Review Sheet for Midterm Test 2 (Math 156) Fall 2011

Question 1 Solution

a) True. Use integration by parts. $\int_0^\infty x(xe^{-x^2})\,dx = x(-\frac{1}{2}e^{-x^2})|_0^\infty - \int_0^\infty -\frac{1}{2}e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$.

b) False. $\bar{x} = \frac{M_y}{m} = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx}$, but $f(x)$ cannot cancel.

c) True, provided the region has uniform density because then half of the mass is on the right and half of the mass is on the left.

d) False. Generally $\text{prob}(X \le \mu) \ne \text{prob}(X \ge \mu)$, unless the p.d.f is symmetric about $x = \mu$. It is the median m which satisfies $\text{prob}(X \le m) = \text{prob}(X \ge m)$.

e) False. A normally distributed random variable is more likely to be closer than away from the mean. One can also calculate: $\text{prob}(\mu \le X \le \mu+\sigma) = \frac{1}{\sqrt{\pi}}\int_0^{\frac{1}{\sqrt{2}}} e^{-x^2}dx = \frac{1}{2}\text{erf}(\frac{1}{\sqrt{2}}) = 0.34$, whereas $\text{prob}(\mu+\sigma \le X \le \mu+2\sigma) = \frac{1}{2}\text{erf}(\frac{2}{\sqrt{2}}) - \frac{1}{2}\text{erf}(\frac{1}{\sqrt{2}}) = 0.14$.

f) False. 'Half-life of 100 years' implies there will be $\frac{1}{2}$kg (initially 1 kg) after 100 years, there will be $\frac{1}{\sqrt{2}}$kg after 50 years, there will be $\frac{1}{4}$kg after 200 years, etc.

g) True. Because first $f(x) \ge 0$ for any x, second $\int_{-1}^{1} \frac{1}{\pi\sqrt{1-x^2}}dx \xrightarrow{x=\sin\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi\sqrt{1-\sin^2\theta}}d\sin\theta$ $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\pi\cos\theta}d\theta = \frac{\theta}{\pi}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1$, and third $f(x)$ has definition everywhere.

h) False. The numerical error decreases by approximately a factor of $\frac{1}{2}$ when h is reduced by $\frac{1}{2}$.

i) True. $\cosh x - \sinh x = \frac{e^x+e^x}{2} - \frac{e^x-e^{-x}}{2} = e^{-x} > 0$ for all x.

j) False. $\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x$, so $\tanh x$ is an odd function.

k) False. $1 + \sinh^2 x = \cosh^2 x$ or $\cosh^2 x - \sinh^2 x = 1$

l) True. $f(0) = 1$ and $f'(0) = 1$ so $T_1(x) = 1 + x$.

m) True. Any convergent sequence is bounded. However, the converse is not true.

n) False. A convergent sequence may converge via damped oscillating, for example, $\{\frac{(-1)^n}{n}\}$.

o) False. Counterexample $a_n = \frac{1}{n}$ and $b_n = n$, $\lim_{n\Rightarrow\infty} a_n = 0$ and $\lim_{n\Rightarrow\infty} b_n = \infty$, but $\lim_{n\Rightarrow\infty} a_n b_n = 1$.

p) False. Counterexample $a_n = \frac{n+1}{2n}$, $0 \le a_n \le 1$ and $a_{n+1} < a_n$, but $\lim_{n\Rightarrow\infty} a_n = \frac{1}{2}$. (The two conditions do imply the sequence converges, but the limit is not necessary zero here.)

r) False. The series is divergent but not <u>geometric</u>.

s) False. Counterexample $a_n = \frac{1}{n}$, $\lim_{n\Rightarrow\infty} a_n = 0$, but $\sum_{n=1}^{\infty} a_n$ diverges.

t) True. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\Rightarrow\infty} a_n = 0$. If $\lim_{n\Rightarrow\infty} a_n \ne 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Question 2 Solution

$\bar{x} = \frac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx}$

Question 3 Solution

$\bar{x} = \frac{\int_a^b x[f(x)-g(x)]dx}{\int_a^b [f(x)-g(x)]dx}$ and $\bar{y} = \frac{1}{2}\frac{\int_a^b [f^2(x)-g^2(x)]dx}{\int_a^b [f(x)-g(x)]dx}$ (in some cases $g(x) = 0$, in other cases $g(x) \neq 0$)

a) $a = 0$, $b = 1$, $g(x) = 0$, $f(x) = x$.

$\bar{x} = \frac{\int_0^1 x^2 dx}{\int_0^1 x dx} = \frac{2}{3}$ and $\bar{y} = \frac{1}{2}\frac{\int_0^1 x^2 dx}{\int_0^1 x dx} = \frac{1}{3}$.

