Math 156 Applied Honors Calculus II Fall 2018

hw7 , due: Tuesday, October 30

1. Sketch the region $R = \{(x,y) : 1 \le x \le 2, 0 \le y \le 1/x\}$, find the center of mass, and indicate it on the sketch.

2. Consider the region between the curves $y = x^m$ and $y = x^n$, for $0 \le x \le 1$, where m, n are integers with $0 \le m < n$. (a) Sketch the region and label the curves for general m and n. (b) Find the center of mass of the region in terms of m and n. (c) Consider the case $n = m+1$. Make a table with the following format; column 1: m, column 2: n, column 3: $\bar{x}$, column 4: $\bar{y}$, column 5: $\bar{x}^m$, column 6: $\bar{x}^n$, column 7: CM lies in region? (yes or no). Take $m = 0, 1, 2, 3$. Show that the CM lies inside the region for $m = 0, 1, 2$ and outside the region for $m = 3$.

3. In class we considered the waiting time in the supermarket checkout line as a random variable T with exponential pdf $f(t)$. Let the average waiting time be $\mu = 5$ minutes and show that the median waiting time is $m = 3.5$ minutes. Explain how it is possible for the average waiting time to be longer than the median waiting time.

4. Evaluate the limit. $\lim\limits_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$

5. The circle $x^2 + (y-a)^2 = r^2$ is rotated about the x-axis. Assume $a > r$, so the resulting shape is a torus. Use the theorem of Pappus to find the volume of the torus.

6. The pdf of a normal distribution is $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, where μ, $\sigma > 0$ are constants.
Show that (a) $\int_{-\infty}^{\infty} f(x)dx = 1$, (b) $\int_{-\infty}^{\infty} x f(x)dx = \mu$, (c) $\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \sigma^2$.
This verifies that μ is the mean and σ is the standard deviation of the given pdf $f(x)$.
(hint: substitute $t = (x-\mu)/\sqrt{2}\sigma$; you may use the fact that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$)

7. Find the antiderivative.
a) $\int \sinh x\, dx$ b) $\int \cosh x\, dx$ c) $\int \tanh x\, dx$ d) $\int \operatorname{sech} x\, dx$, where $\operatorname{sech} x = \frac{1}{\cosh x}$

8. Find the solution $y(t)$ of the differential equation $y' = -y$ satisfying the initial condition $y(0) = c$ for three cases, $c = 1, 2, -1$. Sketch the solutions on the same graph for $t \ge 0$.

9. A bacteria culture starts with 500 cells and grows at a rate proportional to its size. After 3 hours there are 800 cells. a) Find an expression for the number of cells after t hours. b) Find the number of cells after 6 hours. c) When will the cell count reach 2048? Solve this problem two ways, with and without using a calculator.

10. When a function value $f(x)$ is difficult to compute, in some cases it can be approximated by another function value $f(a)$ which is easier to compute, where a is close to x. For example, we can approximate $e^{0.1}$ by $e^0 = 1$ or $\sqrt{10}$ by $\sqrt{9} = 3$. This is an example of Taylor approximation. The error is $|f(x) - f(a)|$ and its size can be analyzed as follows. Using the FTC in the form, $f(x) - f(a) = \int_a^x f'(t)dt$, it follows that

$$|f(x) - f(a)| \le \left|\int_a^x f'(t)dt\right| \le \int_a^x |f'(t)|dt \le \int_a^x M_1 dt \le M_1|x-a|, \tag{1}$$

where $M_1 = \max|f'(t)|$ and t lies between a and x. Equation (1) is called an error bound.
a) Let $f(x) = e^x, a = 0$. Sketch $y = f(x)$ and $y = f(a)$ on the same graph around $x = a$.
b) Make a table with the following format. column 1: $|x-a|$, column 2: $|f(x) - f(a)|$. Take $f(x) = e^x, a = 0$ and fill in the entries for $x = 1, 1/2, 1/4, 1/8$ using a calculator. When $|x-a|$ is reduced by a factor of $\frac{1}{2}$, by what factor is the error $|f(x) - f(a)|$ reduced?