1. The age of organic remains can be determined by **radiocarbon dating**. The principle is as follows. When cosmic rays impact the atmosphere, they convert nitrogen to a radioactive isotope of carbon, $^{14}$C, with a half-life of 5730 years. Living plants and animals continually absorb $^{14}$C through the atmosphere and food chain, and when they die the amount of $^{14}$C present in the remains decreases by radioactive decay. Consider the case of parchment, a thin material made from animal skins used for writing in ancient times. A parchment fragment was discovered having 74% as much $^{14}$C as in current living organic material. Find the age of the fragment.

2. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$. In class we showed that the series converges (by the comparison test), but we didn’t find the sum. To find the sum, first sketch the curves $y = x^n$ for $0 \leq x \leq 1$ and $n = 0, 1, 2, 3, 4$ on a common plot. Let $A_n$ be the area between $y = x^n$ and $y = x^{n+1}$ for any $n \geq 0$, and note the relation between these areas and the area of the unit square. Compute $A_n$ in terms of $n$ and then deduce the sum of the series.

3. Determine whether the series converges or diverges. Justify your answer.
   a) $\sum_{n=1}^{\infty} \frac{1}{3n+1}$  
   b) $\sum_{n=1}^{\infty} \frac{1}{n!}$  
   c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$  
   d) $\sum_{n=1}^{\infty} \frac{1}{n^3}$  
   e) $\sum_{n=1}^{\infty} \frac{n^2}{n^3}$

4. Find the interval of convergence of the power series.  
   a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$  
   b) $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$

5. Consider the power series $\sum_{n=0}^{\infty} (x-4)^n$. Find the values of $x$ for which the series converges and find the sum of the series for those values of $x$.

6. Three sequences of tangent circles approach the vertices of an equilateral triangle. Assume the triangle has sides of length 1. What fraction of the triangle area is occupied by the circles?

7. In each series, how many terms are needed to ensure that the error in the $n$th partial sum, $|s - s_n|$, is less than $10^{-6}$? Use the error bounds derived in class.
   a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$  
   b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

8. This problem continues the discussion of Taylor approximation.
   a) Recall from hw9: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{(x-t)^2}{2} f'''(t) \, dt$.
   b) Define $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$. Note that $T_3(x)$ is a cubic function of $x$; it is called the **Taylor polynomial of degree 3** for $f(x)$ at $x=a$. Show that $T_3(x)$ and $f(x)$ have the same function value, 1st derivative value, 2nd derivative value, and 3rd derivative value at $x=a$.
   c) We view $T_3(x)$ as a **cubic approximation** to $f(x)$. Show that the error satisfies $|f(x) - T_3(x)| \leq \frac{1}{4!} M_4 |x-a|^4$, where $M_4 = \max |f^{(4)}(t)|$. (hint: part (a) implies that $f(x) = T_3(x) + \int_a^x \frac{(x-t)^3}{3!} f^{(4)}(t) \, dt$)
   d) Let $f(x) = e^x$, $a = 0$. Find $T_3(x)$. Sketch $f(x)$ and $T_3(x)$ on the same graph around $x = a$.
   e) Make a table with the following format. column 1: $|x-a|$, column 2: $|f(x) - T_3(x)|$. Take $f(x) = e^x$, $a = 0$ and fill in the entries for $x = 0.2, 0.1, 0.05$ using a calculator. When $|x-a|$ is reduced by a factor of $\frac{1}{2}$, by what factor is the error $|f(x) - T_3(x)|$ reduced?