

## Review Sheet Solutions for Midterm Exam 2 (Math 156) Fall 2017

1. True or False? Justify your answer with a reason or counterexample.

a) True. use integration by parts :  $u = x, dv = xe^{-x^2} \Rightarrow du = dx, v = -\frac{1}{2}e^{-x^2}$

$$\int_0^\infty x^2 e^{-x^2} dx = x\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^\infty - \int_0^\infty -\frac{1}{2}e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-x^2} dx$$

b) False.  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = L \Rightarrow \ln L = \ln \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \ln\left(1 - \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} n \ln\left(1 - \frac{1}{n}\right) = \infty \cdot 0$

$$\text{set } u = \frac{1}{n} \Rightarrow \ln L = \lim_{u \rightarrow 0} \frac{\ln(1-u)}{u} = \frac{0}{0} = \lim_{u \rightarrow 0} \frac{-1}{1-u} = -1 \Rightarrow L = e^{-1}$$

c) True. The region is symmetric about the line  $y = x$  and hence by the symmetry principle the CM lies on the line  $y = x$ .

d) False. A counterexample was given in class on page 34 of the lecture notes. A triangular plate has vertices in the  $xy$ -plane at  $(0, 0), (1, 0), (0, 1)$ . Then  $\bar{x} = \frac{1}{3}$ . However the area to the left of the line  $x = \bar{x}$  is  $\frac{5}{18}$  and the area to the right is  $\frac{4}{18}$ .

e) False. A counterexample is the normal distribution  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ . Then  $f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} > 1$  for  $\sigma < \sqrt{2\pi}$ .

f) True. 1.  $f(x) \geq 0$  for all  $x$

$$\begin{aligned} 2. \int_{-\infty}^{\infty} f(x) dx &= \int_{-1}^1 \frac{1}{\pi\sqrt{1-x^2}} dx, \text{ set } x = \sin \theta, dx = \cos \theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi\sqrt{1-\sin^2 \theta}} \cos \theta d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \theta}{\cos \theta} d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{\theta}{\pi} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 \end{aligned}$$

g) False. It is the median  $m$  which satisfies  $\text{prob}(X \leq m) = \text{prob}(X \geq m)$ . In general,  $\mu \neq m$  and  $\text{prob}(X \leq \mu) \neq \text{prob}(X \geq \mu)$ , unless the pdf is symmetric about  $x = \mu$ .

h) True. The pdf of a normal distribution is  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$  and we know from hw7 that

$$\int_{-\infty}^{\infty} f(x) dx = 1, \int_{-\infty}^{\infty} x f(x) dx = \mu. \text{ So the given integral is } \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} (x - \mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \int_{-\infty}^{\infty} (x - \mu) f(x) dx = \int_{-\infty}^{\infty} x f(x) dx - \mu \int_{-\infty}^{\infty} f(x) dx = \mu - \mu = 0.$$

i) True.  $y(t) = \tanh t \Rightarrow y'(t) = \frac{d}{dt} \tanh t = \frac{d}{dt} \frac{\sinh t}{\cosh t} = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = 1 - \tanh^2 t = 1 - y^2$

j) False. See page 42 of the lecture notes. When a population grows at a rate proportional to itself, the population increases at an exponential rate, not a linear rate. The cell count is  $y(t) = 1000(2.5)^{t/2}$ , so  $y(4) = 6250$  cells.

k) False. If a radioactive material has half-life 100 years and a sample has initial mass 1 kg, then the amount remaining after 100 years is  $\frac{1}{2}$  kg and the amount remaining after only 50 years is more than  $\frac{1}{2}$  kg.

l) False.  $y(t) = 0$  is a constant solution, but it is unstable because if  $y_0 \neq 0$ , then  $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} y_0 e^t \neq 0$ .

m) True. If  $y' = 1 - y^2$ , then the constant solutions are  $y = \pm 1$ , and the phase plane shows that  $y = 1$  is stable and  $y = -1$  is unstable, so if  $y(0) = \frac{1}{2}$ , then  $\lim_{t \rightarrow \infty} y(t) = 1$  since  $y = 1$  is stable.

n) False. As shown in class and on homework, if  $\Delta t$  is reduced by a factor of  $\frac{1}{2}$ , then the error in Euler's method decreases by a factor of approximately  $\frac{1}{2}$ , not  $\frac{1}{4}$ .

o) True.  $\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x} > 0$  for all  $x$

p) False.  $\sinh^2 x + \cosh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4} + \frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \neq 1$  for  $x \neq 0$

q) False.  $\lim_{x \rightarrow \infty} \tanh x = \lim_{x \rightarrow \infty} \frac{\sinh x}{\cosh x} = \lim_{x \rightarrow \infty} \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$

r) True. By definition  $\cosh x = \frac{e^x + e^{-x}}{2}$ , so  $\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x$ , so  $\cosh x$  is an even function.

s) True.  $\frac{d}{dx} \tanh x = \frac{d}{dx} \frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$

t) True.  $2 \sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(e^{2x} - e^{-2x}) = \sinh(2x)$

- u) True.  $T_1(x) = f(a) + f'(a)(x - a)$ , then  $f(x) = e^x, a = 0 \Rightarrow f(0) = 1, f'(0) = 1 \Rightarrow T_1(x) = 1 + x$
- v) False. counterexample :  $a_n = \frac{1}{2} + \frac{1}{2n} \Rightarrow 0 \leq a_n \leq 1$  and  $a_{n+1} < a_n$ , but  $\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \neq 0$
- The two given conditions imply the sequence converges, but the limit is not necessary zero.
- w) False. The 3rd equal sign is not justified, i.e.  $\lim_{n \rightarrow \infty} (n + 1 - n) \neq \lim_{n \rightarrow \infty} (n + 1) - \lim_{n \rightarrow \infty} n$ , because the left side is equal to 1, but the right side is undefined.
- x) False. This is the harmonic series; it diverges, but it is not a geometric series.
- y) True.  $9 + 0.9 + 0.09 + 0.009 + \dots = 9(1 + 0.1 + 0.001 + 0.001 + \dots) = 9 \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = 9 \cdot \frac{1}{1 - \frac{1}{10}} = 10$
- z) False. counterexample :  $a_n = \frac{1}{n}$ , then  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , but  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$  diverges
- aa) True. We showed that if  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ ; so if  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.
- bb) True. The comparison test for series implies that  $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ , and the latter series converges by the  $p$ -test with  $p = 2$ , hence the sum of the  $a_n$  series converges.
- cc) False. The series converges if and only if  $\int_1^{\infty} f(x)dx$  converges and that is not true in general.
- dd) True. The series converges by the AST. The error bound for a convergent alternating series says that  $|s - s_n| < a_{n+1}$ , hence with  $n = 1$  we have  $|s - s_1| < a_2 \Rightarrow |s - 1| < \frac{1}{2} \Rightarrow \frac{1}{2} < s < \frac{3}{2}$ .

