1 Tues 9/6

preview : integrals, differential equations, series

1st goal : prepare for FTC

Appendix E : sigma notation

def : $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$: sum , series

i : index , a_i : terms , $1, n$: limits

ex

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$$

note : A series can be written in different ways.

ex

$$\sum_{i=1}^{5} i = \sum_{j=0}^{4} (j+1) = 1 + 2 + 3 + 4 + 5 = 15$$

set $j = i - 1$

thm

1. $\sum_{i=1}^{n} ca_i = c \cdot \sum_{i=1}^{n} a_i$, where c is any constant

2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

3. $\sum_{i=1}^{n} a_i b_i \neq \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i$

pf

1. $\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c \cdot \sum_{i=1}^{n} a_i$ ok

2. hw

3. counterexample , $n = 2$

$$\sum_{i=1}^{2} a_i b_i = a_1 b_1 + a_2 b_2$$

$$\sum_{i=1}^{2} a_i \cdot \sum_{i=1}^{2} b_i = (a_1 + a_2) \cdot (b_1 + b_2) = a_1 b_1 + \underline{a_1 b_2 + a_2 b_1} + a_2 b_2$$ ok

<u>thm</u>

$$\sum_{i=1}^{n}(a_{i+1}-a_i) \;=\; a_{n+1}-a_1 \quad : \quad \underline{\text{telescoping series}}$$

<u>pf</u>

$$\sum_{i=1}^{n}(a_{i+1}-a_i)$$

$$= (\cancel{a_2}-a_1) + (\cancel{a_3}-\cancel{a_2}) + (\cancel{a_4}-\cancel{a_3}) + \cdots + (\cancel{a_n}-\cancel{a_{n-1}}) + (a_{n+1}-\cancel{a_n})$$

$$= -a_1 + a_{n+1} \qquad \underline{\text{ok}}$$

<u>thm</u>

1. $\displaystyle\sum_{i=1}^{n} 1 \;=\; 1+1+1+\cdots+1 \;=\; n$

2. $\displaystyle\sum_{i=1}^{n} i \;=\; 1+2+3+\cdots+n \;=\; \frac{n(n+1)}{2}$ (e.g. $n=5 \overset{\text{implies}}{\Rightarrow} S=15$)

3. $\displaystyle\sum_{i=1}^{n} i^2 \;=\; 1+4+9+\cdots+n^2 \;=\; \frac{n(n+1)(2n+1)}{6}$ (e.g. $n=5 \Rightarrow S=55$)

<u>pf</u>

1. <u>ok</u>

2. $(i+1)^2-i^2 \;=\; i^2+2i+1-i^2 \;=\; 2i+1$

$$\sum_{i=1}^{n}((i+1)^2-i^2) \;=\; \sum_{i=1}^{n}(2i+1) \;=\; 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \;=\; \boxed{2S+n} \quad , \text{ where } S=\sum_{i=1}^{n} i$$

$$\sum_{i=1}^{n}((i+1)^2-i^2) \;=\; (2^2-1^2)+(3^2-2^2)+(4^2-3^2)+\cdots+((n+1)^2-n^2)$$

$$= (n+1)^2 - 1^2$$

$$= \boxed{n^2+2n}$$

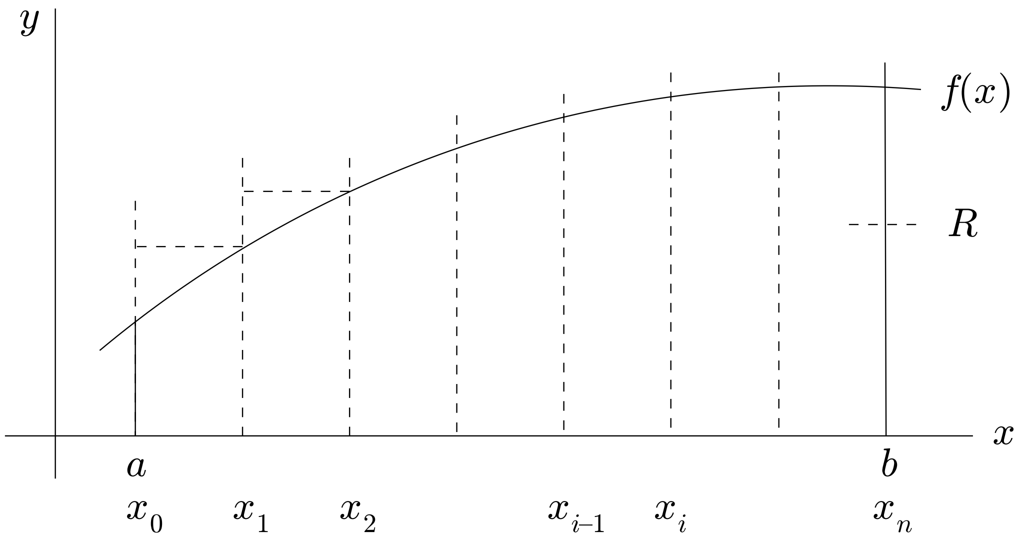
$$\Rightarrow\; 2S+n \;=\; n^2+2n \;\Rightarrow\; 2S \;=\; n^2+n \;=\; n(n+1) \;\Rightarrow\; S=\frac{n(n+1)}{2} \qquad \underline{\text{ok}}$$

3. hw , $(i+1)^3-i^3 \;=\; \cdots$

2 Wed 9/7

5.1 area

Given $f(x) \geq 0$, $a \leq x \leq b$.



R = region in the xy-plane between $y = 0$ and $y = f(x)$ for $a \leq x \leq b$

$= \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$

problem : find the area of R

solution : approximate by rectangles

choose $n \geq 1$, set $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $i = 0, \ldots, n$

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$\ldots$

$x_n = a + n\Delta x = a + b - a = b$

area of ith rectangle $= f(x_i)\Delta x$

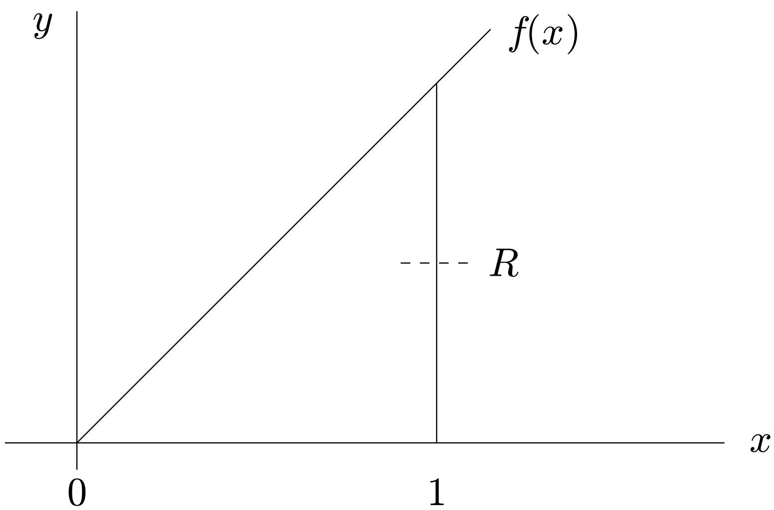
area of region $R \approx \sum_{i=1}^{n} f(x_i)\Delta x$

↑ approximately

area of region $R = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

<u>ex</u>

$f(x) = x \ , \ 0 \le x \le 1$



$R = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$

of course , area $= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2}$

$a = 0 \ , \ b = 1 \ , \ \Delta x = \frac{b-a}{n} = \frac{1}{n}$

$x_i = a + i\Delta x = \frac{i}{n}$

$f(x_i) = x_i = \frac{i}{n}$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}$$ <u>ok</u>

5.2 definite integral

As before, given $f(x)$, $a \le x \le b$, set $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $i = 0, \ldots, n$.
Let x_i^* be any point such that $x_{i-1} \le x_i^* \le x_i$.

Then $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is a Riemann sum.

ex

$x_i^* = x_i$: right-hand RS

$x_i^* = x_{i-1}$: left-hand RS

$x_i^* = \dfrac{x_{i-1}+x_i}{2}$: midpoint RS

$$\text{def}: \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \; : \; \text{definite integral} = \begin{cases} \text{area, volume} \\ \text{work} \\ \text{probability} \\ \ldots \end{cases}$$

ex

$$\int_0^1 dx = \lim_{n\to\infty} \sum_{i=1}^{n} 1 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} 1 = \lim_{n\to\infty} \frac{1}{n} \cdot n = \lim_{n\to\infty} 1 = 1$$

$$a = 0 \; , \; b = 1 \; , \; \Delta x = \frac{b-a}{n} = \frac{1}{n} \; , \; x_i = a + i\Delta x = \frac{i}{n} \; , \; f(x) = 1$$

$$\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2}$$

$$f(x) = x \; , \; f(x_i) = x_i = \frac{i}{n}$$

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}$$

$$f(x) = x^2 \; , \; f(x_i) = x_i^2 = \left(\frac{i}{n}\right)^2$$

$$\int_0^1 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n} \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(e^{1/n}\right)^i \cdot \frac{1}{n} = \cdots = e - 1$$

$f(x) = e^x$, $f(x_i) = e^{x_i} = e^{i/n}$

↑ geometric series (hw2)

3 Fri 9/9

def : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$: right-hand RS , $\lim_{n\to\infty} R_n = \int_a^b f(x)\,dx$

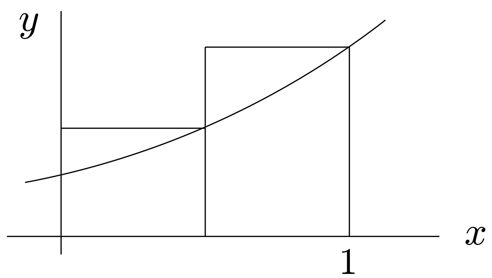
$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$: midpoint RS , $\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx$

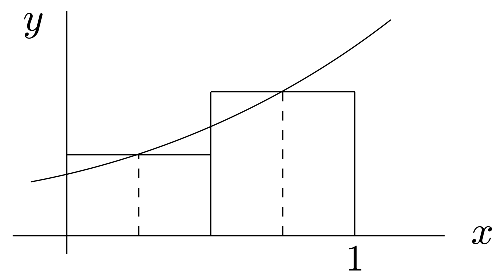
question : For a given value of n, which approximation is more accurate?

ex : $\int_0^1 e^x dx = e - 1 = 1.71828183 = I$

n	Δx	R_n	$\lvert I - R_n\rvert$	M_n	$\lvert I - M_n\rvert$
1	1	2.7183	1.0000	1.6487	0.0696
2	0.5	2.1835	0.4652	1.7005	0.0178
4	0.25	1.9420	0.2237	1.7138	0.0045

Hence the midpoint RS is more accurate than the right-hand RS. Why?





note : If Δx decreases by a factor of $\frac{1}{2}$, then the error in R_n decreases by a factor of approximately $\frac{1}{2}$ and the error in M_n decreases by a factor of approximately $\frac{1}{4}$.

