the rotating axis is \(r\), the length the center of mass traveled is \(2\pi r\). The area of the circle is \(\pi r^2\), the volume of the shape \(V = A \cdot L = 2\pi^2 r^3\).

9. prob(50 ≤ X ≤ 60) = prob(μ - σ ≤ X ≤ μ + σ) ≈ 0.68 = 68% (see page 37 of the lecture notes)

10a) \(\int_{-\infty}^{\infty} f(t) \, dt = \int_{0}^{6} \frac{6}{625} t^2(5 - t^2 - t - 5) \, dt = \frac{6}{625} \int_{0}^{6} (25t^2 - 10t^3 + t^4) \, dt = \frac{6}{625} \left( \frac{25}{3} t^3 - \frac{10}{4} t^4 + \frac{5}{6} t^5 \right) \left|_{0}^{5} \right. = 1 \text{ and } f(t) \geq 0, \text{ thus } f(t) \text{ defines a valid pdf}
\]

mean : \(\mu = \int_{0}^{5} t f(t) \, dt = \frac{6}{625} \int_{0}^{5} (25t^3 - 10t^4 + t^5) \, dt = \cdots = \frac{5}{2}\)

note : \(f(t)\) is symmetric about \(t = \frac{5}{2}\) (show it); this also implies that \(\mu = \frac{5}{2}\)

10b) \(1000 \cdot \text{prob}(T \geq 3) = 1000 \cdot \int_{3}^{5} f(t) \, dt = 1000 \cdot \int_{3}^{5} \frac{6}{625} t^2(5 - t) \, dt = \cdots \approx 317 \text{ batteries}\)

11. \(f(t) = ce^{-ct}\) where \(c = \frac{1}{20}, \ t \) is measured in minutes, \(\mu = \frac{1}{c}, \) thus \(c = \frac{1}{\mu}\).

a) \(\text{Prob}(X \leq 10) = \int_{0}^{10} \frac{1}{20} e^{-\frac{t}{20}} \, dt = 1 - \frac{1}{\sqrt{e}} \approx 0.375\)

b) \(\text{Prob}(X \geq 30) = \int_{30}^{\infty} \frac{1}{20} e^{-\frac{t}{20}} \, dt = e^{-\frac{3}{2}} \approx 0.24\)
c) Solve \( \frac{1}{2} = \int_{m}^{\infty} f(t) dt \Rightarrow \frac{1}{2} = e^{-\frac{m}{c}} \Rightarrow m = 20 \ln 2 \approx 20 \times 0.7 = 14 \) minutes. (The median \( m = \mu \ln 2 < \mu \) for exponential distribution.)

12. 

\[
\text{prob}(\mu - 2\sigma < X < \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \quad (y = (x-\mu)/\sqrt{2}\sigma, dy = dx/\sqrt{2}\sigma)
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{2}} e^{-y^2} dy = \text{erf}(\sqrt{2}) = 0.9545 \approx 95%
\]

Hence for a normal distribution, 95% of the values are within two standard deviations of the mean.

13. The exponential pdf is \( f(t) = \begin{cases} c e^{-ct}, & t \geq 0 \\ 0, & t < 0 \end{cases} \) with mean \( \mu = 1/c. \)

\[
\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx
\]

\[
= \int_{0}^{\infty} (x-\mu)^2 c e^{-cx} dx \quad (\text{set } u = (x-\mu)^2, dv = ce^{-cx} dx \Rightarrow du = 2(x-\mu)dx, v = -e^{-cx})
\]

\[
= -(x-\mu)^2 e^{-cx} \bigg|_{0}^{\infty} + 2 \int_{0}^{\infty} (x-\mu) e^{-cx} dx
\]

\[
= \mu^2 + 2 \int_{0}^{\infty} (x-\mu) e^{-cx} dx \quad (\text{set } u = x-\mu, dv = e^{-cx} dx \Rightarrow du = dx, v = -\frac{1}{c} e^{-cx})
\]

\[
= \mu^2 - 2 \frac{x-\mu}{c} e^{-cx} \bigg|_{0}^{\infty} + \frac{2}{c} \int_{0}^{\infty} e^{-cx} dx = \mu^2 - \frac{2\mu}{c} - \frac{2}{c^2} e^{-cx} \bigg|_{0}^{\infty} = \mu^2 - \frac{2\mu}{c} + \frac{2}{c^2} = \mu^2,
\]

where in the last step we substituted \( c = 1/\mu. \) Hence \( \sigma = \mu. \)