2a.  $S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} dx = \int_0^1 2\pi \sqrt{1 + 9x^4} x^3 dx$   
 set  $u = 1 + 9x^4$ , so  $du = 9 \cdot 4x^3 dx$ , and  $x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 10$ , then we have  
 $S = \int_1^{10} 2\pi \sqrt{u} \frac{1}{9 \cdot 4} du = \frac{1}{18} \pi \int_1^{10} \sqrt{u} du = \frac{1}{18} \pi \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{10} = \frac{\pi}{27} (\sqrt{10}^3 - 1)$

2b.  $S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx = \int_0^1 2\pi \sqrt{1 - x} \sqrt{1 + \left(\frac{-1}{2\sqrt{1-x}}\right)^2} dx$   
 $= \int_0^1 2\pi \sqrt{1 - x} \sqrt{1 + \frac{1}{4(1-x)}} dx = \int_0^1 2\pi \sqrt{1 - x} \sqrt{\frac{5-4x}{4(1-x)}} dx$   
 $= \int_0^1 \pi \sqrt{5 - 4x} dx = \int_0^1 \pi \sqrt{5 - 4x} \left(-\frac{1}{4}\right) d(5 - 4x) = -\frac{\pi}{4} \frac{2}{3} (5 - 4x)^{\frac{3}{2}} \Big|_0^1$   
 $= -\frac{\pi}{6} + \frac{\pi}{6} (\sqrt{5})^3 = \frac{\pi}{6} ((\sqrt{5})^3 - 1)$

2c.  $S = 2\pi \int_0^1 f(x) \sqrt{1 + (f'(x))^2} dx = 2\pi \int_0^1 \cosh x \sqrt{1 + \sinh^2 x} dx = 2\pi \int_0^1 \cosh^2 x dx$   
 $= 2\pi \int_0^1 \left(\frac{e^x + e^{-x}}{2}\right)^2 dx = 2\pi \int_0^1 \left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) dx = \frac{\pi}{2} \left(\frac{e^{2x}}{2} + 2x + \frac{e^{-2x}}{-2}\right) \Big|_0^1$   
 $= \frac{\pi}{2} \left(\frac{e^2}{2} + 2 - \frac{e^{-2}}{2} - \left(\frac{1}{2} - \frac{1}{2}\right)\right) = \frac{\pi}{2} (2 + \sinh 2) = \pi \left(1 + \frac{1}{2} \sinh 2\right)$

3. Assume the two planes are in the right half of the  $xy$ -plane,  $x > 0$ ; otherwise we can either perform a similar calculation for the left half plane, or if they are in both, perform two calculations, one for the left half plane and another for the right half plane. A sphere is generated by rotating the curve  $f(x) = \sqrt{r^2 - x^2}$  around the  $x$  axis. The formula for the surface area of the zone is  $S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$ , where  $a, b$  are the appropriate limits. Here we have  $1 + (f'(x))^2 = \frac{r^2}{r^2 - x^2}$ , so that  $S = 2\pi \int_a^b \sqrt{r^2 - x^2} \frac{r}{\sqrt{r^2 - x^2}} dx = 2\pi r(b - a)$ . Letting  $d = b - a$  be the distance between the planes, we find that  $S = 2\pi r d$ , and hence the surface area of the zone depends only on the distance between the planes and not their location.

4. (ii)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm \sqrt{b^2 - \left(\frac{b}{a}\right)^2 x^2}$  (use the upper half of the ellipse)

$$f(x) = \frac{b}{a} \sqrt{a^2 - x^2} \Rightarrow f'(x) = -\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}} \Rightarrow \sqrt{1 + (f'(x))^2} = \sqrt{1 + \left(\frac{b}{a}\right)^2 \frac{x^2}{a^2 - x^2}} = \frac{\sqrt{a^2 - x^2 + \left(\frac{b}{a}\right)^2 x^2}}{\sqrt{a^2 - x^2}}$$

$$\begin{aligned}
S &= 2\pi \int_{-a}^a f(x) \sqrt{1 + (f'(x))^2} dx = 2\pi \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} \frac{\sqrt{a^2 - x^2 + \left(\frac{b}{a}\right)^2 x^2}}{\sqrt{a^2 - x^2}} dx \\
&= 2\pi \frac{b}{a} \int_{-a}^a \sqrt{a^2 + \left(\frac{b^2 - a^2}{a^2}\right) x^2} dx = 4\pi \frac{b}{a} \int_0^a \sqrt{a^2 + \left(\frac{b^2 - a^2}{a^2}\right) x^2} dx \text{ (the integrand is even)}
\end{aligned}$$

(iii) Now we will show that  $S = 2\pi b(b + a(\sin^{-1} c)/c)$ , where  $c = \sqrt{a^2 - b^2}/a$ .

$$\begin{aligned}
S &= 4\pi \frac{b}{a} \int_0^a \sqrt{a^2 + \left(\frac{b^2 - a^2}{a^2}\right) x^2} dx = 4\pi \frac{b}{a} \int_0^a \sqrt{a^2 - \left(\frac{a^2 - b^2}{a^2}\right) x^2} dx \\
&= 4\pi \frac{b}{a} \int_0^a \sqrt{a^2 - \left(\frac{\sqrt{a^2 - b^2}}{a} x\right)^2} dx = 4\pi \frac{b}{a} \int_0^a \sqrt{a^2 - (cx)^2} dx
\end{aligned}$$

substitute  $u = cx$ ,  $du = cdx$ , change limits  $x = a \rightarrow u = ca$ ,  $x = 0 \rightarrow u = 0$

$$\begin{aligned}
S &= 4\pi \frac{b}{a} \int_0^{ca} \sqrt{a^2 - u^2} \frac{du}{c} = 4\pi \frac{b}{ac} \left[ \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{u}{a}\right) \right]_0^{ca} \text{ (this step uses trig substitution)} \\
&= 2\pi \frac{b}{ac} \left( ca\sqrt{a^2 - c^2 a^2} + a^2 \sin^{-1} c \right) = 2\pi b \left( \sqrt{a^2 - c^2 a^2} + \frac{a}{c} \sin^{-1} c \right)
\end{aligned}$$

note :  $\sqrt{a^2 - c^2 a^2} = \sqrt{a^2 - \left(\frac{a^2 - b^2}{a^2}\right) a^2} = b \Rightarrow S = 2\pi b(b + a(\sin^{-1} c)/c)$

check : when  $a \rightarrow b$ , the ellipse becomes a sphere of radius  $b$ ; in that case we have  $c \rightarrow 0$ , and it can be shown that  $\sin^{-1}(c)/c \rightarrow 1$ , so  $S \rightarrow 4\pi b^2$ , which is the area of a sphere of radius  $b$

$$5. \bar{x} = \frac{M}{m} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{2 \cdot (-10) + 3 \cdot 6 + 1 \cdot x_3}{2 + 3 + 1} = \frac{x_3 - 2}{6} = 0 \Rightarrow x_3 = 2$$

$$6. \bar{x} = \frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}, \bar{y} = \frac{1}{2} \frac{\int_a^b (f(x)^2 - g(x)^2)dx}{\int_a^b (f(x) - g(x))dx}, \text{ in some cases } g(x) = 0$$