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properties of the definite integral

1. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
2. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. If $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

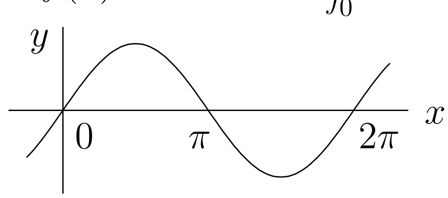
pf

1. $\int_a^b cf(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} cf(x_i)\Delta x = c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = c\int_a^b f(x)\,dx$ ok

2. hw2 , 3. , 4. omit

note : If $f(x)$ changes sign, then $\int_a^b f(x)\,dx$ = signed area.

ex : $f(x) = \sin x \Rightarrow \int_0^{\pi} \sin x\,dx > 0$, $\int_{\pi}^{2\pi} \sin x\,dx < 0$, $\int_0^{2\pi} \sin x\,dx = 0$



group work : $R = \{(x,y) : 1 \le x \le 2, 0 \le y \le x\}$, plot R , find area by RS

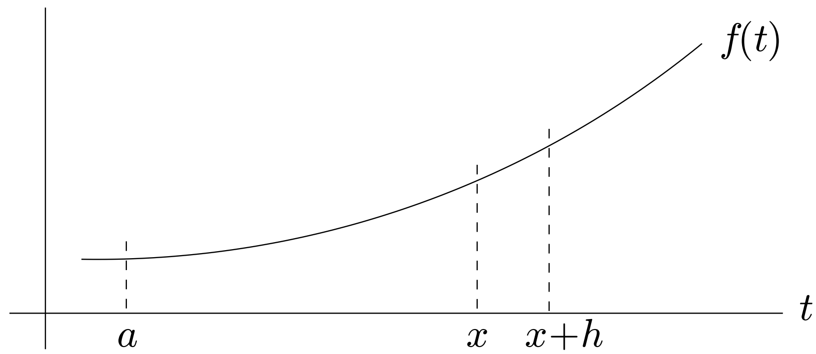
4
Mon
9/12

<u>5.3 FTC , 5.4 antiderivatives</u>

<u>thm</u> (FTC, part 1)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

<u>pf</u>



define : $F(x) = \int_a^x f(t)\,dt$, we need to show that $F'(x) = f(x)$

recall : $F'(x) = \lim\limits_{h\to 0} \dfrac{F(x+h) - F(x)}{h}$

$$F(x+h) = \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt = F(x) + \int_x^{x+h} f(t)\,dt$$

$$\Rightarrow F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt \approx f(t^*)\cdot h \ , \text{ where } x \le t^* \le x+h$$

$$\Rightarrow \frac{F(x+h)-F(x)}{h} \approx f(t^*)$$

$$\Rightarrow F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(t^*) = f(x)$$ <u>ok</u> (if f is continuous)

<u>ex</u>

FTC $\Rightarrow \dfrac{d}{dx}\displaystyle\int_0^x t\,dt = x$

<u>check</u>

$$\int_0^x t\,dt = \lim_{n\to\infty}\sum_{i=1}^n t_i\Delta t = \lim_{n\to\infty}\sum_{i=1}^n \frac{ix}{n}\cdot\frac{x}{n} = \lim_{n\to\infty}\frac{x^2}{n^2}\sum_{i=1}^n i = \lim_{n\to\infty}\frac{x^2}{n^2}\cdot\frac{n(n+1)}{2} = \frac{x^2}{2}$$

$$a = 0\ ,\ b = x\ ,\ \Delta t = \frac{b-a}{n} = \frac{x}{n}\ ,\ t_i = a + i\Delta t = \frac{ix}{n}$$

$$\Rightarrow \frac{d}{dx}\int_0^x t\,dt = \frac{d}{dx}\left(\frac{x^2}{2}\right) = x$$ <u>ok</u>

<u>thm</u> (FTC, part 2)

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

<u>pf</u>

$$\Delta x = \frac{b-a}{n}\ ,\ x_i = a + i\Delta x$$

note : $x_i + \Delta x = a + i\Delta x + \Delta x = a + (i+1)\Delta x = x_{i+1}$

$$\sum_{i=1}^{n} f'(x_i)\Delta x \approx \sum_{i=1}^{n}\left(\frac{f(x_i+\Delta x) - f(x_i)}{\cancel{\Delta x}}\right)\cancel{\Delta x}$$

$$= \sum_{i=1}^{n}(f(x_{i+1}) - f(x_i)) = f(x_{n+1}) - f(x_1) \quad : \text{ telescoping sum}$$

$x_1 = a + \Delta x$

$x_{n+1} = a + (n+1)\Delta x = a + n\Delta x + \Delta x = a + b - a + \Delta x = b + \Delta x$

$$\Rightarrow \int_a^b f'(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f'(x_i)\Delta x = \lim_{n\to\infty}(f(b+\Delta x) - f(a+\Delta x))$$

$$= f(b) - f(a) \quad \underline{\text{ok}} \ \ (\text{if } f \text{ is continuous})$$

<u>note</u>

1. We write $\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$.

2. Suppose we're given an integral of the form $\int_a^b f(x)\,dx$. In order to apply the FTC, we need to find a function $F(x)$ such that $F'(x) = f(x)$, because then we have $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$. In this case we say that $F(x)$ is an <u>antiderivative</u> of $f(x)$ and we also write $\int f(x)\,dx = F(x) + C$.

<u>ex</u>

5 Tues 9/13

1. recall : $\int_0^1 x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\,\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{1}{n^2}\,\frac{n(n+1)}{2} = \frac{1}{2}$.

Now we can use the FTC : $\int_0^1 x\,dx = \int_0^1\left(\frac{x^2}{2}\right)' dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$.

2. $\int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}$

$f(x)$	$F(x)$
x^n	$\dfrac{x^{n+1}}{n+1}$, $n \neq -1$
x^{-1}	$\ln x$
$\ln x$	$x \ln x - x$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\cosh x$ } more later
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\sinh x$ } more later
$\dfrac{1}{x^2+1}$	$\tan^{-1} x$
e^{x^2}	$\int_a^x e^{t^2}\, dt$

<u>final comment on FTC</u>

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$$

<u>pf</u>

Set $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$ and $F(g(x)) = \int_a^{g(x)} f(t)dt$.

$$\Rightarrow \frac{d}{dx}\int_a^{g(x)} f(t)dt = \frac{d}{dx}F(g(x)) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x) \quad \underline{\text{ok}}$$

<u>ex</u>

$$\frac{d}{dx}\int_a^{x^2} e^t dt = e^{x^2} \cdot 2x$$

<u>check</u>

$$\int_a^{x^2} e^t dt = e^t \Big|_a^{x^2} = e^{x^2} - e^a \Rightarrow \frac{d}{dx}\int_a^{x^2} e^t dt = \frac{d}{dx}\left(e^{x^2} - e^a\right) = e^{x^2} \cdot 2x \quad \underline{\text{ok}}$$

6. applications of integration

6.1 area

6.2, 6.3 volume

6.4 work ←

6.5 average value of a function

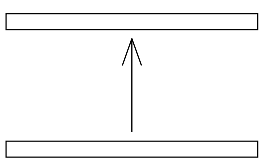
6.4 work

force = mass × acceleration , work = force × distance

units	metric	British
mass	kilogram: kg	slug : slug
distance	meter : m	foot : ft
time	second : s	second : s
force	Newton : N = kg · $\frac{\text{m}}{\text{s}^2}$	pound : lb
work	Joule : J = N·m	foot-pound : ft-lb

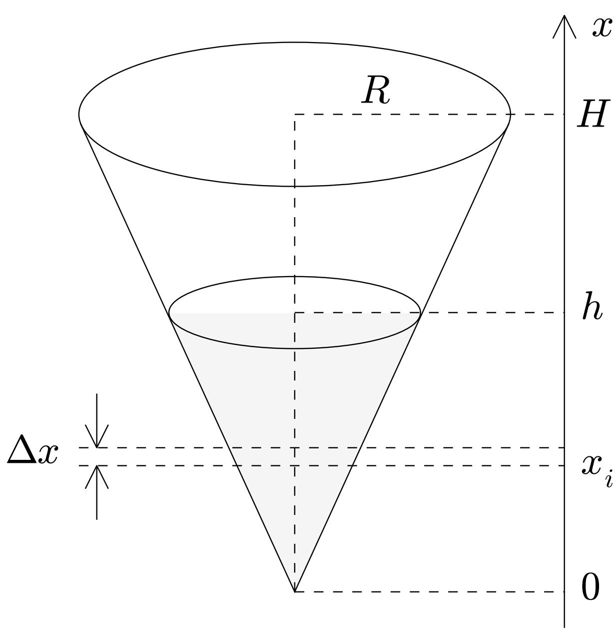
conversion : 1 m = 3.28 ft , 1 N = 0.225 lb ⇒ 1 J = 0.738 ft-lb

ex : Find the work done in lifting a 1 kg book to a height of 1 m off the ground.



$W = \text{force} \times \text{distance} = \text{m}\,\text{g} \times \text{d} = 1\,\text{kg} \times 9.8\,\frac{\text{m}}{\text{s}^2} \times 1\,\text{m} = 9.8\,\text{J}$

ex : A water tank has the shape of an inverted cone.



R : base radius of cone

H : height of cone

h : water level

problem : Find the work done in pumping the water out the top of the tank.

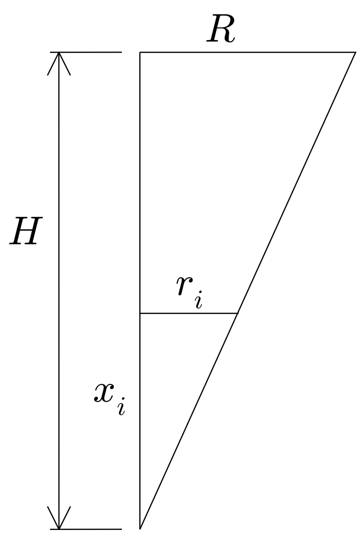
strategy : Think of the water volume as a stack of books.