14. Functions (a), (b), (c), (e) are solutions of the differential equation; functions (d), (f) are not.

15. a) \( y = 1 \) : unstable

15. b) \( y = -1 \) : stable, \( y = 1 \) : unstable

15. c) \( y = 1 \) : one side stable and one side unstable

15. d) \( y = 1 \) : stable, \( y = 2 \) : unstable

15. e) \( y = 2k\pi \) (\( k \) is integer) : unstable, \( y = (2k+1)\pi \) : stable

16. \( E(t) = \frac{1}{2}mv^2 + V(x) \Rightarrow E(t) = \frac{1}{2}m x'(t)^2 + V(x(t)) \)

\( \Rightarrow E'(t) = \frac{1}{2}m \cdot 2x'(t) x''(t) + V'(x(t)) x'(t) = x'(t) [mx''(t) + V'(x(t))] = x'(t) [f(x(t)) - f(x(t))] = 0 \)

\( \Rightarrow E'(t) = 0 \Rightarrow E(t) = \text{constant} \)

Hence if the kinetic energy increases, then the potential energy decreases.

17a) \( y' = y \Rightarrow \frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt \Rightarrow \ln y = t + C \Rightarrow y = e^{C \cdot e^t}, e^C = 1 \) using \( y(0) = 1 \)

The solution is \( y = y(0)e^{t} \Rightarrow y = e^{t}. \)

17b) \( y' = ty \Rightarrow \frac{dy}{dt} = ty \Rightarrow \frac{dy}{y} = t dt \Rightarrow \ln y = \frac{1}{2}t^2 + C \Rightarrow y = e^{C \cdot e^\frac{t^2}{2}}, e^C = 1 \) using \( y(0) = 1 \)
The solution is $y = y(0)e^{\frac{t}{T}} \Rightarrow y = e^{\frac{t}{T}}$.

17c) $y' = y^2 \Rightarrow \frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow -\frac{1}{y} = t + C \Rightarrow y = -\frac{1}{t-C}, C = -1$ using $y(0) = 1$.

The solution is $y = \frac{1}{1-t}$. Check $y' = \frac{1}{(1-t)^2} = y^2$ and $y(0) = 1$ at $t = 0$. What happens when $t \to 1$?

17d) $y' = y(1 - y) \Rightarrow \frac{dy}{dt} = y(1 - y) \Rightarrow \frac{dy}{y(1-y)} = dt \Rightarrow \frac{dy}{y} + \frac{dy}{1-y} = dt \Rightarrow \ln |y| - \ln |1 - y| = t + C \Rightarrow \ln \left|\frac{y}{1-y}\right| = t + C \Rightarrow \frac{y}{1-y} = e^C \cdot e^t$.

One can find the general solution for this equation, but the initial condition $y(0) = 1$ is a constant solution of the system, so the solution will stay at $y = 1$ for all $t \geq 0$, so the solution is $y(t) = 1$, and the general solution is not needed.