Note that the region is symmetric about line $y = 1 - x$, thus $(\bar{x}, \bar{y})$ lies on $y = x$. If you notice this, once $\bar{x}$ is known, $\bar{y}$ can be found without calculation.

b) $a = 0$, $b = 1$, $g(x) = x^2$, $f(x) = \sqrt{x}$.

$\bar{x} = \frac{\int_0^1 x(\sqrt{x}-x^2)dx}{\int_0^1 (\sqrt{x}-x^2)dx} = \frac{9}{20}$ and $\bar{y} = \frac{1}{2}\frac{\int_0^1 [(\sqrt{x})^2-(x^2)^2]dx}{\int_0^1 (\sqrt{x}-x^2)dx} = \frac{9}{20}$.

c) $a = 0$, $b = 2$, $g(x) = 0$, $f(x) = \sqrt{x(2-x)}$.

The region is symmetric about $x = 1$ ($\Rightarrow \bar{x} = 1$). To see this more clearly, let $t = x - 1 \Rightarrow x = 1 + t$, the region is, in t versus y plane, becomes $-1 \leq t \leq 1$, $0 \leq y \leq \sqrt{(1+t)(1-t)} = \sqrt{1-t^2} \equiv f(t)$, the region is symmetric about $t = 0$, equivalent to the original region is symmetric about $x = 1$, both cases yield the same $\bar{y}$, (the t versus y scheme, easier).

$\bar{y} = \frac{1}{2}\frac{\int_{-1}^1 (\sqrt{1-t^2})^2 dt}{\int_{-1}^1 \sqrt{1-t^2} dt} = \frac{1}{2}\frac{\int_{-1}^1 (1-t^2)dt}{\frac{\pi}{2}} = \frac{4}{3\pi}$ (The denominator gives the area of a half circle).

$\bar{x} = 1$ and $\bar{y} = \frac{4}{3\pi}$.

d) $a = -1$, $b = 1$, $g(x) = 0$, $f(x) = \cosh x$, since $\cosh x$ is an even function, thus the region is symmetric about $x = 0 \Rightarrow \bar{x} = 0$.

$\bar{y} = \frac{1}{2}\frac{\int_{-1}^1 \cosh^2 x dx}{\int_{-1}^1 \cosh x dx} = \frac{1}{2}\frac{\frac{1}{2}(\cosh x \sinh x + 1)|_{-1}^1}{\sinh x|_{-1}^1} = \frac{1}{4}\frac{\cosh 1 \sinh 1 + 1}{\sinh 1}$ (where $\int \cosh^2 x = \frac{1}{2}\cosh x \sinh x + \frac{1}{2}x + C$ see Question 24c).

e) $a = -2$, $b = 2$, $f(x) = \sqrt{4-x^2}$, but $g(x) = \begin{cases} 0 & \text{for } -2 \leq x \leq -1 \\ \sqrt{1-x^2} & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } 1 \leq x \leq 2 \end{cases}$

The region is symmetric about $x = 0 \Rightarrow \bar{x} = 0$.

$\bar{y} = \frac{M_x}{m} = \frac{1}{2}\frac{\int_{-2}^{-1} (\sqrt{4-x^2})^2 dx + \int_{-1}^1 [(\sqrt{4-x^2})^2 - (\sqrt{1-x^2})^2]dx + \int_1^2 (\sqrt{4-x^2})^2 dx}{2\pi - \frac{1}{2}\pi} = \frac{1}{2}\frac{\int_{-2}^2 4-x^2 dx - \int_{-1}^1 1-x^2 dx}{2\pi - \frac{1}{2}\pi} = \frac{28}{9\pi}$ (since it is a circular region, the area m can be found directly.)

f) $a = 1$, $b = 2$, $g(x) = 0$, $f(x) = x^{-1}$.

$\bar{x} = \frac{\int_1^2 x \cdot x^{-1} dx}{\int_1^2 x^{-1} dx} = \frac{1}{\ln x|_1^2} = \frac{1}{\ln 2}$ and $\bar{y} = \frac{1}{2}\frac{\int_1^2 (x^{-1})^2 dx}{\int_1^2 x^{-1} dx} = \frac{1}{2}\frac{-x^{-1}|_1^2}{\ln 2} = \frac{1}{4\ln 2}$.