6
Wed
9/14

$\Delta x = \dfrac{h}{n}$: width of a water layer

$x_i = i\Delta x$: height of ith layer ($x_0 = 0$, $x_n = h$)

$r_i = \ ?$: radius of ith layer



$$\frac{R}{H} = \frac{r_i}{x_i} \ \Rightarrow\ r_i = x_i \frac{R}{H}$$

check : $x_i = 0 \ \Rightarrow\ r_i = 0$

$x_i = H \ \Rightarrow\ r_i = R$ ok

work = force × distance = mass × acceleration × distance

mass of water in ith layer = density × volume $\approx \rho \cdot \pi r_i^2 \Delta x$

force acting on ith layer $= \rho\pi r_i^2 \Delta x \cdot g$

work done in raising ith layer $= \rho g \pi r_i^2 \Delta x \cdot (H - x_i)$

work done in raising entire water volume

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \rho g \pi r_i^2 (H - x_i)\Delta x = \int_0^h \rho g \pi x^2 \frac{R^2}{H^2}(H - x)dx$$

$$\int_0^h x^2(H-x)dx = \int_0^h (x^2H - x^3)dx = \left(\frac{x^3}{3}H - \frac{x^4}{4}\right)\Big|_0^h = \frac{h^3}{3}H - \frac{h^4}{4} = \frac{h^3}{12}(4H - 3h)$$

$W = \rho g \pi \dfrac{R^2}{H^2} \cdot \dfrac{h^3}{12}(4H - 3h)$, check : $h = 0 \Rightarrow W = 0$ ok

plug in numbers

$R = 4\,\mathrm{m}$, $H = 10\,\mathrm{m}$, $h = 8\,\mathrm{m}$, $\rho = 1000\,\mathrm{kg/m^3}$, $g = 9.8\,\mathrm{m/s^2}$

$$W = 10^3\,(9.8)\,(3.14)\,\frac{16}{10^2} \cdot \frac{8^3}{12}\,(40 - 24)\,\frac{\mathrm{kg}}{\cancel{\mathrm{m}}^3}\,\frac{\mathrm{m}}{\mathrm{s}^2}\,\frac{\cancel{\mathrm{m}}^2}{\cancel{\mathrm{m}}^2}\,\cancel{\mathrm{m}}^3 \cdot \mathrm{m}$$

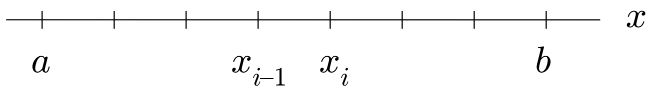
$$\approx 10 \cdot 10 \cdot \cancel{3} \cdot \frac{4}{\cancel{3}} \cdot 5 \cdot 10^2 \cdot 16\,\mathrm{J} \approx 3.2 \cdot 10^6\,\mathrm{J}$$

<u>note</u>

In the previous examples, the force (due to gravity) is assumed to be constant as the mass is raised (book, water layer), but in general the force may depend on the displacement of the mass.

<u>ex</u>

Compute the work done in moving an object from $x = a$ to $x = b$, subject to a force $f(x)$.



, $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$

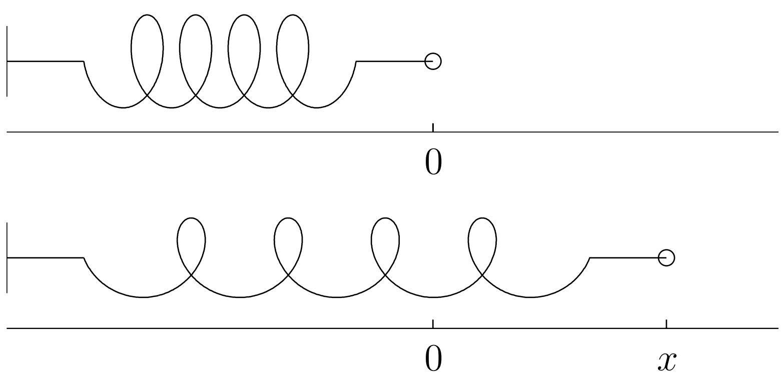
work done in moving object from x_{i-1} to $x_i \approx f(x_i)\Delta x$

. . . . " . . . . from a to $b \approx \sum_{i=1}^{n} f(x_i)\Delta x$

$$\Rightarrow W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

<u>ex</u> 7 Fri 9/16

A spring is stretched x units from its natural length.



<u>Hooke's law</u> : The force needed to maintain a spring in its stretched state is proportional to the displacement x from its natural length.

force $= f(x) = kx$, $k > 0$: <u>spring constant</u>

<u>ex</u>

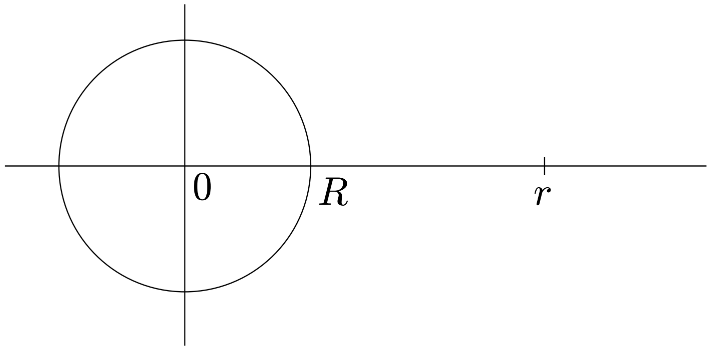
A spring has natural length 10 cm and a force of 40 N is needed to maintain the spring when it is stretched to a length of 15 cm. Find the work done in stretching the spring from 15 cm to 20 cm.

$$40\,\text{N} = k \cdot (15\,\text{cm} - 10\,\text{cm}) = k \cdot 5\,\text{cm} \;\Rightarrow\; k = 8\,\text{N/cm}$$

$$W = \int_a^b f(x)dx = \int_5^{10} kx\,dx = k\frac{x^2}{2}\bigg|_5^{10} = 8\left(\frac{100}{2} - \frac{25}{2}\right) = 300\frac{\text{N}}{\text{cm}}\,\text{cm}^2 = 3\,\text{N}\,\text{m} = 3\,\text{J}$$

<u>ex</u>

Find the work done in moving a particle from the Earth's surface to ∞.



R : radius of Earth

r : distance from particle to center of Earth , $R \le r < \infty$

$f(r) = \dfrac{GMm}{r^2}$: force on particle due to gravity (Newton)

G : gravitational constant

M : mass of Earth

m : mass of particle

$$W = \int_R^\infty f(r)dr = GMm \int_R^\infty \frac{dr}{r^2} = GMm \cdot \frac{-1}{r}\Big|_R^\infty = GMm \cdot \left(0 - \frac{-1}{R}\right) = \frac{GMm}{R}$$

$\uparrow$ <u>improper integral</u> , more later

note : We can compute the <u>escape velocity</u> of the particle.

initial kinetic energy - final ~~kinetic energy~~ (= 0) = work

$$\frac{mv_{\text{esc}}^2}{2} = \frac{GMm}{R} \;\Rightarrow\; v_{\text{esc}} = \left(\frac{2GM}{R}\right)^{1/2}$$

$$v_{\text{esc}} = \left(2 \cdot \frac{6.67 \cdot 10^{-11}\text{N} \cdot \text{m}^2/\text{kg}^2 \times 5.98 \cdot 10^{24}\text{kg}}{6.37 \cdot 10^6\,\text{m}}\right)^{1/2} \approx 11\frac{\text{km}}{\text{s}} \approx 30 \cdot \text{sound speed (air)}$$

note : black hole $\Rightarrow M \to \infty , R \to 0 \Rightarrow v_{\text{esc}} \to \infty$: impossible (Einstein)

$\Rightarrow$ nothing can escape the gravitational field of a black hole

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8.8 improper integrals

def

$\int_a^b f(x)\,dx$ is a proper integral if $\begin{cases} (a,b) \text{ is a bounded interval} \\ \text{and} \\ f(x) \text{ is a bounded function for } a \le x \le b \end{cases}$

otherwise, $\int_a^b f(x)\,dx$ is an improper integral, i.e. if $\begin{cases} a = -\infty \text{ or } b = \infty \\ \text{or} \\ f(x) \to \pm\infty \text{ in } (a,b) \end{cases}$

ex

$\int_0^1 x\,dx$: proper

$\int_1^\infty x\,dx$: improper (because $b = \infty$)

$\int_1^\infty \frac{dx}{x}$: improper

$\int_0^1 \frac{dx}{x}$: improper (because $\frac{1}{x} \to \infty$ as $x \to 0$)

$\int_1^2 \frac{dx}{x}$: proper

note : An improper integral is evaluated by taking a limit of proper integrals. If the limit is finite, the integral converges; otherwise, it diverges.

ex : $\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1$: converges

$\Rightarrow$ the area under the graph of $y = \frac{1}{x^2}$ from $x = 1$ to $x = \infty$ is finite

short cut : $\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty} + 1 = 1$

---

ex : $\int_1^\infty \frac{dx}{x} = \ln x\Big|_1^\infty = \ln\infty - \ln 1 = \infty$: diverges

$\Rightarrow$ the area under the graph of $y = \frac{1}{x}$ from $x = 1$ to $x = \infty$ is infinite