18. Let $y(t)$ be the amount of salt (kg) in the tank at time $t$ (min).

rate of change of amount = (rate coming in) - (rate going out)
rate coming in = $0.05 \frac{\text{kg}}{L} \cdot 5 \frac{\text{L}}{\text{min}} = 0.25 \frac{\text{kg}}{\text{min}}$, rate going out = $\frac{y \text{kg}}{1000 \text{L}} \cdot 5 \frac{\text{L}}{\text{min}} = 0.005 y \frac{\text{kg}}{\text{min}}$

$y' = 0.25 - 0.005 \Rightarrow y' = -0.005(y - 50)$

Note that we have written the equation in the form $y' = k(y - T)$, which is Newton’s law of cooling/heating, with $k = -0.005, T = 50$. The solution is $y(t) = T + (y_0 - T)e^{kt} = 50 + (0 - 50)e^{-0.005t} = 50(1 - e^{-0.005t})$, using the initial condition $y(0) = y_0 = 0$, since the tank initially contains pure water.

a) After one hour = 60 minutes we have $y(60) = 50(1 - e^{-0.005\cdot60}) = 50(1 - e^{-0.3}) \approx 13$ kg.

b) $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} 50(1 - e^{-0.005t}) = 50$ kg

Note that this is the quantity of salt present in the tank if the tank contained only sea water.

19a) $x(t) = \text{value of new currency in circulation at time } t, x'(t) = \text{rate of change of new currency in circulation = (rate going in) - (rate going out)} = (0.05) - \left(\frac{x}{10} \cdot 0.05\right) \Rightarrow x' = 0.05(1 - \frac{x}{10})$.

19b) The equation can be written as $x' = -0.005(x - 10)$, which is the same as Newton’s law of cooling/heating, $x' = k(x - T)$, with $k = -0.005$ and $T = 10$. Hence $x(t) = T + (x_0 - T)e^{-kt} = 10 + (0 - 10)e^{-0.005t} = 10(1 - e^{-0.005t})$, where we used the initial condition $x_0 = x(0) = 0$.

19c) 90% means $x(t) = 9 \Rightarrow 9 = 10(1 - e^{-0.005t}) \Rightarrow t = 200 \ln 10 \approx 460$ days.

20. $c' = k(a - c)(b - c)$, where $k > 0, a = 1, b = 2 \Rightarrow c' = k(1 - c)(2 - c)$, it has two constant solutions $c = 1$ and $c = 2$ (ie., let $c' = 0$, solve the equation), $c = 1$ is stable while $c = 2$ is unstable, since for $0 \leq c < 1$, $c' > 0$, $c(t)$ increases (towards $c = 1$), for $1 < c < 2$, $c' < 0$, $c(t)$ decreases (towards $c = 1$), for $c > 2$, $c' > 0$, $c(t)$ increases (away from $c = 2$).

To find the product concentration, we solve $c' = k(1 - c)(2 - c)$ by separation of variables, $\frac{dc}{(1-c)(2-c)} = kdt$. Using partial fractions, we write $\frac{1}{(1-c)(2-c)} = \frac{1}{1-c} - \frac{1}{2-c}$, and integrate to obtain
\[
\ln \left| \frac{2-c}{1-c} \right| = kt + d, \text{ or } \frac{2-c}{1-c} = De^{kt}, \text{ for some constant } D \text{ (using the initial condition } c(0) = 0, \text{ we find this constant to be } D = 2). \text{ We now solve the equation for } c(t), \text{ writing } 2 - c = (1 - c)De^{kt}. \text{ Rearranging terms and simplifying, we get } c(t) = 1 - \frac{1}{2e^{kt}-1}.
\]

Given that } c(0) = 0 \text{ at } t = 0, \text{ the asymptotic value of } c(t) \text{ is } c = 1 \text{ as } t \to \infty. \text{ The asymptotic value will not change if } c(0) = 1.5 \text{ mole/L. This can also be seen using the phase plane.}

21. differential equation : \[ mc_p T' = -e\sigma T^4 \Rightarrow \int \frac{T'}{T^4} = -\frac{e\sigma}{mc_p} \Rightarrow \int \frac{T'}{T^4}dt = \int -\frac{e\sigma}{mc_p}dt \Rightarrow -\frac{1}{3T^3} = -\frac{e\sigma}{mc_p}t + C \]

initial condition : \[ T(0) = T_0 \Rightarrow C = -\frac{1}{3T^3_0} \Rightarrow -\frac{1}{3T^3} = -\frac{e\sigma}{mc_p}t - \frac{1}{3T^3_0} \Rightarrow T(t) = \left(3\frac{e\sigma}{mc_p}t + \frac{1}{T^3_0}\right)^{-1/3} \]

In the limit } t \to \infty, T(t) \sim t^{-1/3}, \text{ so } \alpha = \frac{1}{3}.