6a)  $a = 0, b = 1, g(x) = 0, f(x) = x$

$$\bar{x} = \frac{\int_0^1 x^2 dx}{\int_0^1 x dx} = \frac{2}{3} \text{ and } \bar{y} = \frac{1}{2} \frac{\int_0^1 x^2 dx}{\int_0^1 x dx} = \frac{1}{3}.$$

Note that the region is symmetric about line  $y = 1 - x$ , thus  $(\bar{x}, \bar{y})$  lies on  $y = 1 - x$ . If you notice this, once  $\bar{x}$  is known, then  $\bar{y}$  can be found without extra calculation.

6b)  $a = 0, b = 2, g(x) = 0, f(x) = \sqrt{x(2-x)}$

First consider the curve  $y = \sqrt{x(2-x)}$ .

Then  $y^2 = x(2-x) = 2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1 - 1) = -(x-1)^2 + 1 \Rightarrow (x-1)^2 + y^2 = 1$ , hence the curve is a circle centered at  $(x, y) = (1, 0)$  with radius 1. By symmetry we have  $\bar{x} = 1$  and then we compute  $\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2} \int_0^2 (\sqrt{x(2-x)})^2 dx}{\int_0^2 \sqrt{x(2-x)} dx} = \frac{\frac{1}{2} \int_0^2 x(2-x) dx}{\frac{\pi}{2}} = \frac{2}{\pi} \frac{1}{2} (x^2 - \frac{1}{3} x^3) \Big|_0^2 = \frac{1}{\pi} (4 - \frac{8}{3}) = \frac{4}{3\pi}$ .

(The denominator is the area of a half circle).  $\bar{x} = 1$  and  $\bar{y} = \frac{4}{3\pi}$

6c)  $a = -1, b = 1, g(x) = 0, f(x) = \cosh x$ , since  $\cosh x$  is an even function, thus the region is symmetric about  $x = 0 \Rightarrow \bar{x} = 0$ .

$$\bar{y} = \frac{1}{2} \frac{\int_{-1}^1 \cosh^2 x dx}{\int_{-1}^1 \cosh x dx} = \frac{1}{2} \frac{\frac{1}{2} (\cosh x \sinh x + 1) \Big|_{-1}^1}{\sinh x \Big|_{-1}^1} = \frac{1}{4} \frac{\cosh 1 \sinh 1 + 1}{\sinh 1} \text{ (where } \int \cosh^2 x = \frac{1}{2} \cosh x \sinh x + \frac{1}{2} x \text{)}$$

$$6d) a = -2, b = 2, f(x) = \sqrt{4 - x^2}, \text{ but } g(x) = \begin{cases} 0 & \text{for } -2 \leq x \leq -1 \\ \sqrt{1 - x^2} & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } 1 \leq x \leq 2 \end{cases}$$

The region is symmetric about  $x = 0 \Rightarrow \bar{x} = 0$ .

$\bar{y} = \frac{M_x}{m} = \frac{1}{2} \frac{\int_{-2}^{-1} (\sqrt{4-x^2})^2 dx + \int_{-1}^1 [(\sqrt{4-x^2})^2 - (\sqrt{1-x^2})^2] dx + \int_1^2 (\sqrt{4-x^2})^2 dx}{2\pi - \frac{1}{2}\pi} = \frac{1}{2} \frac{\int_{-2}^2 4-x^2 dx - \int_{-1}^1 1-x^2 dx}{2\pi - \frac{1}{2}\pi} = \frac{28}{9\pi}$  (since it is a circular region, the area  $m$  can be found directly.)

6e)  $a = 1, b = 2, g(x) = 0, f(x) = x^{-1}$   
 $\bar{x} = \frac{\int_1^2 x \cdot x^{-1} dx}{\int_1^2 x^{-1} dx} = \frac{1}{\ln 2} = \frac{1}{\ln 2}$  and  $\bar{y} = \frac{1}{2} \frac{\int_1^2 (x^{-1})^2 dx}{\int_1^2 x^{-1} dx} = \frac{1}{2} \frac{-x^{-1}|_1^2}{\ln 2} = \frac{1}{4 \ln 2}$

7a) The region is a half circle with radius  $r$ , the area is  $\frac{1}{2}\pi r^2$ .

The length the center of mass (of the region) traveled is  $2\pi\bar{y}$ .

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2} \int_{-r}^r (\sqrt{r^2-x^2})^2 dx}{\frac{1}{2}\pi r^2} = \frac{4r}{3\pi}$$

$$\text{Thus } V = A \cdot L = \frac{1}{2}\pi r^2 \cdot 2\pi \frac{4r}{3\pi} = \frac{4}{3}\pi r^3$$

7b) Let the axis (height) of the cone on  $y$  axis, the cone can be regards as a triangle (area  $A = \frac{1}{2}hr$ ) revolving around  $y$  axis.  $V = A \cdot L$  where  $L$  is the length that the center of mass  $(\bar{x}, \bar{y})$  of the triangle travels (circumference) around  $y$ -axis, ie,  $2\pi\bar{x}$  here.

The equation of the hypotenuse is  $y = f(x) = h - \frac{h}{r}x$

$$\bar{x} = \frac{\int_0^r x f(x) dx}{\int_0^r f(x) dx} = \frac{\int_0^r x(h - \frac{h}{r}x) dx}{\int_0^r (h - \frac{h}{r}x) dx} = \frac{\frac{1}{6}hr^2}{\frac{1}{2}hr} = \frac{1}{3}r \quad (\text{Note that the denominator is actually the area})$$

$$V = A \cdot L = \frac{1}{2}hr \cdot 2\pi \frac{1}{3}r = \frac{1}{3}\pi hr^2 \quad (\text{known result, } 1/3 \text{ volume of a cylinder}).$$

7c) The region is a circle, the center of mass is the center of the circle, from the center of the circle to the rotating axis is  $r$ , the length the center of mass traveled is  $2\pi r$ . The area of the circle is  $\pi r^2$ , the volume of the shape  $V = A \cdot L = 2\pi^2 r^3$ .