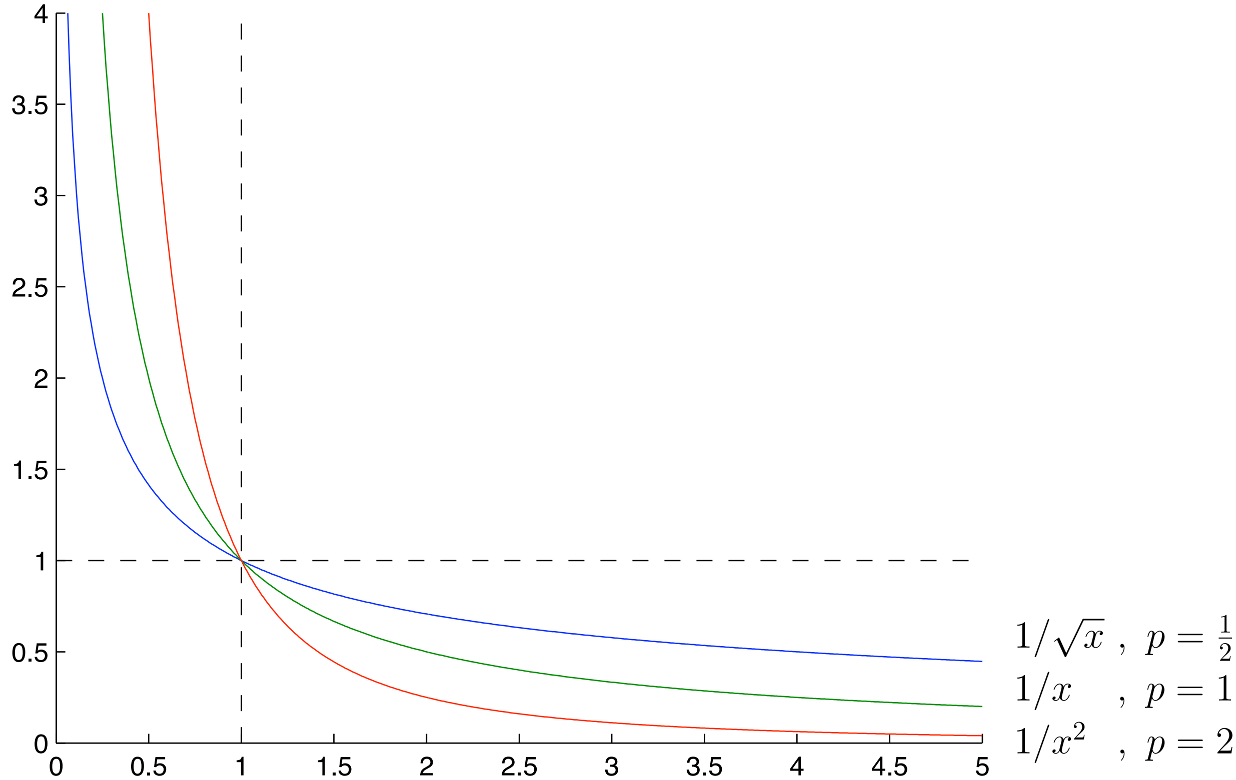
---

ex : $\int_1^\infty \frac{dx}{\sqrt{x}}$: diverges

pf 1 : $\int_1^\infty \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_1^\infty = 2\sqrt{\infty} - 2 = \infty$: diverges

pf 2 : $\frac{1}{x} \le \frac{1}{\sqrt{x}}$ for $x \ge 1 \Rightarrow \int_1^\infty \frac{dx}{x} \le \int_1^\infty \frac{dx}{\sqrt{x}}$: diverges (comparison test ...)

comparison of $y = \frac{1}{x^2}$, $\frac{1}{x}$, $\frac{1}{\sqrt{x}}$, general form : $\frac{1}{x^p}$



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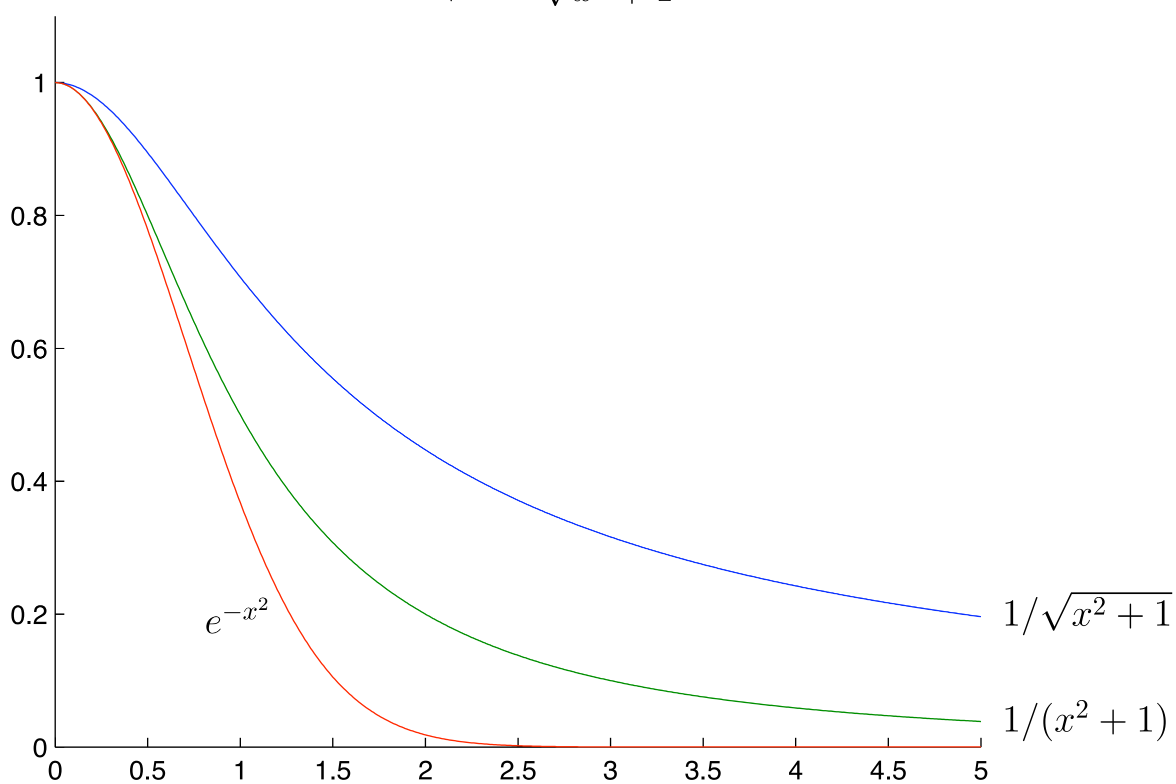
comparison of $y = e^{-x^2}$, $\frac{1}{x^2+1}$, $\frac{1}{\sqrt{x^2+1}}$



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<u>note</u> : So far we considered $\int_1^\infty f(x)\,dx$ where $f(x) \to 0$ as $x \to \infty$.

Now consider $\int_0^1 f(x)\,dx$ where $f(x) \to \infty$ as $x \to 0$.

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2 - 0 = 2 \ : \ \text{converges} \ , \ p = \frac{1}{2}$$

$$\int_0^1 \frac{dx}{x} = \ln x\Big|_0^1 = \ln 1 - \ln 0 = 0 - (-\infty) = \infty \ : \ \text{diverges} \ , \ p = 1$$

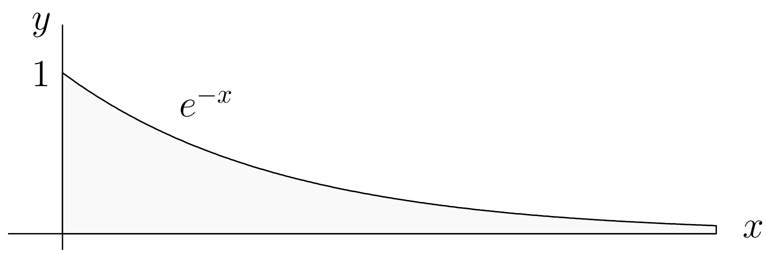
$$\int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^1 = -\frac{1}{1} - \Big(-\frac{1}{0}\Big) = -1 + \infty = \infty \ : \ \text{diverges} \ , \ p = 2$$

<u>note</u> : The comparison test can also be used in these examples.

<u>summary</u> (p-test)

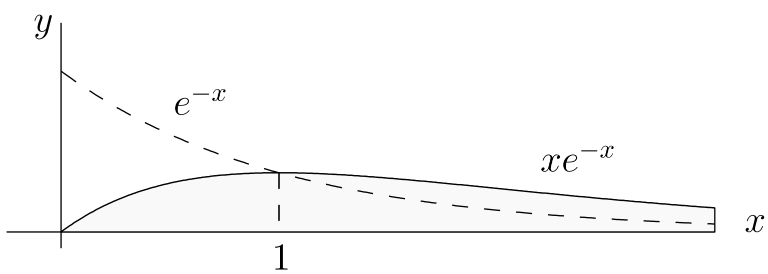
$$\int_1^\infty \frac{dx}{x^p} : \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases} , \quad \int_0^1 \frac{dx}{x^p} : \begin{cases} \text{converges if } p < 1 \\ \text{diverges if } p \ge 1 \end{cases} , \quad \underline{\text{pf}}: \text{ omit}$$

<u>ex</u> : $\int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -e^{-\infty} - (-e^0) = 0 - (-1) = 1$: converges



<u>ex</u> : $\int_0^\infty x e^{-x} dx = 1$: converges

$$\lim_{x\to\infty} x e^{-x} = \infty \cdot 0 = \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \quad \text{(l'Hopital's rule, pf later)}$$



$\int_0^\infty x e^{-x} dx = \ ?$, integration by parts : $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$, <u>pf</u>: soon

choose : $u = x$, $dv = e^{-x} dx \ \Rightarrow \ du = dx$, $v = -e^{-x}$

$$\Rightarrow \int_0^\infty x e^{-x} dx = \cancel{-xe^{-x}}\Big|_0^\infty + \int_0^\infty e^{-x} dx = 0 + 1 = 1$$

<u>question</u> : What happens if we choose $u = e^{-x}$, $dv = x\,dx$? ...

<u>integration by parts</u>

$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x)$

$(uv)' = uv' + u'v$

$\Rightarrow \int_a^b (uv)'dx \;=\; \int_a^b uv'dx \;+\; \int_a^b u'vdx$

$\Rightarrow \int_a^b uv'dx \;=\; uv\Big|_a^b - \int_a^b vu'dx$

$v'dx = \dfrac{dv}{dx} \cdot dx = dv \;,\; u'dx = \dfrac{du}{dx} \cdot dx = du$

$\Rightarrow \int_a^b udv \;=\; uv\Big|_a^b - \int_a^b vdu$ <u>ok</u>

<u>ex</u>

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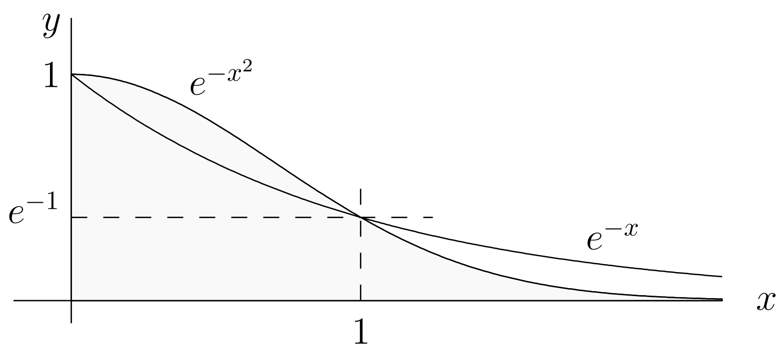
$\int_0^\infty e^{-x^2}dx$: converges



The antiderivative is not an elementary function, and integration by parts doesn't help in this case (try it!), so we will use a different approach.