22a) The decay rate of radioactive sample satisfies } y' = -ky. \text{ By separation of variables, we find the solution } y(t) = y_0e^{-kt} \text{ where } y_0 \text{ is initial mass at } t = 0. \text{ It is given that } y(2) = 128 \text{ after 2 hours and } y(5) = 2 \text{ after 5 hours, } \Rightarrow \begin{cases} 128 = y_0e^{-2k} \\ 2 = y_0e^{-5k} \end{cases}. \text{ Dividing the first equation by the second equation yields } 64 = e^{3k} \Rightarrow e^k = 64^{1/3} = 4. \text{ Substituting in the first equation gives } 128 = y_04^{-2} \Rightarrow y_0 = 128 \cdot 16 = 2048.

22b) Let } y(t) = 1 \text{ kg. Then } 1 = 2048 \cdot 4^{-t} \Rightarrow 4^t = 2048 \Rightarrow t = \frac{\ln 2048}{\ln 4} = \frac{\ln 2^{11}}{\ln 2^2} = \frac{11}{2} \Rightarrow t = 5.5 \text{ hours.}

23. The initial value of the investment is } x = $500 \text{ and the annual interest rate is } r = 12\% = 0.12.

a) If the interest is compounded annually, then the value of the investment after 10 years is } x(1 + r)^n = 500(1.12)^{10} \approx$1552.92.

b) If the interest is compounded continuously, then the value of the investment after 10 years is } xe^{rt} = 500e^{0.12(10)} \approx$1660.06.

c) The equivalent annual interest rate } r_{eq} \text{ is found by solving the equation } e^r = 1 + r_{eq}. \text{ Since } e^r = e^{0.12} \approx 1.127 = 1 + 0.127, \text{ the equivalent annual interest rate is } r_{eq} = 0.127. \text{ In other words, 12\% compounded continuously is the same as 12.7\% compounded annually.}

24. Newton’s Law of cooling /heating implies } y(t) = T + (y_0 - T)e^{-kt}, \text{ where } y_0 = 21 \text{ at } t = 0.

We know } y(1) = 27 \text{ and } y(2) = 30, \text{ thus we have } \begin{cases} 27 = T + (21 - T)e^{-k} \\ 30 = T + (21 - T)e^{-2k} \end{cases} \text{ Combining these two equations, we find } T = 33.

25a) The correct equation is (iii) } m' = r - km.

25b) } r - km = 0 \Rightarrow m = \frac{r}{k} = M \text{ (phase plane, done in class, stable)}

25c) The equation in a) can be rewritten as } m' = -k(m - \frac{r}{k}). \text{ This is the same as Newton’s Law of cooling/heating. The solution is } m(t) = \frac{r}{k} + (m_0 - \frac{r}{k})e^{-kt}.

The endowment dropped to } \frac{1}{2}M \text{ last year, so we take } m_0 = \frac{1}{2}M. \text{ The solution is}
\[ m(t) = \frac{a}{k} + \left(\frac{1}{2}M - \frac{a}{k}\right)e^{-kt} = M - \frac{1}{2}Me^{-kt}. \]

To find how long it will take to recover to \( \frac{3}{4}M \), we set
\[ m(t) = \frac{3}{4}M \Rightarrow M - \frac{1}{2}Me^{-kt} \Rightarrow \frac{3}{4}M = \frac{1}{4}M \Rightarrow M = \frac{1}{4} \Rightarrow t = -\frac{1}{k} \ln \frac{1}{2} = \frac{1}{k} \ln 2. \]

For \( k = \frac{1}{20} \text{ year}^{-1} \) and \( \ln 2 \approx 0.7 \), we have \( t \approx 14 \text{ years} \).