8. To show that  $f(x)$  is minimized when  $x = \bar{x}$ , we need to show that  $f'(\bar{x}) = 0, f''(\bar{x}) > 0$ .

$$\begin{aligned} f(x) = \sum_{i=1}^n m_i (x - x_i)^2 &\Rightarrow f'(x) = \sum_{i=1}^n 2m_i (x - x_i) = \sum_{i=1}^n 2m_i x - \sum_{i=1}^n 2m_i x_i = 2x \sum_{i=1}^n m_i - 2 \sum_{i=1}^n m_i x_i \\ &= 2xm - 2M, \text{ where } m = \sum_{i=1}^n m_i, M = \sum_{i=1}^n m_i x_i \end{aligned}$$

then  $f'(x) = 0 \Rightarrow 2xm - 2M = 0 \Rightarrow x = \frac{M}{m} = \bar{x}$  and  $f''(x) = 2m > 0$  for all  $x$

9.  $\text{prob}(50 \leq X \leq 60) = \text{prob}(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68 = 68\%$  (see page 37 of the lecture notes)

10a)  $\int_{-\infty}^{\infty} f(t) dt = \int_0^5 \frac{6}{625} t^2 (5-t)^2 dt = \frac{6}{625} \int_0^5 t^2 (25 - 10t + t^2) dt = \frac{6}{625} \int_0^5 (25t^2 - 10t^3 + t^4) dt = \frac{6}{625} \left( \frac{25}{3} t^3 - \frac{10}{4} t^4 + \frac{t^5}{5} \right) \Big|_0^5 = 1$  and  $f(t) \geq 0$ , thus  $f(t)$  defines a valid pdf

mean :  $\mu = \int_0^5 t f(t) dt = \frac{6}{625} \int_0^5 (25t^3 - 10t^4 + t^5) dt = \dots = \frac{5}{2}$

note :  $f(t)$  is symmetric about  $t = \frac{5}{2}$  (show it); this also implies that  $\mu = \frac{5}{2}$

10b)  $1000 \cdot \text{prob}(T \geq 3) = 1000 \cdot \int_3^5 f(t) dt = 1000 \cdot \int_3^5 \frac{6}{625} t^2 (5-t)^2 dt = \dots \approx 317$  batteries

11.  $f(t) = ce^{-ct}$  where  $c = \frac{1}{20}$ ,  $t$  is measured in minutes,  $\mu = \frac{1}{c}$ , thus  $c = \frac{1}{\mu}$ .

a)  $\text{prob}(X \leq 10) = \int_0^{10} \frac{1}{20} e^{-\frac{t}{20}} dt = 1 - \frac{1}{\sqrt{e}} \approx 0.375$

b)  $\text{prob}(X \geq 30) = \int_{30}^{\infty} \frac{1}{20} e^{-\frac{t}{20}} dt = e^{-\frac{3}{2}} \approx 0.24$

c) Solve  $\frac{1}{2} = \int_m^{\infty} f(t) dt \Rightarrow \frac{1}{2} = e^{-\frac{m}{20}} \Rightarrow m = 20 \ln 2 \approx 20 \times 0.7 = 14$  minutes. (The median  $m = \mu \ln 2 < \mu$  for exponential distribution.)

12.  $\text{prob}(\mu - 2\sigma < X < \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$  (set  $y = (x - \mu)/\sqrt{2}\sigma, dy = dx/\sqrt{2}\sigma$ )

$$= \frac{1}{\sqrt{\pi}} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{2}} e^{-y^2} dy = \operatorname{erf}(\sqrt{2}) = 0.9545 \approx 95\%$$

Hence for a normal distribution, 95% of the values are within two standard deviations of the mean.

13. The exponential pdf is  $f(t) = \begin{cases} ce^{-ct}, & t \geq 0 \\ 0, & t < 0 \end{cases}$  with mean  $\mu = 1/c$ .

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\ &= \int_0^{\infty} (x - \mu)^2 ce^{-cx} dx \quad (\text{set } u = (x - \mu)^2, dv = ce^{-cx} dx \Rightarrow du = 2(x - \mu)dx, v = -e^{-cx}) \\ &= -(x - \mu)^2 e^{-cx} \Big|_0^{\infty} + 2 \int_0^{\infty} (x - \mu) e^{-cx} dx \\ &= \mu^2 + 2 \int_0^{\infty} (x - \mu) e^{-cx} dx \quad (\text{set } u = x - \mu, dv = e^{-cx} dx \Rightarrow du = dx, v = -\frac{1}{c} e^{-cx}) \\ &= \mu^2 - 2 \frac{x - \mu}{c} e^{-cx} \Big|_0^{\infty} + \frac{2}{c} \int_0^{\infty} e^{-cx} dx = \mu^2 - \frac{2\mu}{c} - \frac{2}{c^2} e^{-cx} \Big|_0^{\infty} = \mu^2 - \frac{2\mu}{c} + \frac{2}{c^2} = \mu^2, \end{aligned}$$

where in the last step we substituted  $c = 1/\mu$ . Hence  $\sigma = \mu$ .

14. Functions (a), (b), (c), (e) are solutions of the differential equation; functions (d), (f) are not.

15. a)  $y = 1$  : unstable

15. b)  $y = -1$  : stable ,  $y = 1$  : unstable

15. c)  $y = 1$  : one side stable and one side unstable

15. d)  $y = 1$  : stable ,  $y = 2$  : unstable

15. e)  $y = 2k\pi$  ( $k$  is integer) : unstable ,  $y = (2k + 1)\pi$  : stable

16.  $E(t) = \frac{1}{2}mv^2 + V(x) \Rightarrow E(t) = \frac{1}{2}mx'(t)^2 + V(x(t))$

$$\Rightarrow E'(t) = \frac{1}{2}m \cdot 2x'(t)x''(t) + V'(x(t))x'(t) = x'(t)[mx''(t) + V'(x(t))] = x'(t)[f(x(t)) - f(x(t))] = 0$$

$$\Rightarrow E'(t) = 0 \Rightarrow E(t) = \text{constant}$$

Hence if the kinetic energy increases, then the potential energy decreases.

17a)  $y' = y \Rightarrow \frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt \Rightarrow \ln y = t + c \Rightarrow y = e^c \cdot e^t, e^c = 1$  using  $y(0) = 1$

The solution is  $y = y(0)e^t \Rightarrow y = e^t$ .