From the graph, one can see that since $f(x)$ is decreasing, if one make $f(x)$ less steep, this will increase $\bar{x}$. In the case $f(x) = \ln 2$, the region becomes a rectangle and the area is reserved, but $\bar{x} = \frac{3}{2}$ must be larger than the case $f(x) = \frac{1}{x}$ where $\bar{x} = \frac{1}{\ln 2}$. Thus $\frac{1}{\ln 2} < \frac{3}{2} \Rightarrow \ln 2 > \frac{2}{3}$.

Question 4 Solution

a) The region is a half circle with radius r, the area is $\frac{1}{2}\pi r^2$.

The length the center of mass (of the region) traveled is $2\pi\bar{y}$.

$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_{-r}^{r}(\sqrt{r^2-x^2})^2 dx}{\frac{1}{2}\pi r^2} = \frac{4r}{3\pi}$

Thus $V = A \cdot L = \frac{1}{2}\pi r^2 \cdot 2\pi \frac{4r}{3\pi} = \frac{4}{3}\pi r^3$

b) Let the axis (height) of the cone on y axis, the cone can be regards as a triangle (area $A = \frac{1}{2}hr$) revolving around y axis. $V = A \cdot L$ where L is the length that the center of mass ($\bar{x}$, $\bar{y}$) of the triangle travels (circumference) around y-axis, ie, $2\pi\bar{x}$ here.

The equation of the hypotenuse is $y = f(x) = h - \frac{h}{r}x$

$\bar{x} = \frac{\int_0^r xf(x)dx}{\int_0^r f(x)dx} = \frac{\int_0^r x(h-\frac{h}{r}x)dx}{\int_0^r (h-\frac{h}{r}x)dx} = \frac{\frac{1}{6}hr^2}{\frac{1}{2}hr} = \frac{1}{3}r$ (Note that the denominator is actually the area)

$V = A \cdot L = \frac{1}{2}hr \cdot 2\pi\frac{1}{3}r = \frac{1}{3}\pi hr^2$ (known result, 1/3 volume of a cylinder).

c) The region is a circle, the center of mass is the center of the circle, from the center of the circle to the rotating axis is r, the length the center of mass traveled is $2\pi r$. The area of the circle is πr^2, the volume of the shape $V = A \cdot L = 2\pi^2 r^3$.

Question 5 Solution

To show that $f(x)$ is minimized when $x = \bar{x} \Leftrightarrow f'(\bar{x}) = 0$ and $f''(\bar{x}) > 0$.

From $f(x) = \sum_{i=1}^{n} m_i(x - x_i)^2$ we have

$f'(x) = \sum_{i=1}^{n} 2m_i(x - x_i) = \sum_{i=1}^{n} 2m_i x - \sum_{i=1}^{n} 2m_i x_i = 2x\sum_{i=1}^{n} m_i - 2\sum_{i=1}^{n} m_i x_i.$

Let $f'(x) = 0 \Rightarrow x = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \bar{x}$. By the definition for the center of mass, $f'(x) = 0$ when $x = \bar{x}$.

Furthermore $f''(x) = 2\sum_{i=1}^{n} m_i > 0$ for any x.

Question 6 Solution

Prob$(255 \le X \le 285) = \int_{255}^{285} f(x)dx$ where $f(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-270)^2}{2\times 15^2}}$.

Note that Prob$(\mu - \sigma \le X \le \mu + \sigma) \approx 0.68$ for normal distribution. The probability of being greater than three standard deviations away from the mean is approximately 0.13%.

Question 7 Solution

a) $\int_{-\infty}^{\infty} f(t)dt = \int_0^5 \frac{6}{625}t^2(5-t)^2 dt = \frac{6}{625}\int_0^5 t^2(25 - 10t + t^2)dt = \frac{6}{625}\int_0^5 (25t^2 - 10t^3 + t^4)dt = \frac{6}{625}\left(\frac{25}{3}t^3 - \frac{10}{4}t^4 + \frac{t^5}{5}\right)\Big|_0^5 = 1$ and $f(t) \ge 0$, thus $f(t)$ defines a valid pdf.

The mean $\mu = \int_0^5 tf(t)dt = \frac{6}{625}\int_0^5 (25t^3 - 10t^4 + t^5)dt = \frac{5}{2}$, actually the $f(t)$ is symmetric about $x = \frac{5}{2}$.

b) $1000 \cdot$ Prob$(t \ge 3) = 1000\int_3^5 f(t)dt \approx 317$.