26. For Bob, his coffee temperature after adding the cold milk is \( T_{Bob}(0) = (1-a)T_h + aT_c \), where \( a = \frac{1}{8} + \frac{1}{5} = \frac{1}{3} \) is the volume fraction of cold milk in the mixture. After that, the coffee temperature follows Newton’s Law of cooling/heating \( y' = -k(y - T_a) \), where \( k > 0 \) is a rate constant and \( T_a \) is the ambient cafe temperature. The solution is \( y(t) = T_a + (y_0 - T_a)e^{-kt} \). Thus, after 2 minutes, \( T_{Bob}(2) = T_a + [(1-a)T_h + aT_c - T_a]e^{-2k} \).

For Ray, the difference is that \( y(0) = T_h \), because he did not add cold milk at the beginning. So at \( t = 2 \) minutes, his coffee temperature is \( y(2) = T_a + (T_h - T_a)e^{-2k} \), but then adding 1oz of cold milk yields \( T_{Ray}(2) = (1-a)y(2) + aT_c = (1-a)[T_a + (T_h - T_a)e^{-2k}] + aT_c \).

\[ T_{Ray}(2) - T_{Bob}(2) = (1-a)[T_a + (T_h - T_a)e^{-2k}] + aT_c - T_a - [(1-a)T_h + aT_c - T_a]e^{-2k} \]
\[ = T_a[(1-a) - (1-a)e^{-2k} + 1 + e^{-2k}] + T_c(a - ae^{-2k}) + T_h[(1-a)e^{-2k} - (1-a)e^{-2k}] \]
\[ = T_a(ae^{-2k} - a) + T_c(a - ae^{-2k}) = a(1 - e^{-2k})(T_c - T_a) < 0, \]
since \( a > 0, k > 0, T_c < T - a. \)

So Ray ends up with colder coffee.

This can be understood as follows. The larger the temperature difference between the object and its surroundings, the faster the temperature changes. Since Bob’s strategy reduced the initial temperature difference, he reduced the temperature rate of change, which was not a wise strategy for cooling the coffee. See the plots of \( T_{Bob}(t), T_{Ray}(t) \).

27. Let \( y(t) \) be the fraction of the population who have already heard the rumor at time \( t \). Then \( 1 - y(t) \) is the fraction of the population who have not already heard the rumor at time \( t \).

The rate of change of the fraction of the population who have heard the rumor is \( y' = ky(1-y) \), for some \( k > 0 \). This makes sense because if \( y = 0 \) or \( y = 1 \), then \( y' = 0 \).

This is a logistic equation with general solution \( y(t) = \frac{y_0}{y_0 + (1-y_0)e^{-kt}} \). Set \( t = 0 \) at 8am, then \( y_0 = \frac{10}{1000} = 0.01 \). Then at 9am, \( t = 1 \text{ hour} \), \( y(1) = \frac{20}{1000} = 0.02 \). Substituting these into the
solution yields $0.02 = \frac{0.01}{0.01+0.99e^{-k}} \implies 0.01 + 0.99e^{-k} = \frac{1}{2} \implies e^{-k} = \frac{49}{99}$. The solution becomes $y(t) = \frac{0.01}{0.01+0.99e^{-kt}}$; note that $e^{-kt} = (e^{-k})^t$. Then $y(t) = \frac{1}{2} \implies 0.01 + 0.99(\frac{49}{99})^t = 0.02 \implies (\frac{49}{99})^t = \frac{1}{99} \implies t = \ln(99)/(\ln(99) - \ln(49)) = 6.5337$, i.e. around 2:30pm.