17b)  $y' = ty \Rightarrow \frac{dy}{dt} = ty \Rightarrow \frac{dy}{y} = tdt \Rightarrow \ln y = \frac{1}{2}t^2 + c \Rightarrow y = e^c \cdot e^{\frac{t^2}{2}}, e^c = 1$  using  $y(0) = 1$

The solution is  $y = y(0)e^{\frac{t^2}{2}} \Rightarrow y = e^{\frac{t^2}{2}}$ .

17c)  $y' = y^2 \Rightarrow \frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow -\frac{1}{y} = t + c \Rightarrow y = \frac{1}{-t-c}, c = -1$  using  $y(0) = 1$

The solution is  $y = \frac{1}{1-t}$ . The graph has a vertical asymptote at  $t = 1$ .

17d)  $y' = y(1 - y) \Rightarrow \frac{dy}{dt} = y(1 - y) \Rightarrow \frac{dy}{y(1-y)} = dt \Rightarrow \frac{dy}{y} + \frac{dy}{1-y} = dt \Rightarrow \ln y - \ln(1 - y) = t + c \Rightarrow \ln \frac{y}{(1-y)} = t + c \Rightarrow \frac{y}{(1-y)} = e^t \cdot e^c \dots$

One can find the general solution  $y(t)$ , but the initial condition  $y(0) = 1$  is a constant solution, so the solution will stay at  $y = 1$  for all  $t \geq 0$ , so the solution in this case is  $y(t) = 1$ , and the general solution is not needed.

18.  $y(t)$  = amount of salt (kg) in the tank at time  $t$  (min)

$y'(t)$  = rate of change of amount = (rate coming in) - (rate going out)

$$\text{rate coming in} = 0.05 \frac{\text{kg}}{\text{L}} \cdot 5 \frac{\text{L}}{\text{min}} = 0.25 \frac{\text{kg}}{\text{min}}, \quad \text{rate going out} = \frac{y \text{ kg}}{1000 \text{ L}} \cdot 5 \frac{\text{L}}{\text{min}} = 0.005y \frac{\text{kg}}{\text{min}}$$

$$y' = 0.25 - 0.005y \Rightarrow y' = -0.005(y - 50)$$

Note that we have written the equation in the form  $y' = k(y - T)$ , which is Newton's law of cooling/heating, with  $k = -0.005$ ,  $T = 50$ . The solution is  $y(t) = T + (y_0 - T)e^{kt} = 50 + (0 - 50)e^{-0.005t} = 50(1 - e^{-0.005t})$ , using the initial condition  $y(0) = y_0 = 0$ , since the tank initially contains pure water.

a) After one hour = 60 minutes we have  $y(60) = 50(1 - e^{-0.005 \cdot 60}) = 50(1 - e^{-0.3}) \approx 13$  kg.

b)  $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} 50(1 - e^{-0.005t}) = 50$  kg

Note that this is the quantity of salt present in the tank if the tank contained only sea water.

19a)  $x(t)$  = value of new-type bills (in units of \$1B) in circulation at time  $t$  (days)

$x'(t)$  = rate of change of new-type bills in circulation = (rate going in) - (rate going out)

rate going in =  $50 \frac{\text{M}}{\text{day}} = 0.05 \frac{\text{B}}{\text{day}}$ , rate going out =  $\frac{x}{10} \cdot 0.05 \frac{\text{B}}{\text{day}} \Rightarrow x' = 0.05 - 0.005x$

19b) The equation can be written as  $x' = -0.005(x - 10)$ , which is the same as Newton's law of cooling/heating,  $x' = k(x - T)$ , with  $k = -0.005$  and  $T = 10$ . Hence  $x(t) = T + (x_0 - T)e^{-kt} = 10 + (0 - 10)e^{-0.005t} = 10(1 - e^{-0.005t})$ , where we used the initial condition  $x_0 = x(0) = 0$ .

19c) 90% means  $x(t) = 9 \Rightarrow 9 = 10(1 - e^{-0.005t}) \Rightarrow \dots \Rightarrow t = 200 \ln 10 \approx 200 \cdot 2.3 = 460$  days

20.  $c' = k(a - c)(b - c)$ , where  $k > 0$ ,  $a = 1$ ,  $b = 2 \Rightarrow c' = k(1 - c)(2 - c)$ , it has two constant solutions  $c = 1$  and  $c = 2$  (ie., let  $c' = 0$ , solve the equation),  $c = 1$  is stable while  $c = 2$  is unstable, since for  $0 \leq c < 1$ ,  $c' > 0$ ,  $c(t)$  increases (towards  $c = 1$ ), for  $1 < c < 2$ ,  $c' < 0$ ,  $c(t)$  decreases (towards  $c = 1$ ), for  $c > 2$ ,  $c' > 0$ ,  $c(t)$  increases (away from  $c = 2$ ).

To find the product concentration, we solve  $c' = k(1 - c)(2 - c)$  by separation of variables,  $\frac{dc}{(1-c)(2-c)} = kdt$ . Using partial fractions, we write  $\frac{1}{(1-c)(2-c)} = \frac{1}{1-c} - \frac{1}{2-c}$ , and integrate to obtain  $\ln \left| \frac{2-c}{1-c} \right| = kt + d$ , or  $\frac{2-c}{1-c} = De^{kt}$ , for some constant  $D$  (using the initial condition  $c(0) = 0$ , we find this constant to be  $D = 2$ ). We now solve the equation for  $c(t)$ , writing  $2 - c = (1 - c)De^{kt}$ . Rearranging terms and simplifying, we get  $c(t) = 1 - \frac{1}{2e^{kt} - 1}$ .

Given that  $c(0) = 0$  at  $t = 0$ , the asymptotic value of  $c(t)$  is  $c = 1$  as  $t \rightarrow \infty$ . The asymptotic value will not change if  $c(0) = 1.5$  mole/L. This can also be seen using the phase plane.

21.  $mc_p T' = -e\sigma T^4 \Rightarrow \frac{T'}{T^4} = -\frac{e\sigma}{mc_p} \Rightarrow \int \frac{T'}{T^4} dt = \int -\frac{e\sigma}{mc_p} dt \Rightarrow -\frac{1}{3T^3} = -\frac{e\sigma}{mc_p} t + c$

initial condition :  $T(0) = T_0 \Rightarrow c = -\frac{1}{3T_0^3} \Rightarrow -\frac{1}{3T^3} = -\frac{e\sigma}{mc_p} t - \frac{1}{3T_0^3} \Rightarrow T(t) = \left( 3\frac{e\sigma}{mc_p} t + \frac{1}{T_0^3} \right)^{-1/3}$

In the limit  $t \rightarrow \infty$ ,  $T(t) \sim t^{-1/3}$ , so  $\alpha = \frac{1}{3}$ .