$\int_0^\infty e^{-x^2}dx \;=\; \int_0^1 e^{-x^2}dx \;+\; \int_1^\infty e^{-x^2}dx$

$\int_0^1 e^{-x^2}dx$: converges (proper integral)

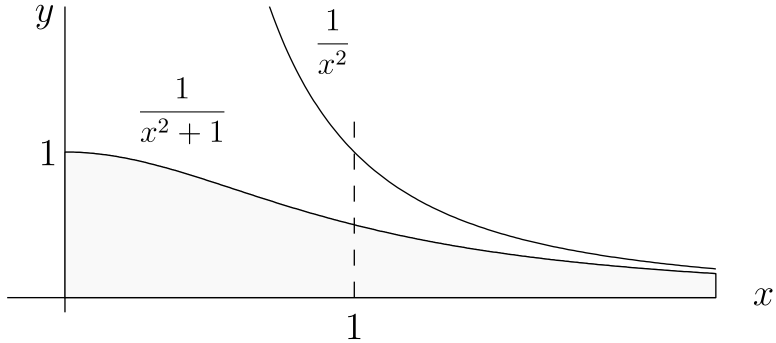
$x \geq 1 \;\Rightarrow\; x^2 \geq x \;\Rightarrow\; -x^2 \leq -x \;\Rightarrow\; e^{-x^2} \leq e^{-x}$ for $x \geq 1$

(this is because e^x is an <u>increasing</u> function, i.e. $a \leq b \Rightarrow e^a \leq e^b$)

$\Rightarrow \int_1^\infty e^{-x^2}dx \leq \int_1^\infty e^{-x}dx$: converges (comparison test) <u>ok</u>

note : $\int_0^\infty e^{-x^2}dx \;=\; \dfrac{\sqrt{\pi}}{2}$: Math 255 (multivariable calculus) ex

$$\int_0^\infty \frac{dx}{x^2+1} \;:\; \text{converges}$$



$$\text{note}: \; \frac{1}{x^2+1} \le \frac{1}{x^2} \Rightarrow \int_0^\infty \frac{dx}{x^2+1} \le \int_0^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^\infty = -\frac{1}{\infty} + \frac{1}{0} = \infty \;:\; \text{diverges}$$

Hence the comparison test yields no information, but $\dfrac{1}{x^2+1} \sim \dfrac{1}{x^2}$ for $x \to \infty$, so we expect that the given integral converges.

$\uparrow$ asymptotic

$$\int_0^\infty \frac{dx}{x^2+1} = \int_0^1 \frac{dx}{x^2+1} + \int_1^\infty \frac{dx}{x^2+1}$$

$\uparrow$ proper

$\uparrow$ converges because $\displaystyle\int_1^\infty \frac{dx}{x^2+1} \le \int_1^\infty \frac{dx}{x^2}$: converges ok

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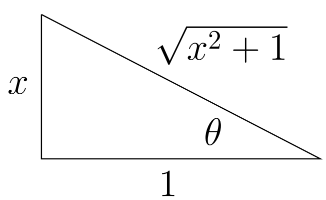
alternative : $\displaystyle\int \frac{dx}{x^2+1} = \tan^{-1}x = \arctan x$

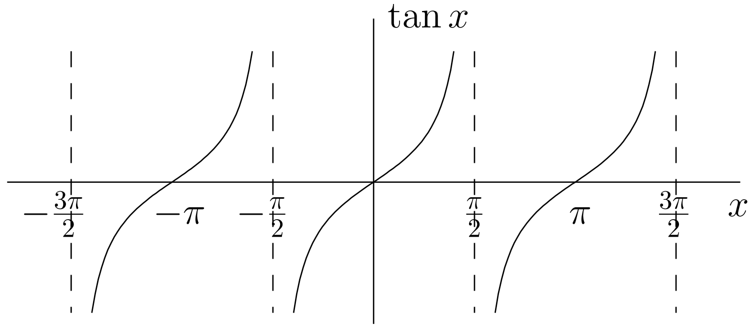


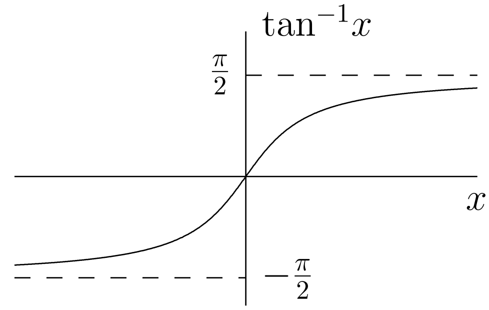
$\tan\theta = x$: trigonometric substitution

$\sec^2\theta \, d\theta = dx$

$\sec\theta = \sqrt{x^2+1}$

$$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta \, d\theta}{\sec^2\theta} = \int d\theta = \theta = \tan^{-1}x \quad \text{ok}$$





$$\int_0^\infty \frac{dx}{x^2+1} = \tan^{-1}x\Big|_0^\infty = \tan^{-1}\infty - \tan^{-1}0 = \frac{\pi}{2} - 0 = \frac{\pi}{2} \;:\; \text{converges} \quad \text{ok}$$

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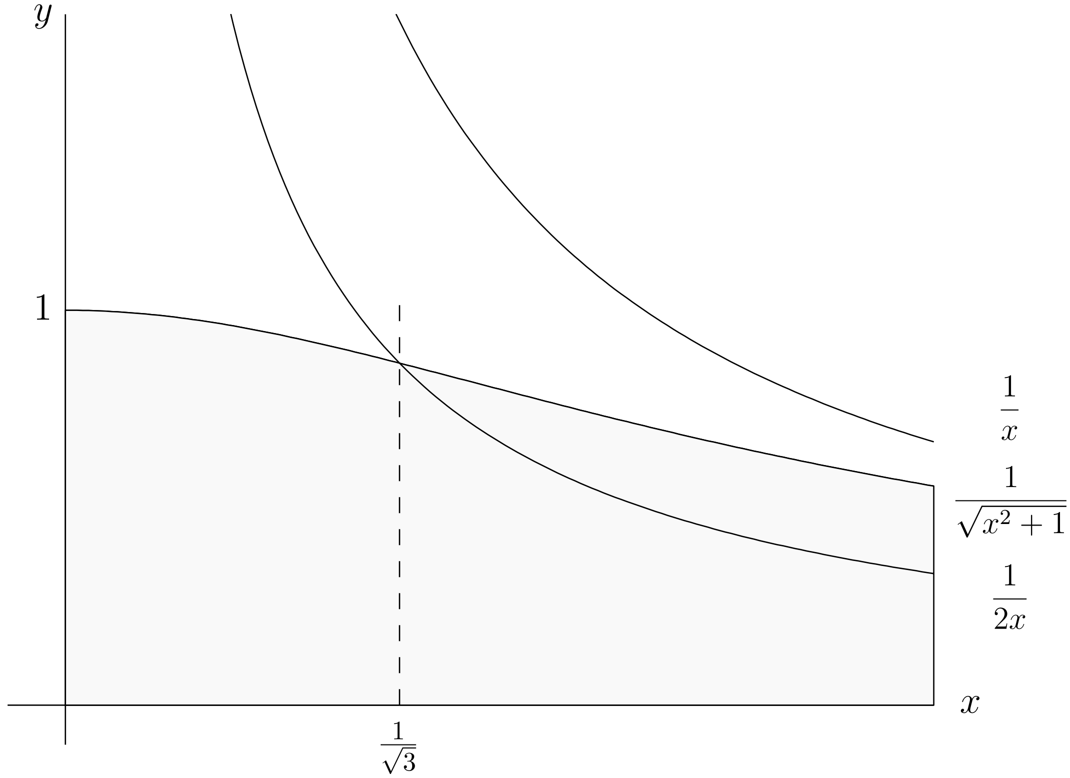
ex

$$\int_0^\infty \frac{dx}{\sqrt{x^2+1}} \ : \ \text{diverges}$$



idea : $\dfrac{1}{\sqrt{x^2+1}} \sim \dfrac{1}{x}$ for $x \to \infty$, so we expect that the integral diverges

Let's make this more rigorous.

$$\frac{1}{\sqrt{x^2+1}} \le \frac{1}{x} \ \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^2+1}} \le \int_0^\infty \frac{dx}{x} = \ln x \Big|_0^\infty = \ln\infty - \ln 0 = \infty \ : \ \text{diverges}$$

But this yields no information about the given integral; instead we need a reverse inequality and there is a trick for that.

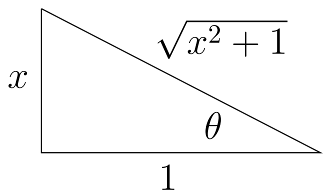
$$\frac{1}{\sqrt{x^2+1}} \ge \frac{1}{2x} \ \Leftrightarrow \ \sqrt{x^2+1} \le 2x \ \Leftrightarrow \ x^2+1 \le 4x^2 \ \Leftrightarrow \ 1 \le 3x^2 \ \Leftrightarrow \ x \ge \frac{1}{\sqrt{3}}$$

$$\int_0^\infty \frac{dx}{\sqrt{x^2+1}} = \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{x^2+1}} + \int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{\sqrt{x^2+1}}$$

1st integral : proper

2nd integral : $\displaystyle\int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{\sqrt{x^2+1}} \ge \int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{2x} = \frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{x}$: diverges <u>ok</u>

alternative



$$\tan\theta = x$$
$$\sec^2\theta\, d\theta = dx$$
$$\sec\theta = \sqrt{x^2+1}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\sec^2\theta}{\sec\theta} d\theta = \int \sec\theta d\theta = \ ?$$

$$\int \sec\theta d\theta = \int \frac{1}{\cos\theta} d\theta = \int \frac{\cos\theta}{\cos^2\theta} d\theta = \int \frac{\cos\theta}{1-\sin^2\theta} d\theta = \int \frac{du}{1-u^2} = \ ?$$

$$u = \sin\theta \ \Rightarrow\ du = \cos\theta d\theta$$

partial fractions

idea : $\dfrac{3}{10} = \dfrac{3}{2\cdot 5} = \dfrac{1}{2} - \dfrac{1}{5}$

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{a}{1+u} + \frac{b}{1-u} = \frac{a(1-u)+b(1+u)}{(1+u)(1-u)}$$

$$= \frac{(a+b)+u(-a+b)}{1-u^2} \quad \Rightarrow \quad \left.\begin{array}{r} a+b=1 \\ -a+b=0 \end{array}\right\} \ \Rightarrow \ a=b=\frac{1}{2}$$

$$\Rightarrow\ \frac{1}{1-u^2} = \frac{1}{2(1+u)} + \frac{1}{2(1-u)} \quad , \quad \text{check} \ldots$$ 12 Tues 9/27

$$\int \frac{du}{1-u^2} = \int \frac{du}{2(1+u)} + \int \frac{du}{2(1-u)} = \frac{1}{2}\ln(1+u) - \frac{1}{2}\ln(1-u) = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$

recall : $\ln a + \ln b = \ln(ab)$, $\ln a - \ln b = \ln(a/b)$, $a\ln b = \ln(b^a)$

$$\int \sec\theta d\theta = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\right) = \frac{1}{2}\ln\left(\frac{(1+\sin\theta)^2}{(1-\sin^2\theta)}\right)$$

$$= \ln\left(\frac{1+\sin\theta}{\cos\theta}\right) = \ln(\sec\theta + \tan\theta)$$