29. We observe that $\sum_{n=0}^{\infty} (2x - 1)^n$ is a geometric series with ratio $r = 2x - 1$; hence the series converges if and only if $-1 < r < 1 \iff -1 < 2x - 1 < 1 \iff 0 < 2x < 2 \iff 0 < x < 1$. If $0 < x < 1$, then the sum of the series is $\sum_{n=0}^{\infty} (2x - 1)^n = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} = \frac{1}{1-(2x-1)} = \frac{1}{2-2x}$.

30a) $\int \cosh 2xdx = \frac{1}{2} \sinh 2x$

30b) $\int \tanh 2xdx = \frac{1}{2} \int \tanh(2x)d(2x) = \frac{1}{2} \int \frac{\sinh 2x}{\cosh 2x}d(2x) = \frac{1}{2} \int \frac{1}{\cosh 2x}d\cosh 2x = \frac{1}{2} \ln(\cosh 2x)$

30c) $\int \cosh^2 2xdx = \int (e^x + e^{-x})^2dx = \int e^{2x} + 2e^{-2x} + e^{2x}dx = \frac{1}{4}(e^{2x} + 2x - \frac{1}{2}e^{-2x}) = \frac{1}{4}\sinh 2x + \frac{1}{2}x$

31. $f(x) = \cosh x, f'(x) = \sinh x, 1 + [f'(x)]^2 = 1 + \sinh^2 x = \cosh^2 x$

$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx = \int_1^2 2\pi \cosh^2 2xdx = 2\pi \int_1^2 (\cosh x)^2dx = 2\pi \int_1^2 \frac{e^{2x} + e^{-2x}}{4}dx = \frac{\pi}{2}(\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x})\bigg|_1^2 = \frac{\pi}{2}(2\sinh 2x + 2x)\bigg|_1^2 = \frac{\pi}{2}(2\sinh 2 + 2) = \pi(\sinh 2 + 1)$

32. recall : $\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}$

a) LHS = $\sinh(x + y) = \frac{2}{e^{x+y} - e^{-x-y}}$

RHS = $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} - e^{y-x} + e^{x-y} - e^{y-x}}{4} + \frac{e^{x+y} + e^{y-x} - e^{y-x} - e^{-x-y}}{4} = \frac{e^{x+y} - e^{-x-y}}{2}$

LHS

b) One can prove it by the definitions of sinh, cosh, as in (a), or by using $\sinh(-y) = -\sinh y, \cosh(-y) = \cosh y$ (i.e. sinh is an odd function, cosh is an even function) and based on a):

$\sinh(x-y) = \sinh[x + (-y)] \overset{(i)}{=} \sinh(x) \cosh(-y) + \cosh(x) \sinh(-y) = \sinh x \cosh y - \cosh x \sinh y$

Note : For parts c) and d), one can prove them by the definitions of sinh, cosh similar to a), or by using the relationships ($\sinh x)' = \cosh x, (\cosh x)' = \sinh x$ and the identity a) (see below).

c) Taking derivative for both sides of a) with respect to $x$, regarding $y$ as a constant, we have:

$cosh(x + y) = [\sinh(x + y)]' = [\sinh x \cosh y + \cosh x \sinh y]' = (\sinh x)' \cosh y + (\cosh x)' \sinh y$

$= \cosh x \cosh y + \sinh x \sinh y$

d) Differentiating both sides of b) with respect to $x$, regarding $y$ as constant yields the result.
a) \( f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2}, f''(x) = 2x^{-3} \)
\[ a = 1 \Rightarrow f(1) = 1, f'(1) = -1, f''(1) = 2 \]
\[ T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - (x - 1) = 2 - x \]

b) \( f(x) = \sin x \Rightarrow f'(x) = \cos x, f''(x) = -\sin x \)
\[ a = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi}{2}\right) = 1, f'\left(\frac{\pi}{2}\right) = 0, f''\left(\frac{\pi}{2}\right) = -1 \]
\[ T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - 0(x - \frac{\pi}{2}) = 1 \]

c) \( f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}, f''(x) = -\frac{1}{4x^{3/2}} \)
\[ a = 4 \Rightarrow f(4) = \sqrt{4} = 2, f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}, f''(4) = -\frac{1}{4\sqrt{4}^3} = -\frac{1}{32} \]
\[ T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 2 + \frac{1}{4}(x - 4) \]

approximations of \( \sqrt{5} = 2.2361 \) (viewing \( \sqrt{5} \) as the value of \( \sqrt{x} \) at \( x = 5 \))
\[ T_1(5) = 2 + \frac{1}{4}(5 - 4) = \frac{9}{4} = 2.25 \]