22a) The decay rate of radioactive sample satisfies  $y' = -ky$ . By separation of variables, we find the solution  $y(t) = y_0 e^{-kt}$  where  $y_0$  is initial mass at  $t = 0$ . It is given that  $y(2) = 128$

after 2 hours and  $y(5) = 2$  after 5 hours,  $\Rightarrow \begin{cases} 128 = y_0 e^{-2k} \\ 2 = y_0 e^{-5k} \end{cases}$ . Dividing the first equation by

the second equation yields  $64 = e^{3k} \Rightarrow e^k = 64^{1/3} = 4$ . Substituting in the first equation gives  $128 = y_0 4^{-2} \Rightarrow y_0 = 128 \cdot 16 = 2048$ .

22b) Let  $y(t) = 1$  kg. Then  $1 = 2048 \cdot 4^{-t} \Rightarrow 4^t = 2048 \Rightarrow t = \frac{\ln 2048}{\ln 4} = \frac{\ln 2^{11}}{\ln 2^2} = \frac{11}{2} \Rightarrow t = 5.5$  hours.

23. The initial value of the investment is  $x = \$500$  and the annual interest rate is  $r = 12\% = 0.12$ .

a) If the interest is compounded annually, then the value of the investment after 10 years is  $x(1 + r)^n = 500(1.12)^{10} \approx \$1552.92$ .

b) If the interest is compounded continuously, then the value of the investment after 10 years is  $xe^{rt} = 500e^{0.12(10)} \approx \$1660.06$ .

c) The equivalent annual interest rate  $r_{eq}$  is found by solving the equation  $e^r = 1 + r_{eq}$ . Since  $e^r = e^{0.12} \approx 1.127 = 1 + 0.127$ , the equivalent annual interest rate is  $r_{eq} = 0.127$ . In other words, 12% compounded continuously is the same as 12.7% compounded annually.

24. Newton's Law of cooling /heating implies  $y(t) = T + (y_0 - T)e^{-kt}$ , where  $y_0 = 21$  at  $t = 0$ .

We know  $y(1) = 27$  and  $y(2) = 30$ , thus we have 
$$\begin{cases} 27 = T + (21 - T)e^{-k}, \\ 30 = T + (21 - T)e^{-2k}. \end{cases}$$

Combining these two equations, we find  $T = 33$ .

25a) The correct equation is (iii)  $m' = r - km$ .

25b)  $r - km = 0 \Rightarrow m = \frac{r}{k} = M$  (phase plane, done in class, stable)

25c) The equation in a) can be rewritten as  $m' = -k(m - \frac{r}{k})$ . This is the same as Newton's Law of cooling/heating. The solution is  $m(t) = \frac{r}{k} + (m_0 - \frac{r}{k})e^{-kt}$ .

The endowment dropped to  $\frac{1}{2}M$  last year, so we take  $m_0 = \frac{1}{2}M$ . The solution is

$$m(t) = \frac{r}{k} + (\frac{1}{2}M - \frac{r}{k})e^{-kt} = M - \frac{1}{2}Me^{-kt}.$$

To find how long it will take to recover to  $\frac{3}{4}M$ , we set

$$m(t) = \frac{3}{4}M \Rightarrow M - \frac{1}{2}Me^{-kt} = \frac{3}{4}M \Rightarrow -\frac{1}{2}Me^{-kt} = -\frac{1}{4}M \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow t = -\frac{1}{k} \ln \frac{1}{2} = \frac{1}{k} \ln 2.$$

For  $k = \frac{1}{20} \text{ year}^{-1}$  and  $\ln 2 \approx 0.7$ , we have  $t \approx 14$  years.

26. Newton's law of cooling/heating :  $T' = k(T - T_a) \Rightarrow T(t) = T_a + (T(0) - T_a)e^{kt}$ ,  $k < 0$

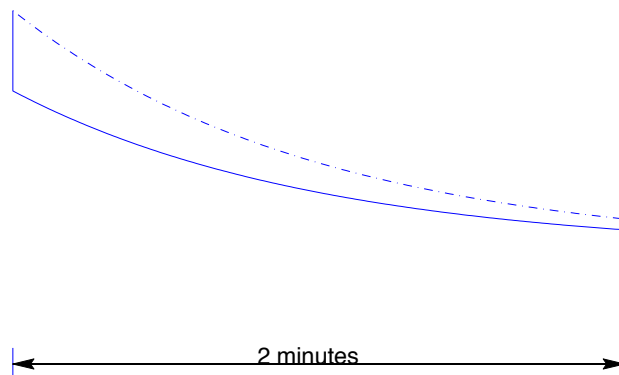
Bob : adds milk, then waits 2 mins , Ray : waits 2 mins, then adds milk

initial condition :  $T_{Bob}(0) = \frac{8T_h + T_c}{9}$ ,  $T_{Ray}(0) = T_h$

$$\begin{aligned} T_{Bob} - T_{Ray} &= \left( T_a + \left( \frac{8T_h + T_c}{9} - T_a \right) e^{2k} \right) - \left( \frac{8(T_a + (T_h - T_a)e^{2k}) + T_c}{9} \right) \\ &= T_a - \frac{8}{9}T_a - \frac{1}{9}T_c + \left( \frac{8T_h + T_c}{9} - T_a - \frac{8}{9}(T_h - T_a) \right) e^{2k} \\ &= \frac{1}{9}(T_a - T_c) + \frac{1}{9}(T_c - T_a)e^{2k} = \frac{1}{9}(T_a - T_c)(1 - e^{2k}) > 0 \Rightarrow T_{Bob} > T_{Ray} \end{aligned}$$

$\text{---} T_{Ray}$   
 $\text{---} T_{Bob}$

So Ray ends up with the colder coffee. This can be understood as follows. According to Newton's law, the larger the temperature difference between the object and its surroundings, the faster the temperature changes. Since Bob's strategy reduced the initial temperature difference, he reduced the temperature rate of change, so his coffee cools more slowly during the 2 minutes. See the plots of  $T_{Bob}(t), T_{Ray}(t)$ .



27. Let  $y(t)$  be the fraction of the population who have already heard the rumor at time  $t$ . Then  $1 - y(t)$  is the fraction of the population who have not already heard the rumor at time  $t$ .

The rate of change of the fraction of the population who have heard the rumor is  $y' = ky(1 - y)$ , for some  $k > 0$ . This makes sense because if  $y = 0$  or  $y = 1$ , then  $y' = 0$ .