Question 8 Solution

$f(t) = ce^{-ct}$ where $c = \frac{1}{20}$, t is measured in minutes, $\mu = \frac{1}{c}$, thus $c = \frac{1}{\mu}$.

a) $\text{Prob}(X \le 10) = \int_0^{10} \frac{1}{20} e^{-\frac{t}{20}} dt = 1 - \frac{1}{\sqrt{e}} \approx 0.375$

b) $\text{Prob}(X \ge 30) = \int_{30}^{\infty} \frac{1}{20} e^{-\frac{t}{20}} dt = e^{-\frac{3}{2}} \approx 0.24$

c) Solve $\frac{1}{2} = \int_m^{\infty} f(t)dt \Rightarrow \frac{1}{2} = e^{-\frac{m}{20}} \Rightarrow m = 20 \ln 2 \approx 20 \times 0.7 = 14$ minutes. (The median $m = \mu \ln 2 < \mu$ for exponential distribution.)

Question 9 Solution

Note that second order differential equation must have two linearly independent different solutions. The answers are (a), (b) and (d).

For (a), $y = e^{-t}$, $y' = -e^{-t}$ and $y'' = e^{-t}$, $y'' + 2y' + y = e^{-t} - 2e^{-t} + e^{-t} = 0$, satisfies the equation.

For (b), $y = 2e^{-t}$, $y' = -2e^{-t}$ and $y'' = 2e^{-t}$, $2(y'' + 2y' + y) = 2(e^{-t} - 2e^{-t} + e^{-t}) = 0$, satisfies the equation.

For (c), $y'' = 4y$, $y' = -2y$ and so $4y - 4y + y \ne 0$.

For (d), $y = te^{-t}$, $y' = e^{-t} - te^{-t}$ (product rule), $y'' = -2e^{-t} + te^{-t}$, $y'' + 2y' + y = -2e^{-t} + te^{-t} + 2(e^{-t} - te^{-t}) + te^{-t} = 0$, satisfies the equation.

For (e) $y' = 2te^{-t} + t^2 e^{-t}$ and $y" = 2e^{-t} - t^2 e^{-t}$ and so the equation is not satisfied.

Question 10 Solution

a) $y = 1$ unstable

b) $y = -1$ stable and $y = 1$ unstable

c) $y = 1$ one side stable and one side unstable

d) $y = 1$ stable and $y = 2$ unstable

e) $y = 2k\pi$ (k is integer) unstable, $y = (2k+1)\pi$ stable

Note that a stable/unstable solution corresponds to a -/+ slope in the phase plane.

Question 11 Solution

Show $E(t) = \frac{1}{2}mv^2 + V(x)$ is constant, ie, $\frac{1}{2}mv^2 + V(x) = C$, taking derivative (on t) on both sides $\left(\frac{1}{2}mv^2 + V(x)\right)' = 0 \Rightarrow mv(t)v'(t) + V'(x)x'(t) = 0 \Rightarrow mx'(t)x''(t) + V'(x)x'(t) = 0$ (note that $v(t) = x'(t)$, $v'(t) = x''(t)$, and $\frac{dV(x)}{dt} = V'(x)x'(t)$ by chain rule.)

$\Rightarrow x'(t)[mx''(t) + V'(x)] = 0 \Rightarrow mx''(t) + V'(x) = 0$ (since otherwise $x'(t) \equiv 0$ not physical) $\Rightarrow mx'' = f(x)$ where $f(x) = -V'(x)$. ok.

Question 12 Solution

a) $y' = y \Rightarrow \frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt \Rightarrow \ln y = t + C \Rightarrow y = e^C \cdot e^t$, $e^C = 1$ using $y(0) = 1$

The solution is $y = y(0)e^t \Rightarrow y = e^t$.

b) $y' = ty \Rightarrow \frac{dy}{dt} = ty \Rightarrow \frac{dy}{y} = tdt \Rightarrow \ln y = \frac{1}{2}t^2 + C \Rightarrow y = e^C \cdot e^{\frac{t^2}{2}}$, $e^C = 1$ using $y(0) = 1$

The solution is $y = y(0)e^{\frac{t^2}{2}} \Rightarrow y = e^{\frac{t^2}{2}}$.

c) $y' = y^2 \Rightarrow \frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow -\frac{1}{y} = t + C \Rightarrow y = \frac{1}{-t-C}$, $C = -1$ using $y(0) = 1$

The solution is $y = \frac{1}{1-t}$. Check $y' = \frac{1}{(1-t)^2} = y^2$ and $y(0) = 1$ at $t = 0$. What happens when $t \to 1$?

d) $y' = y(1-y) \Rightarrow \frac{dy}{dt} = y(1-y) \Rightarrow \frac{dy}{y(1-y)} = dt \Rightarrow \frac{dy}{y} + \frac{dy}{1-y} = dt \Rightarrow \ln|y| - \ln|1-y| = t + C \Rightarrow \ln\frac{|y|}{|1-y|} = t + C \Rightarrow \frac{|y|}{|1-y|} = e^C \cdot e^t$

One can find the general solution for this equation, but the initial condition $y(0) = 1$ is a constant solution of the system, so the solution will stay at $y = 1$ for all $t \geq 0$, so the solution is $y(t) = 1$, and the general solution is no needed.