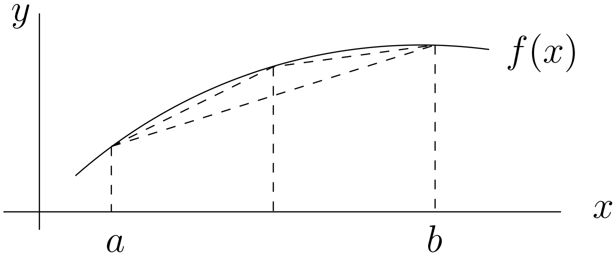
$$\int \sec\theta d\theta = \ln(\sec\theta+\tan\theta) \quad , \quad \text{check} \ldots$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln(\sqrt{x^2+1}+x) \quad , \quad \text{check} \ldots$$

$$\int_0^\infty \frac{dx}{\sqrt{x^2+1}} = \ln(\sqrt{x^2+1}+x)\Big|_0^\infty = \ln\infty - \ln 1 : \text{diverges} \qquad \underline{\text{ok}}$$

9.1 arclength

problem : compute the length of the graph of a function



1st approximation : $\sqrt{(b-a)^2 + (f(b) - f(a))^2}$

2nd approximation : set $x_0 = a$, $x_1 = \dfrac{a+b}{2}$, $x_2 = b$

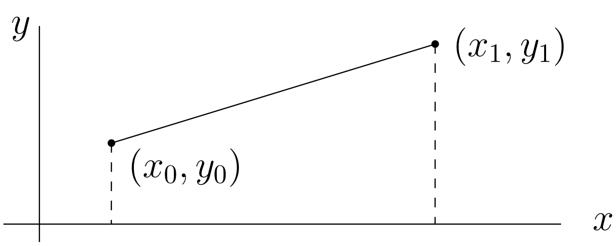
$$\sqrt{(x_1 - x_0)^2 + (f(x_1) - f(x_0))^2} + \sqrt{(x_2 - x_1)^2 + (f(x_2) - f(x_1))^2}$$

nth approximation : $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $x_i - x_{i-1} = \Delta x$

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 \left(1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2\right)}$$

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2} \cdot \Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

ex : straight line



$y = f(x) = mx + b$

alternative : $y = f(x) = m(x - x_0) + y_0 = \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) \cdot (x - x_0) + y_0$

check : $f(x_0) = y_0$, $f(x_1) = x_1$ ok

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2} dx = \sqrt{1 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2} \cdot \int_{x_0}^{x_1} dx$$

$$= \sqrt{1 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2} \cdot (x_1 - x_0) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \quad \text{ok}$$

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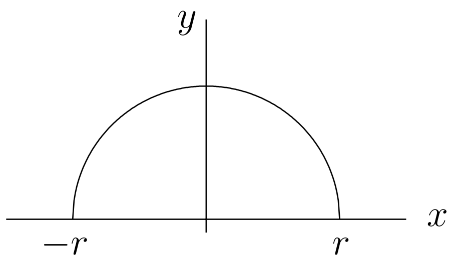
ex : circumference of a circle of radius r, $L = 2\pi r$



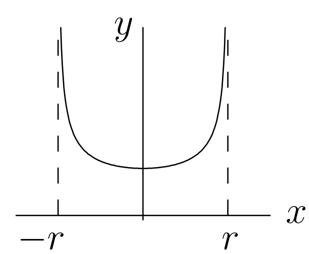
$$L = 2\int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

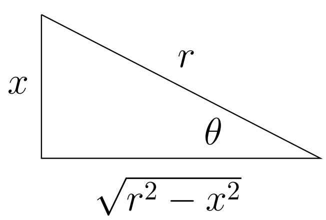
$$x^2 + y^2 = r^2 \;\Rightarrow\; f(x) = (r^2 - x^2)^{1/2}$$

$$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

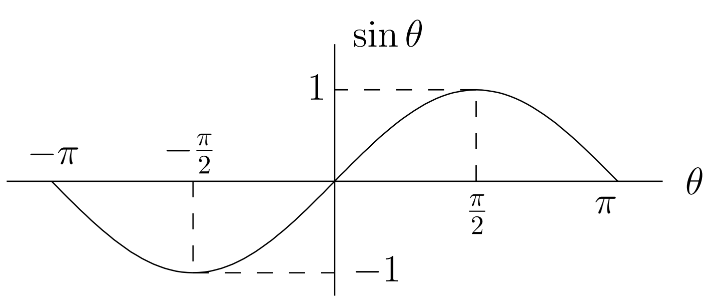
$$1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$L \;=\; 2\int_a^b \sqrt{1 + (f'(x))^2}\, dx \;=\; 2\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx \;:\; \text{improper}$$





$$\sin\theta = \frac{x}{r} \;\Rightarrow\; \cos\theta\, d\theta = \frac{dx}{r}\;,\;\; \cos\theta = \frac{\sqrt{r^2 - x^2}}{r}$$

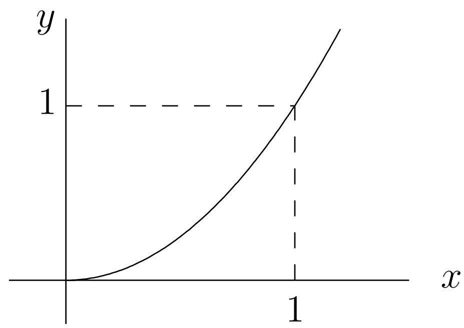


$$x = r \;\Rightarrow\; \sin\theta = 1 \;\Rightarrow\; \theta = \tfrac{\pi}{2}$$

$$x = -r \;\Rightarrow\; \sin\theta = -1 \;\Rightarrow\; \theta = -\tfrac{\pi}{2}$$

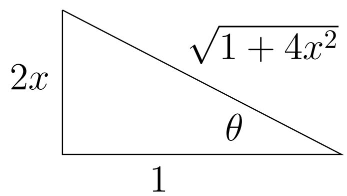
$$L \;=\; 2\int_{-\pi/2}^{\pi/2} \sec\theta \cdot r\cos\theta\, d\theta \;=\; 2r\int_{-\pi/2}^{\pi/2} d\theta \;=\; 2r\cdot\theta\Big|_{-\pi/2}^{\pi/2} \;=\; 2\pi r \qquad \underline{\text{ok}}$$

<u>ex</u> : parabola , $y = x^2$, $0 \le x \le 1$



note : $L > \sqrt{2} = 1.4142$

$$f(x) = x^2 \;\Rightarrow\; f'(x) = 2x \;\Rightarrow\; L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_0^1 \sqrt{1 + 4x^2}\,dx$$



$$\tan\theta = 2x \;\Rightarrow\; \sec^2\theta d\theta = 2dx$$

$$\sec\theta = \sqrt{1 + 4x^2}$$

$$\int \sqrt{1 + 4x^2}\,dx = \int \sec\theta \cdot \tfrac{1}{2}\sec^2\theta d\theta = \tfrac{1}{2}\int \sec^3\theta d\theta$$

$$\int \sec^3\theta d\theta = \int \frac{d\theta}{\cos^3\theta} = \int \frac{\cos\theta}{\cos^4\theta} d\theta = \int \frac{\cos\theta d\theta}{(1 - \sin^2\theta)^2} = \int \frac{du}{(1 - u^2)^2}$$

$$u = \sin\theta \;\Rightarrow\; du = \cos\theta d\theta$$

$$(1 - u^2)^2 = ((1 - u)(1 + u))^2 = (1 - u)^2(1 + u)^2$$

$$\frac{1}{(1 - u^2)^2} = \frac{a}{1 + u} + \frac{b}{(1 + u)^2} + \frac{c}{1 - u} + \frac{d}{(1 - u)^2} = \cdots$$

<u>alternative</u>

$$\int \sec^3\theta d\theta = \int \sec\theta \cdot \sec^2\theta d\theta$$

$$u = \sec\theta\,,\; dv = \sec^2\theta d\theta \;\Rightarrow\; du = \sec\theta\tan\theta d\theta\,,\; v = \tan\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta = \sec\theta\tan\theta - \int \sec\theta \cdot \frac{\sin^2\theta}{\cos^2\theta} d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta(1 - \cos^2\theta) d\theta = \sec\theta\tan\theta - \int \sec^3\theta + \int \sec\theta d\theta$$

$$\Rightarrow \int \sec^3\theta d\theta = \tfrac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))$$

$$\int \sqrt{1 + 4x^2}\,dx = \tfrac{1}{4}\Big(2x\sqrt{1 + 4x^2} + \ln\Big(2x + \sqrt{1 + 4x^2}\Big)\Big)$$

$$\int_0^1 \sqrt{1 + 4x^2}\,dx = \tfrac{1}{4}\Big(2x\sqrt{1 + 4x^2} + \ln\Big(2x + \sqrt{1 + 4x^2}\Big)\Big)\Big|_0^1$$

$$= \tfrac{1}{4}(2\sqrt{5} + \ln(2 + \sqrt{5}\,)) = 1.4789$$

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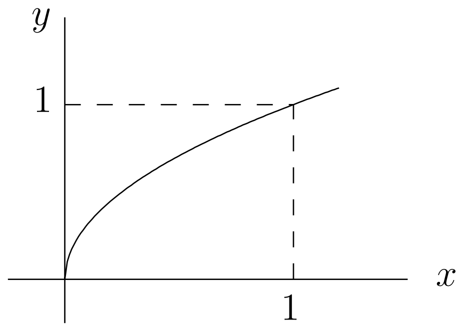
<u>ex</u> : $y = \sqrt{x}$, $0 \le x \le 1$



$$f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = \frac{1}{2\sqrt{x}}$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx \; : \; \text{improper , converges}$$

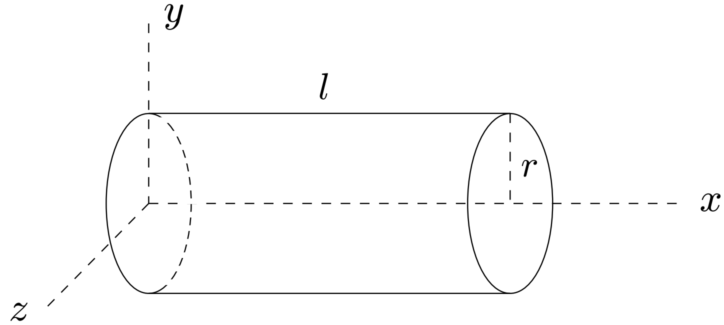
substitute : $y = \sqrt{x} \Rightarrow dy = \dfrac{dx}{2\sqrt{x}} = \dfrac{dx}{2y}$

$$L = \int_0^1 \sqrt{1 + \frac{1}{4y^2}} \cdot 2y\, dy = \int_0^1 \sqrt{4y^2 + 1}\, dy = \text{arclength of a parabola}$$ <u>ok</u>

<u>9.2 surface area</u>

<u>def</u> : A <u>surface of revolution</u> is formed by rotating a curve about an axis.