34. recall : if \( f(-x) = f(x) \), then \( f \) is even, and if \( f(-x) = -f(x) \), then \( f \) is odd

a) \( f(x) = \sqrt{1 + x^2} : \sqrt{1 + (-x)^2} = \sqrt{1 + x^2} \), so \( \sqrt{1 + x^2} \) is even

b) \( f(x) = \sin 2x : \sin 2(-x) = -\sin 2x \), so \( \sin 2x \) is odd

c) \( f(x) = 1 + x \):
\[ 1 + (-x) = 1 - x \neq 1 + x \text{, so } 1 + x \text{ is neither even nor odd} \]

d) \( f(x) = |x| : | - x | = |x| \), so \( |x| \) is even

e) \( f(x) = \tan x : \tan(-x) = \sin(-x)/\cos(-x) = -\sin x/\cos x = -\tan x \), so \( \tan x \) is odd

f) \( f(x) = \sinh x : \sinh(-x) = \frac{e^{-x} - e^{-( -x)}}{2} = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh x \), so \( \sinh x \) is odd

g) \( f(x) = \cosh x : \cosh(-x) = \frac{e^{-x} + e^{-( -x)}}{2} = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \), so \( \cosh x \) is even

35. Let \( L = \lim_{n \to \infty} n((1 + \frac{1}{n})^n - e) \). This limit \( L \) has the form \( \infty \cdot 0 \), so to apply \( \text{Hôpital's rule} \) we move \( n \) to the denominator, \( L = \lim_{n \to \infty} \frac{(1 + \frac{1}{n})^n - e}{\frac{1}{n}} \). Now we have a limit of the type \( \frac{0}{0} \), for which we can apply \( \text{Hôpital's rule} \). However it is more convenient to substitute \( u = \frac{1}{n} \).

\[ L = \lim_{n \to \infty} \frac{(1 + \frac{1}{n})^n - e}{\frac{1}{n}} = \lim_{u \to 0} \frac{(1 + u)^{1/u} - e}{u} \]

note : \( (1 + u)^{1/u} = e^{\ln[(1+u)^{1/u}]} = e^{\frac{\ln(1+u)}{u}} \)
\[ \Rightarrow \frac{d}{du} (1 + u)^{1/u} = e^{\frac{\ln(1+u)}{u}} \cdot \frac{1}{1+u} - \ln(1 + u) \]
\[ \Rightarrow L = \lim_{u \to 0} \frac{(1 + u)^{1/u} - e}{u} = \lim_{u \to 0} \frac{e^{\frac{\ln(1+u)}{u}}}{u} \cdot \frac{1}{1+u} - \ln(1 + u) \]
\[ = \lim_{u \to 0} \frac{(1 + u)^{1/u} \cdot \frac{1}{1+u} - \ln(1 + u)}{u^2} = \lim_{u \to 0} \frac{e^{\frac{\ln(1+u)}{u}}}{u} \cdot \frac{1}{1+u} - \ln(1 + u) \]
\[ = \lim_{u \to 0} \frac{1 - \left[\frac{1+u}{1+u} + \ln(1+u)\right]}{(1+u) \cdot 2u + u^2} = e \cdot \lim_{u \to 0} \frac{-\ln(1+u)}{2u + 3u^2} = e \cdot \lim_{u \to 0} \frac{-1}{2 + 6u} = e \cdot \frac{-1}{2} = -\frac{e}{2} \quad \text{ok} \]