Question 13 Solution

Since the question asks concentration, denote concentration by y (kg/L), the amount of salt is $1000y$(kg) (since the volume is 1000(L)).

rate of change of amount = the rate of input−the rate of output, so $(1000y)' = 0.05 \cdot 5 - y \cdot 5$, i.e., $y' = \frac{5}{1000}(0.05 - y)$ (it is $y' = k(T - y)$ form, where $k = \frac{5}{1000}$ and $T = 0.05$, similar to Newton's law of cooling or heating). The solution is $y(t) = T + (y_0 - T)e^{-kt} = 0.05 + (0 - 0.05)e^{-0.005t}$ where initial condition $y(0) = y_0 = 0$, since initially it is pure water.

After one hour, i.e., 60 minutes (note the time unit in the equation is minutes), $y(60) = 0.05 - 0.05e^{-0.005\times 60} \approx 0.013$ (kg/L).

Question 14 Solution

a) $x(t)$ = value of new currency in circulation at time t, $x'(t)$ = rate of change of new currency in circulation = (rate going in) - (rate going out) = $(0.05) - (\frac{x}{10} \cdot 0.05) \Rightarrow x' = 0.05(1 - \frac{x}{10})$

b) The equation can be written as $x' = -0.005(x - 10)$, which is the same as Newton's law of cooling/heating, $x' = k(x - T)$, with $k = -0.005$ and $T = 10$. Hence $x(t) = T + (x_0 - T)e^{-kt} = 10 + (0 - 10)e^{-0.005t} = 10(1 - e^{-0.005t})$, where we used the initial condition $x_0 = x(0) = 0$.

c) 90% means $x(t) = 9 \Rightarrow 9 = 10(1 - e^{-0.005t}) \Rightarrow t = 200 \ln 10 \approx 460$ days

Question 15 Solution

$c' = k(a - c)(b - c)$, where $k > 0$, $a = 1$, $b = 2 \Rightarrow c' = k(1 - c)(2 - c)$, it has two constant solutions $c = 1$ and $c = 2$ (ie., let $c' = 0$, solve the equation), $c = 1$ is stable while $c = 2$ is unstable, since for $0 \le c < 1$, $c' > 0$, $c(t)$ increases (towards $c = 1$), for $1 < c < 2$, $c' < 0$, $c(t)$ decreases (towards $c = 1$), for $c > 2$, $c' > 0$, $c(t)$ increases (away from $c = 2$).

To find the product concentration, we solve $c' = k(1-c)(2-c)$ by separation of variables, $\frac{dc}{(1-c)(2-c)} = kdt$. Using partial fractions, we write $\frac{1}{(1-c)(2-c)} = \frac{1}{1-c} - \frac{1}{2-c}$, and integrate to obtain $\ln\left|\frac{2-c}{1-c}\right| = kt + d$, or $\frac{2-c}{1-c} = De^{kt}$, for some constant D (using the initial condition $c(0) = 0$, we find this constant to be $D = 2$). We now solve the equation for $c(t)$, writing $2 - c = (1-c)De^{kt}$. Rearranging terms and simplifying, we get $c(t) = 1 - \frac{1}{2e^{kt}-1}$.

Given that $c(0) = 0$ at $t = 0$, the asymptotic value of $c(t)$ is $c = 1$ as $t \Rightarrow \infty$. The asymptotic value will not change if $c(0) = 1.5$ mole/L. This can also be seen using the phase plane.

Question 16 Solution

$mc_pT' = -e\sigma T^4 \Rightarrow \frac{T'}{T^4} = -\frac{e\sigma}{mc_p} \Rightarrow \int \frac{T'}{T^4}dt = \int -\frac{e\sigma}{mc_p}dt \Rightarrow -\frac{1}{3T^3} = -\frac{e\sigma}{mc_p}t + C$ where C is constant, which can be determined by initial condition, $T(0) = T_0$ at $t = 0$, $\Rightarrow C = -\frac{1}{3T_0^3}$. Thus $-\frac{1}{3T^3} = -\frac{e\sigma}{mc_p}t - \frac{1}{3T_0^3} \Rightarrow T(t) = \left(3\frac{e\sigma}{mc_p}t + \frac{1}{T_0^3}\right)^{-\frac{1}{3}}$

In the limit $t \Rightarrow \infty$, $3\frac{e\sigma}{mc_p}t \gg \frac{1}{T_0^3}$ (where '$\gg$' means much larger), $\alpha = -\frac{1}{3}$.