This is a logistic equation with general solution  $y(t) = \frac{y_0}{y_0 + (1 - y_0)e^{-kt}}$ . Set  $t = 0$  at 8am, then  $y_0 = \frac{10}{1000} = 0.01$ . Then at 9am,  $t = 1$  hour,  $y(1) = \frac{20}{1000} = 0.02$ . Substituting these into the solution yields  $0.02 = \frac{0.01}{0.01 + 0.99e^{-k}} \Rightarrow 0.01 + 0.99e^{-k} = \frac{1}{2} \Rightarrow e^{-k} = \frac{49}{99}$ . The solution becomes  $y(t) = \frac{0.01}{0.01 + 0.99(\frac{49}{99})^t}$ ; note that  $e^{-kt} = (e^{-k})^t$ . Then  $y(t) = \frac{1}{2} \Rightarrow \frac{0.01}{0.01 + 0.99(\frac{49}{99})^t} = \frac{1}{2} \Rightarrow 0.01 + 0.99(\frac{49}{99})^t = 0.02 \Rightarrow (\frac{49}{99})^t = \frac{1}{99} \Rightarrow t = \ln(99)/(\ln(99) - \ln(49)) = 6.5337$ , i.e. around 2:30pm.

$$28a) 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - (\frac{1}{2})} = 2$$

$$28b) 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \dots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 - (-\frac{1}{2})} = \frac{2}{3}$$

$$28c) 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots = (1 + \frac{1}{2} - \frac{1}{4}) + (\frac{1}{8} + \frac{1}{16} - \frac{1}{32}) + (\frac{1}{64} + \frac{1}{128} - \frac{1}{256}) + \dots$$

$$= \frac{5}{4} + \frac{5}{32} + \frac{5}{256} + \dots = \frac{5}{4}(1 + \frac{1}{8} + \frac{1}{64} + \dots) = \frac{5}{4} \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n = \frac{5}{4} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{5}{4} \cdot \frac{8}{7} = \frac{10}{7}$$

29. (a)  $\sum_{n=0}^{\infty} (2x - 1)^n$  is a geometric series with  $r = 2x - 1$ ; hence the series converges if and only if  $-1 < r < 1 \Leftrightarrow -1 < 2x - 1 < 1 \Leftrightarrow 0 < 2x < 2 \Leftrightarrow 0 < x < 1$ .

(b) If  $0 < x < 1$ , then we have  $\sum_{n=0}^{\infty} (2x - 1)^n = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots = \frac{1}{1 - r} = \frac{1}{1 - (2x - 1)} = \frac{1}{2 - 2x}$ .

30. Determine whether the series converges or diverges. Justify your answer.

a)  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  : diverges,  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ , a necessary condition for convergence is  $\lim_{n \rightarrow \infty} a_n = 0$

b)  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$  : diverges,  $\lim_{n \rightarrow \infty} a_n$  does not exist, diverges for same reason as (a)

c)  $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$  : diverges by integral test,  $\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{x}{x^2+1} dx = \frac{1}{2} \ln(x^2 + 1) \Big|_1^{\infty} = \infty - \frac{1}{2} \ln 2 = \infty$

d)  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$  : converges by alternating series test,  $a_n = \frac{n}{n^2+1}$ , (1)  $a_n > 0$ , (2)  $\{a_n\}$  is decreasing because  $a_n = f(n)$ , and  $f(x) = \frac{x}{x^2+1}$  is decreasing for  $x \geq 1$  because  $f'(x) = \frac{(x^2+1) \cdot 1 - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$  for  $x \geq 1$ , (3)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0$

e)  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$  : converges by comparison test,  $\frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} \Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  which converges by the  $p$ -test with  $p = \frac{3}{2}$

f)  $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n^2+1}$  : converges by alternating series test,  $a_n = \frac{\sqrt{n}}{n^2+1}$ , (1)  $a_n > 0$ , (2)  $\{a_n\}$  is decreasing because  $a_n = f(n)$ , and  $f(x) = \frac{\sqrt{x}}{x^2+1}$  is decreasing for  $x \geq 1$  because  $f'(x) = \frac{(x^2+1) \cdot \frac{1}{2}x^{-1/2} - \sqrt{x} \cdot 2x}{(x^2+1)^2} = \frac{(x^2+1) \cdot \frac{1}{2} - x \cdot 2x}{(x^2+1)^2} = \frac{\frac{1}{2} - x^2}{\sqrt{x}(x^2+1)^2} \leq 0$  for  $x \geq 1$ , (3)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n^2+1} = 0$

31. For each series, if the sum  $s$  is approximated by the  $n$ th partial sum  $s_n$ , how large must  $n$  be to ensure the error  $|s - s_n|$  is less than 0.02?

a)  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  :  $a_n = f(n)$ , where  $f(x) = \frac{1}{x^2}$  is positive and decreasing, so by a result derived in class we have  $0 < s - s_n < \int_n^{\infty} f(x) dx = \int_1^{\infty} \frac{dx}{x^2} = -\frac{1}{x} \Big|_n^{\infty} = \frac{1}{n}$ , so we take  $\frac{1}{n} = 0.02 = \frac{1}{50}$  or  $n = 50$



b)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  : alternating series test applies, so by a result derived in class we have  $|s - s_n| < a_{n+1}$ , so we take  $a_{n+1} = \frac{1}{(n+1)^2} = 0.02 = \frac{1}{50}$  or  $(n+1)^2 = 50 \Rightarrow n+1 \approx \sqrt{50} \approx 7$ , so we can take  $n = 7$

32a)  $\int \cosh 2x dx = \frac{1}{2} \sinh 2x$

32b)  $\int \tanh 2x dx = \frac{1}{2} \int \tanh(2x) d(2x) = \frac{1}{2} \int \frac{\sinh 2x}{\cosh 2x} d(2x) = \frac{1}{2} \int \frac{1}{\cosh 2x} d \cosh 2x = \frac{1}{2} \ln(\cosh 2x)$

32c)  $\int \cosh^2 x dx = \int \left(\frac{e^x + e^{-x}}{2}\right)^2 dx = \int \frac{e^{2x} + 2 + e^{-2x}}{4} dx = \frac{1}{4} \left(\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x}\right) = \frac{1}{4} \sinh 2x + \frac{1}{2} x$

33.  $f(x) = \cosh x, f'(x) = \sinh x, 1 + [f'(x)]^2 = 1 + \sinh^2 x = \cosh^2 x$

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx = \int_{-1}^1 2\pi \cosh^2 x dx = 2\pi \int_{-1}^1 \left(\frac{e^x + e^{-x}}{2}\right)^2 dx = 2\pi \int_{-1}^1 \frac{e^{2x} + 2 + e^{-2x}}{4} dx = \frac{\pi}{2} \left(\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x}\right) \Big|_{-1}^1 = \frac{\pi}{2} (\sinh 2x + 2x) \Big|_{-1}^1 = \frac{\pi}{2} (2 \sinh 2 + 4) = \pi(\sinh 2 + 2)$$