<u>ex</u> : cylinder

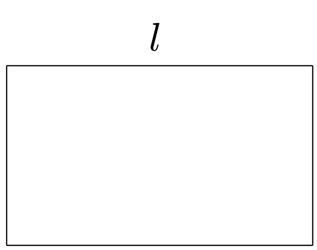


l : length

r : radius

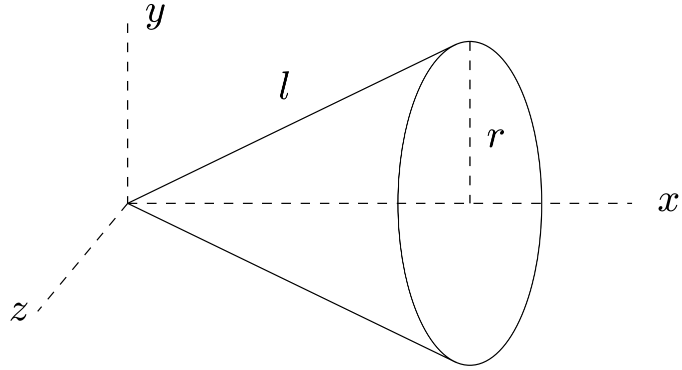
S : surface area

To find S, cut the cylinder and spread it flat to form a rectangle.



$2\pi r \;\Rightarrow\; S = 2\pi r l$

<u>ex</u> : cone

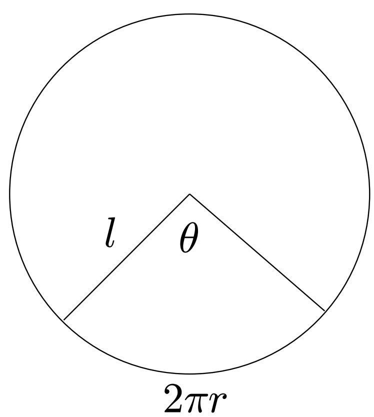


l : <u>slant length</u>

r : radius

S : surface area

To find S, cut the cone and spread it flat to form a circular sector.



l : radius of circle

θ : angle of sector

$2\pi r$: length of edge of sector

S : area of sector = area of cone

<u>claim</u>

$$\left.\begin{array}{l} \text{a) } 2\pi r = l\theta \\ \text{b) } S = \frac{1}{2}l^2\theta \end{array}\right\} \Rightarrow S = \tfrac{1}{2}l \cdot l\theta = \tfrac{1}{2}l \cdot 2\pi r = \pi r l$$

<u>pf</u>

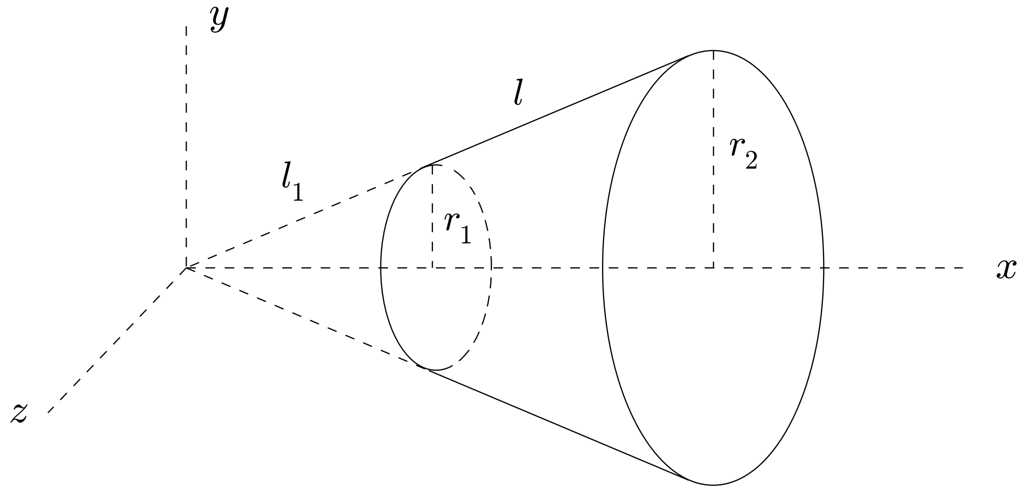
a) $\dfrac{2\pi r}{2\pi l} = \dfrac{\theta}{2\pi} \Rightarrow 2\pi r = 2\pi l \cdot \dfrac{\theta}{2\pi} = l\theta$ <u>ok</u>

b) $\dfrac{S}{\pi l^2} = \dfrac{\theta}{2\pi} \Rightarrow S = \pi l^2 \cdot \dfrac{\theta}{2\pi} = \tfrac{1}{2}l^2\theta$ <u>ok</u>

<u>note</u>

Another proof is given on hw5 using $A = \int_a^b f(x)dx$, $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$.

<u>ex</u> : truncated cone



l, r_1, r_2 = slant length and radii of truncated cone

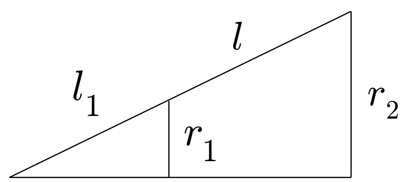
S = surface area of truncated cone

S_1 = ······" ······ small cone = $\pi r_1 l_1$

S_2 = ······" ······ large cone = $\pi r_2(l_1 + l)$

$S = S_2 - S_1 = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi l_1(r_2 - r_1) + \pi r_2 l$

we can eliminate l_1



$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2} \;\Rightarrow\; l_1 r_2 = r_1(l_1 + l) \;\Rightarrow\; l_1(r_2 - r_1) = r_1 l$$

$$\Rightarrow\; S = \pi r_1 l + \pi r_2 l = \pi(r_1 + r_2)l$$

<u>note</u>

1. $S = 2\pi\left(\frac{r_1 + r_2}{2}\right)l = 2\pi\bar{r}l$, where $\bar{r} = \frac{r_1 + r_2}{2}$ = average radius

2. special cases

$r_1 = r_2 \Rightarrow$ truncated cone becomes a cylinder , $S = 2\pi rl$

$r_1 = 0 \Rightarrow$ ············" ············ cone , $S = \pi rl$

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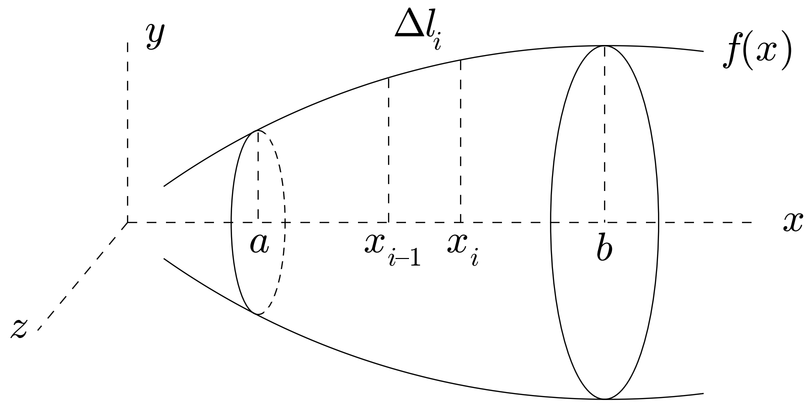
<u>general case</u>



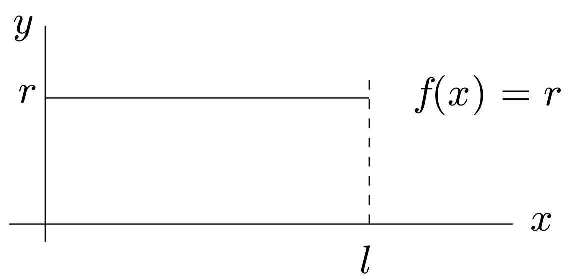
$\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$

surface area of ith slice $\approx \pi(f(x_{i-1}) + f(x_i))\Delta l_i$

$\Delta l_i \approx \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \approx \sqrt{1 + (f'(x_i))^2} \cdot \Delta x$

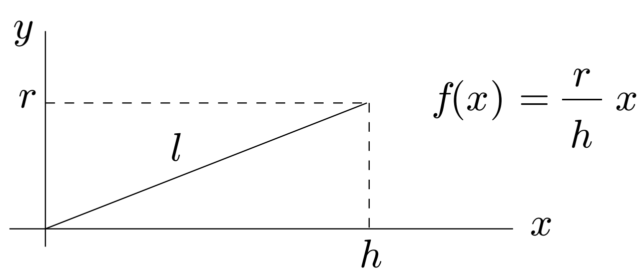
$$S = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(f(x_{i-1}) + f(x_i))\sqrt{1 + (f'(x_i))^2} \cdot \Delta x = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$$

<u>check 1</u> : cylinder



$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx = \int_0^l 2\pi r dx = 2\pi r \int_0^l dx = 2\pi r l \quad \underline{\text{ok}}$$