Question 17 Solution

The decay rate of radioactive sample satisfies $y' = -ky$ (by separation of variables) $\Rightarrow y(t) = y_0e^{-kt}$ where y_0 is initial mass at $t = 0$. It is given that $y(2) = 128$ after 2 hours and $y(5) = 2$ after 5 hours, $\Rightarrow \begin{cases} 128 = y_0e^{-2k} \\ 2 = y_0e^{-5k} \end{cases}$. Dividing the first equation by the second equation yields $64 = e^{3k} \Rightarrow e^k = 4$. Substituting in the first equation gives $128 = y_0 4^{-2} \Rightarrow y_0 = 128 \cdot 16 = 2048$.

Let $y(t) = 1$ kg, $1 = 2048 \cdot 4^{-t} \Rightarrow t = 5.5$ hours.

From 2048 kg to 2 kg needs 5 hours, from 2048 kg to 1 kg needs 5.5 hours, thus from 2 kg to 1 kg needs half hour.

Question 18 Solution

Newton's Law of cooling or heating, $y(t) = T + (y_0 - T)e^{-kt}$ where $y_0 = 21$ at $t = 0$. We know $y(1) = 27$ and $y(2) = 30$, thus we have $\begin{cases} 27 = T + (21-T)e^{-k} \\ 30 = T + (21-T)e^{-2k} \end{cases}$ Combining these two equations, we find $T = 33$.

Question 19 Solution

a) $y' = r - \alpha y$, where $y = y(t)$.

b) Let $r - \alpha y = 0 \Rightarrow y = \frac{r}{\alpha} \equiv M$ (phase plane, done in class, stable)

c) The equation in a) can be rewritten as $y' = -\alpha(y - \frac{r}{\alpha})$. This is the same as Newton's Law of heating/cooling $y' = k(y - T)$ with $T = \frac{r}{\alpha}, k = -\alpha$.

The endowment dropped to $\frac{1}{2}M$ last year, so we take $y_0 = \frac{1}{2}M$. The solution is

$$y(t) = T + (y_0 - T)e^{kt} = \frac{r}{\alpha} + (\frac{1}{2}M - \frac{r}{\alpha})e^{-\alpha t} = M - \frac{1}{2}Me^{-\alpha t}$$

To find how it will take to recover to $\frac{3}{4}M$, we set

$$y(t) = \frac{3}{4}M \Rightarrow M - \frac{1}{2}Me^{-\alpha t} = \frac{3}{4}M \Rightarrow -\frac{1}{2}Me^{-\alpha t} = -\frac{1}{4}M \Rightarrow e^{-\alpha t} = \frac{1}{2} \Rightarrow t = -\frac{1}{\alpha}\ln\frac{1}{2} = \frac{1}{\alpha}\ln 2.$$

<u>Question 20 Solution</u>

For Bob, his coffee temperature after adding the cold milk is $T_{Bob}(0) = (1-a)T_h + aT_c$, where $a = \frac{1}{8+1} = \frac{1}{9}$ is the volume fraction of cold milk in the mixture. After that, the coffee temperature follows Newton's Law of cooling/heating $y' = -k(y - T_a)$, where $k > 0$ is a rate constant and T_a is the ambient cafe temperature. The solution is $y(t) = T_a + (y_0 - T_a)e^{-kt}$. Thus, after 2 minutes, $T_{Bob}(2) = T_a + [(1-a)T_h + aT_c - T_a]e^{-2k}$.

For Ray, the difference is that $y(0) = T_h$, because he did not add cold milk at the beginning. So at $t = 2$ minutes, his coffee temperature is $y(2) = T_a + (T_h - T_a)e^{-2k}$, but then adding 1oz of cold milk yields $T_{Ray}(2) = (1-a)y(2) + aT_c = (1-a)[T_a + (T_h - T_a)e^{-2k}] + aT_c$.