34. recall :  $\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}$

a) LHS =  $\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2}$

$$\begin{aligned} \text{RHS} &= \sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\ &= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4} = \frac{2e^{x+y} - 2e^{-x-y}}{4} = \text{LHS} \end{aligned}$$

b) One can prove it by the definitions of sinh, cosh, as in (a), or by using  $\sinh(-y) = -\sinh y, \cosh(-y) = \cosh y$  (i.e. sinh is an odd function, cosh is an even function) and based on a):

$$\sinh(x-y) = \sinh[x+(-y)] \stackrel{a)}{=} \sinh(x) \cosh(-y) + \cosh(x) \sinh(-y) = \sinh x \cosh y - \cosh x \sinh y$$

Note : For parts c) and d), one can prove them by the definitions of sinh, cosh similar to a), or by using the relationships  $(\sinh x)' = \cosh x, (\cosh x)' = \sinh x$  and the identity a) (see below).

c) Taking derivative for both sides of a) with respect to  $x$ , regarding  $y$  as a constant, we have:

$$\begin{aligned} \cosh(x+y) &= [\sinh(x+y)]' = [\sinh x \cosh y + \cosh x \sinh y]' = (\sinh x)' \cosh y + (\cosh x)' \sinh y \\ &= \cosh x \cosh y + \sinh x \sinh y \end{aligned}$$

d) Differentiating both sides of b) with respect to  $x$ , regarding  $y$  as constant yields the result.

35. a)  $\lim_{x \rightarrow \pm\infty} \operatorname{sech} x = 0$     b)  $\frac{d}{dx} \operatorname{sech} x = \frac{d}{dx} \frac{1}{\cosh x} = \frac{-\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$     c) like a bell curve

36. a)  $f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2}, f''(x) = 2x^{-3}$

$$a = 1 \Rightarrow f(1) = 1, f'(1) = -1, f''(1) = 2$$

$$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - (x-1) = 2 - x$$

b)  $f(x) = \sin x \Rightarrow f'(x) = \cos x, f''(x) = -\sin x$

$$a = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi}{2}\right) = 1, f'\left(\frac{\pi}{2}\right) = 0, f''\left(\frac{\pi}{2}\right) = -1$$

$$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - 0(x - \frac{\pi}{2}) = 1$$

c)  $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}, f''(x) = -\frac{1}{4\sqrt{x}^3}$

$$a = 4 \Rightarrow f(4) = \sqrt{4} = 2, f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}, f''(4) = -\frac{1}{4\sqrt{4}^3} = -\frac{1}{32}$$

$$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 2 + \frac{1}{4}(x-4)$$

approximation of  $\sqrt{5} = 2.2361$  (viewing  $\sqrt{5}$  as the value of  $\sqrt{x}$  at  $x = 5$ )

$$T_1(5) = 2 + \frac{1}{4}(5-4) = \frac{9}{4} = 2.25, \text{ the error is } |f(5) - T_1(5)| = |2.2361 - 2.25| = 0.0139$$

37. Let  $L = \lim_{n \rightarrow \infty} n \left( \left(1 + \frac{1}{n}\right)^n - e \right)$ . This limit  $L$  has the form  $\infty \cdot 0$ , so to apply l'Hôpital's rule we move  $n$  to the denominator,  $L = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n - e}{\frac{1}{n}}$ . Now we have a limit of the type  $\frac{0}{0}$ , for which we can apply l'Hôpital's rule. However it is more convenient to substitute  $u = \frac{1}{n}$ .

$$L = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n - e}{\frac{1}{n}} = \lim_{u \rightarrow 0} \frac{(1+u)^{1/u} - e}{u}$$

Now differentiate the numerator, but note that  $(1+u)^{1/u} = e^{\ln((1+u)^{1/u})} = e^{\frac{\ln(1+u)}{u}}$ .

$$\frac{d}{du} (1+u)^{1/u} = e^{\frac{\ln(1+u)}{u}} \cdot \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2}$$

$$\begin{aligned} \Rightarrow L &= \lim_{u \rightarrow 0} \frac{(1+u)^{1/u} - e}{u} = \lim_{u \rightarrow 0} e^{\frac{\ln(1+u)}{u}} \cdot \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2} = \lim_{u \rightarrow 0} e^{\frac{\ln(1+u)}{u}} \cdot \lim_{u \rightarrow 0} \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2} \\ &= \lim_{u \rightarrow 0} (1+u)^{1/u} \cdot \lim_{u \rightarrow 0} \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \cdot \lim_{u \rightarrow 0} \frac{u - (1+u) \ln(1+u)}{(1+u)u^2} \\ &= e \cdot \lim_{u \rightarrow 0} \frac{1 - \left[\frac{1+u}{1+u} + \ln(1+u)\right]}{(1+u) \cdot 2u + u^2} = e \cdot \lim_{u \rightarrow 0} \frac{-\ln(1+u)}{2u + 3u^2} = e \cdot \lim_{u \rightarrow 0} \frac{-\frac{1}{1+u}}{2 + 6u} = e \cdot -\frac{1}{2} = -\frac{e}{2} \quad \text{ok} \end{aligned}$$

This confirms a result about Euler's method in the Table on page 49 of the lecture notes.

38. The alternating series test says that if  $\{a_n\}$  satisfies (1)  $a_n > 0$ , (2)  $a_{n+1} < a_n$ , (3)  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\sum_{n=0}^{\infty} (-1)^n a_n$  converges. Show by example that if conditions (1) and (3) hold, but (2) doesn't, then the series may converge or diverge.

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 example 1 :  $\frac{1}{2} - 1 + \frac{1}{8} - \frac{1}{4} + \frac{1}{32} - \frac{1}{16} + \dots = -\frac{2}{3}$

Conditions (1) and (3) hold, but (2) doesn't hold; to show the series converges, see below.

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots = \frac{1}{1 - \frac{1}{2}} = \frac{2}{3}, \text{ geometric series, } r = -\frac{1}{2}$$

The example series is obtained by switching the order of each consecutive pair of terms in the geometric series; the partial sums are just the negative of what they were before, so the series with switched order terms converges to the negative of the sum of the geometric series.

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 example 2. :  $2 - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{6} + \frac{2}{7} - \frac{1}{8} + \dots$  : diverges

Conditions (1) and (3) hold, but (2) doesn't hold; to show the series diverges, see below.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots : \text{converges (alternating harmonic series, AST)}$$

$$1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + 0 + \frac{1}{7} + 0 + \dots : \text{diverges (integral test, } a_n = \frac{1}{2n-1}, f(x) = \frac{1}{2x-1})$$

Adding the series term by term gives the series in the example and shows that it diverges.