$T_{Ray}(2) - T_{Bob}(2)$

$= (1-a)[T_a + (T_h - T_a)e^{-2k}] + aT_c - T_a - [(1-a)T_h + aT_c - T_a]e^{-2k}$

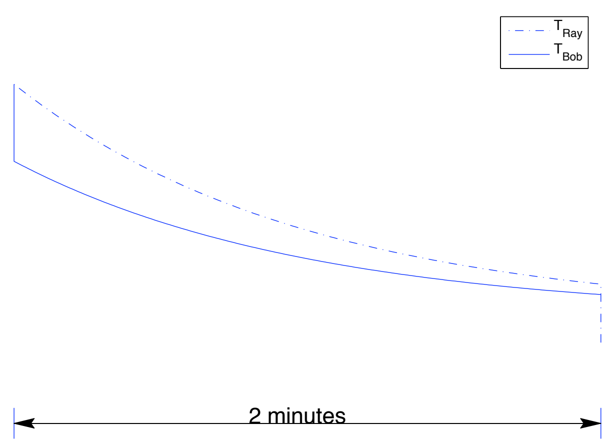
$= T_a[(1-a) - (1-a)e^{-2k} - 1 + e^{-2k}] + T_c(a - ae^{-2k}) + T_h[(1-a)e^{-2k} - (1-a)e^{-2k}]$

$= T_a(ae^{-2k} - a) + T_c(a - ae^{-2k}) = a(1 - e^{-2k})(T_c - T_a) < 0,$

since $a > 0, k > 0, T_c < T - a$.

So Ray ends up with colder coffee.

This can be understood as follows. The <u>larger</u> the temperature difference between the object and its surroundings, the <u>faster</u> the temperature changes. Since Bob's strategy reduced the initial temperature difference, he reduced the temperature rate of change, which was not a wise strategy for cooling the coffee. See the plots of $T_{Bob}(t), T_{Ray}(t)$.



Question 21 Solution

Let $y(t)$ be the fraction of the population who have already heard the rumor at time t. Then $1 - y(t)$ is the fraction of the population who have not already heard the rumor at time t.

The rate of change of the fraction of the population who have heard the rumor is $y' = ky(1-y)$, for some $k > 0$. This makes sense because if $y = 0$ or $y = 1$, then $y' = 0$.

This is a logistic equation with general solution $y(t) = \frac{y_0}{y_0+(1-y_0)e^{-kt}}$. Set $t = 0$ at 8am, then $y_0 = \frac{10}{1000} = 0.01$. Then, at 9am, $t = 1$ hour, $y(1) = \frac{20}{1000} = 0.02$. Substituting these into the solution yields $0.02 = \frac{0.01}{0.01+0.99e^{-k}} \Rightarrow 0.01 + 0.99e^{-k} = \frac{1}{2} \Rightarrow e^{-k} = \frac{49}{99}$. The solution becomes $y(t) = \frac{0.01}{0.01+0.99(\frac{49}{99})^t}$; note that $e^{-kt} = (e^{-k})^t$. Then $y(t) = \frac{1}{2} \Rightarrow \frac{0.01}{0.01+0.99(\frac{49}{99})^t} = \frac{1}{2} \Rightarrow$ $0.01 + 0.99(\frac{49}{99})^t = 0.02 \Rightarrow (\frac{49}{99})^t = \frac{1}{99} \Rightarrow t = \ln(99)/(\ln(99) - \ln(49)) = 6.5337$, i.e. around 2:30pm.

Question 22 Solution

a) $\lim\limits_{n\Rightarrow\infty} a_n = \lim\limits_{n\Rightarrow\infty} \frac{n}{n+1} = \lim\limits_{n\Rightarrow\infty} \frac{1}{1+1/n} = 1$
b) $\lim\limits_{n\Rightarrow\infty} a_n = \lim\limits_{n\Rightarrow\infty} \frac{(-1)^n}{n} = 0$
c) $\lim\limits_{n\Rightarrow\infty} a_n = \lim\limits_{n\Rightarrow\infty} \frac{2^n}{n} = \infty$
d) $\lim\limits_{n\Rightarrow\infty} a_n = \lim\limits_{n\Rightarrow\infty} 1 - \frac{1}{n} = 1$
e) $\lim\limits_{n\Rightarrow\infty} a_n = \lim\limits_{n\Rightarrow\infty} (1 - \frac{1}{n}) = e^{-1}$

Question 23 Solution

a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \cdots = \sum\limits_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$

b) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots = \frac{2}{3} \sum\limits_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} \cdot \frac{1}{1-\frac{2}{3}} = 2$

c) $1 + 0.1 + 0.01 + .001 + \cdots = \sum\limits_{n=0}^{\infty} (0.1)^n = \frac{1}{1-0.1} = \frac{10}{9} = 1.1111\ldots$

Question 24 Solution

a) $\int \cosh 2x dx = \frac{1}{2} \sinh 2x$
b) $\int \tanh 2x dx = \frac{1}{2} \int \tanh(2x) d(2x) = \frac{1}{2} \int \frac{\sinh 2x}{\cosh 2x} d(2x) = \frac{1}{2} \int \frac{1}{\cosh 2x} d \cosh 2x = \frac{1}{2} \ln(\cosh 2x)$
c) $\int \cosh^2 x dx = \int (\frac{e^x+e^{-x}}{2})^2 dx = \int \frac{e^{2x}+2+e^{-2x}}{4} dx = \frac{1}{4}\left(\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right) = \frac{1}{4}\sinh 2x + \frac{1}{2}x$

Question 25 Solution

a) $f(x) = \cosh x, f'(x) = \sinh x, 1 + [f'(x)]^2 = 1 + \sinh^2 x = \cosh^2 x$
$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_{-1}^1 \cosh x dx = \sinh x|_{-1}^1 = \sinh 1 - \sinh(-1) = 2\sinh 1$
b) $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2} dx = \int_{-1}^1 2\pi \cosh^2 x dx = 2\pi \int_{-1}^1 (\frac{e^x+e^{-x}}{2})^2 dx = 2\pi \int_{-1}^1 \frac{e^{2x}+2+e^{-2x}}{4} dx =$ $\frac{\pi}{2}\left(\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right)\Big|_{-1}^1 = \frac{\pi}{2}(\sinh 2x + 2x)|_{-1}^1 = \frac{\pi}{2}(2\sinh 2 + 2) = \pi(\sinh 2 + 1)$

Question 26 Solution

$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}$

a) LHS $= \sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2}$

RHS $= \sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y}}{4} + \frac{e^{x+y} + e^{y-x} - e^{x-y} - e^{-x-y}}{4} = \frac{2e^{x+y} - 2e^{-x-y}}{4} = \frac{e^{x+y} - e^{-x-y}}{2} = \text{LHS}$

b) One can prove it by the definitions of sinh, cosh, as in (a), or by using $\sinh(-y) = -\sinh y, \cosh(-y) = \cosh y$ (i.e. sinh is an odd function, cosh is an even function) and based on a):

$$\sinh(x-y) = \sinh[x+(-y)] \overset{a)}{=} \sinh(x)\cosh(-y) + \cosh(x)\sinh(-y) = \sinh x \cosh y - \cosh x \sinh y$$

For c), d), one can prove them by the definitions of $\sinh, \cosh$ similar to (a) or by using the relationships $(\sinh x)' = \cosh(x), (\cosh x)' = \sinh x$ and the identity a). Namely,

c) Taking derivative for both sides of a) with respect to x, regarding y as a constant, we have:

$\cosh(x+y) = [\sinh(x+y)]' = [\sinh x \cosh y + \cosh x \sinh y]' = (\sinh x)' \cosh y + (\cosh x)' \sinh y = \cosh x \cosh y + \sinh x \sinh y$

d) Differentiating both sides of b) with respect to x, regarding y as constant yields the result.

Question 27 Solution

(i) $f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2}, f''(x) = 2x^{-3}$

$a = 1 \Rightarrow f(1) = 1,\ f'(1) = -1, f''(1) = 2$

$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - (x-1) = 2 - x$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = T_1(x) + \frac{2}{2}(x-1)^2 = 2 - x + (x-1)^2 = x^2 - 3x + 3$

(i) $f(x) = \sin x \Rightarrow f'(x) = \cos x, f''(x) = -\sin x$

$a = \frac{\pi}{2} \Rightarrow f(\frac{\pi}{2}) = 1,\ f'(\frac{\pi}{2}) = 0, f''(\frac{\pi}{2}) = -1$

$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - 0(x - \frac{\pi}{2}) = 1$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = T_1(x) + \frac{-1}{2}(x - \frac{\pi}{2})^2 = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$

(iii) $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}, f''(x) = -\frac{1}{4\sqrt{x^3}}$

$a = 4 \Rightarrow f(4) = \sqrt{4} = 2, f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}, f''(4) = -\frac{1}{4\sqrt{4^3}} = -\frac{1}{32}$

$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 2 + \frac{1}{4}(x-4)$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = T_1(x) + \frac{-\frac{1}{32}}{2}(x-4)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-4)^2$

approximations of $\sqrt{5} = 2.2361$ (viewing $\sqrt{5}$ as the value of $\sqrt{x}$ at $x = 5$)

$T_1(5) = 2 + \frac{1}{4}(5-4) = 2.25$

$T_2(5) = 2 + \frac{1}{4}(5-4) - \frac{1}{64}(5-4)^2 = 2.2344$

$T_2(5)$ gives a more accurate approximation for $\sqrt{5}$ because the 2nd degree Taylor approximation gives a more accurate approximation to the function