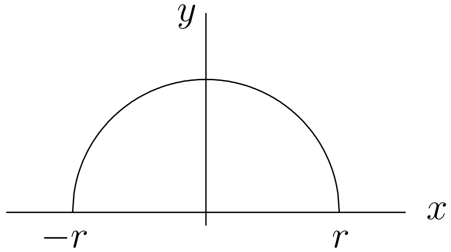
<u>check 2</u> : cone



$$S = \int_0^h 2\pi \frac{r}{h} x \sqrt{1 + \frac{r^2}{h^2}}\, dx = 2\pi \frac{r}{h}\sqrt{1 + \frac{r^2}{h^2}} \cdot \frac{h^2}{2} = \pi r\sqrt{h^2 + r^2} = \pi r l \quad \underline{\text{ok}}$$

<u>check 3</u> : truncated cone , review sheet

<u>ex</u>: sphere



$$f(x) = \sqrt{r^2 - x^2} = (r^2 - x^2)^{1/2} \Rightarrow f'(x) = \tfrac{1}{2}(r^2 - x^2)^{-1/2} \cdot -2x = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}}\, dx = 2\pi r \int_{-r}^{r} dx = 2\pi r \cdot 2r$$

$S = 4\pi r^2$ <u>ok</u>

<u>ex</u>: torus

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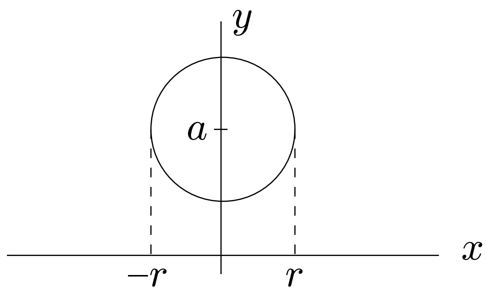


assume $a \geq r$, equation of circle : $x^2 + (y - a)^2 = r^2 \Rightarrow y = a \pm \sqrt{r^2 - x^2}$

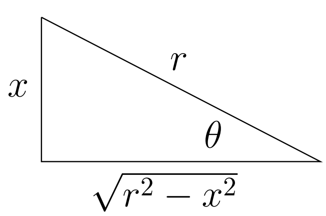
upper semicircle : $f_+(x) = a + \sqrt{r^2 - x^2}$

lower semicircle : $f_-(x) = a - \sqrt{r^2 - x^2}$

$$S = S_+ + S_- = \int_{-r}^{r} 2\pi f_+(x)\sqrt{1 + (f_+'(x))^2}\, dx + \int_{-r}^{r} 2\pi f_-(x)\sqrt{1 + (f_-'(x))^2}\, dx$$

$1 + (f_+'(x))^2 = \dfrac{r^2}{r^2 - x^2} = 1 + (f_-'(x))^2$, because $f_-'(x) = -f_+'(x)$

$$S = 2\pi \int_{-r}^{r} (a + \sqrt{r^2 - x^2} + a - \sqrt{r^2 - x^2}) \frac{r}{\sqrt{r^2 - x^2}}\, dx = 2\pi \cdot 2a \cdot r \int_{-r}^{r} \frac{dx}{\sqrt{r^2 - x^2}}$$



$$\sin\theta = \frac{x}{r} \Rightarrow \cos\theta\, d\theta = \frac{dx}{r}$$

$$\cos\theta = \frac{\sqrt{r^2 - x^2}}{r}$$

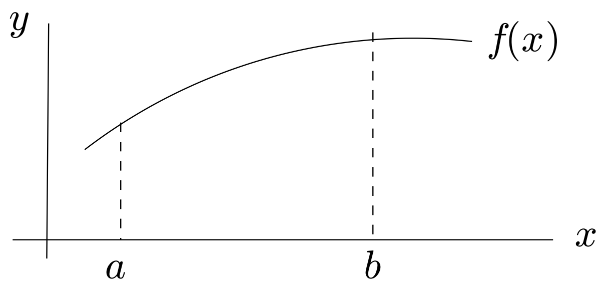
$$\Rightarrow S = 4\pi a r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r\cos\theta}{r\cos\theta} d\theta = 4\pi a r \cdot \pi = 4\pi^2 a r$$

<u>note</u>

1. $S_{\text{torus}} = 4\pi^2 ar = 2\pi a \cdot 2\pi r =$ product of circumferences of two circles

2. For the cylinder and cone, we can find S by cutting the surface, spreading it flat, and computing the area of the resulting shape. Does this work for the sphere and torus? No. (Math 433 , differential geometry)

<u>summary</u>



area under graph of $y = f(x)$ on $a \le x \le b$: $A = \int_a^b f(x)dx$

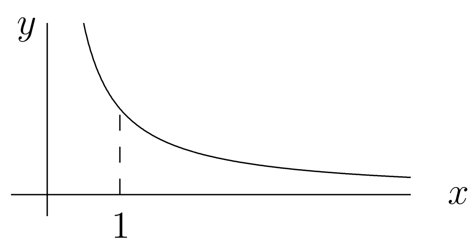
arclength ································ : $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

surface area ······························ : $S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$

volume ································· : $V = \int_a^b \pi f(x)^2 dx$

<u>ex</u>

$f(x) = \dfrac{1}{x}$, $1 \le x < \infty$



$A = \int_1^\infty \dfrac{dx}{x}$: diverges, $p = 1$

$L = \int_1^\infty \sqrt{1 + \dfrac{1}{x^4}}\, dx$: diverges , comparison test , $\sqrt{1 + \frac{1}{x^4}} \ge 1$

$S = \int_1^\infty 2\pi \dfrac{1}{x}\sqrt{1 + \dfrac{1}{x^4}}\, dx$: diverges , comparison test , $\frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \ge \frac{1}{x}$

$V = \int_1^\infty \dfrac{\pi}{x^2} dx = -\dfrac{\pi}{x}\Big|_1^\infty = \pi$: converges , $p = 2$

This shape is called <u>Gabriel's horn</u>; it has finite volume and infinite surface area. (see 595/25)

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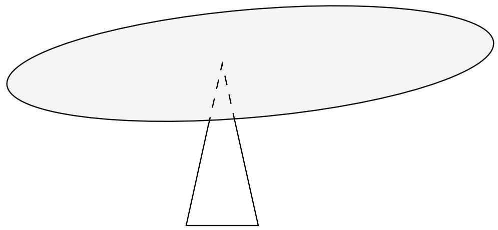
9.3 center of mass

problem : Find the point at which a thin plate balances.



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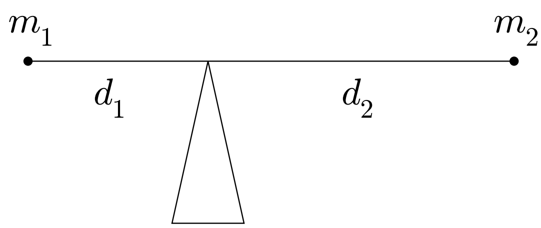
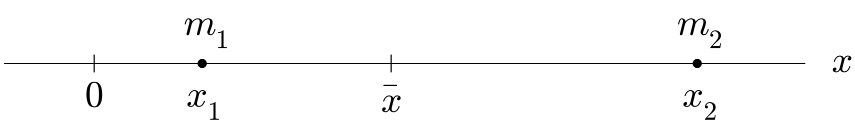
ex : two masses m_1, m_2 connected by a rod of negligible mass



balance of moments (prevents tipping) : $m_1 d_1 = m_2 d_2$



x_1 , x_2 , $\bar{x}$: coordinates of m_1 , m_2 , CM on x-axis

$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \Rightarrow (m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2 \Rightarrow \bar{x} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$m_i x_i$: moment of mass i about $x = 0$, units are mass $\times$ distance

note : The balance of moments can be written as $m_1(\bar{x} - x_1) + m_2(\bar{x} - x_2) = 0$.

---

ex : n masses $m_1, \ldots, m_n$ connected by a rod of negligible mass

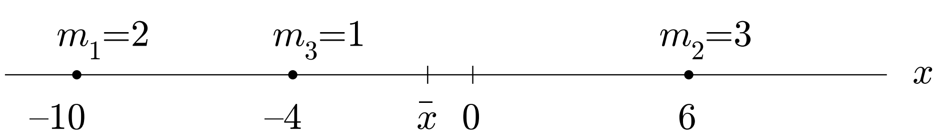
balance of moments $\Rightarrow \sum_{i=1}^{n} m_i(\bar{x} - x_i) = 0 \Rightarrow \sum_{i=1}^{n} m_i \bar{x} = \sum_{i=1}^{n} m_i x_i$

$$\Rightarrow \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M}{m} \; , \; M = \sum_{i=1}^{n} m_i x_i : \text{total moment} \; , \; m = \sum_{i=1}^{n} m_i : \text{total mass}$$

note : If $n = 2$, this agrees with the formula above.

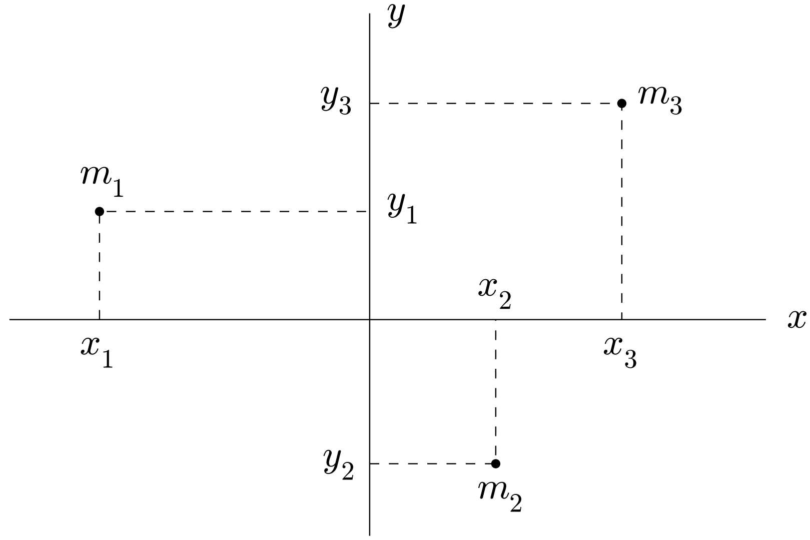
---

ex : $m_1 = 2$, $m_2 = 3$, $m_3 = 1$, $x_1 = -10$, $x_2 = 6$, $x_3 = -4$: find CM



$$\left.\begin{array}{l} M = m_1 x_1 + m_2 x_2 + m_3 x_3 = -20 + 18 - 4 = -6 \\ m = m_1 + m_2 + m_3 = 2 + 3 + 1 = 6 \end{array}\right\} \Rightarrow \bar{x} = \frac{M}{m} = \frac{-6}{6} = -1$$

<u>two-dimensional case</u>



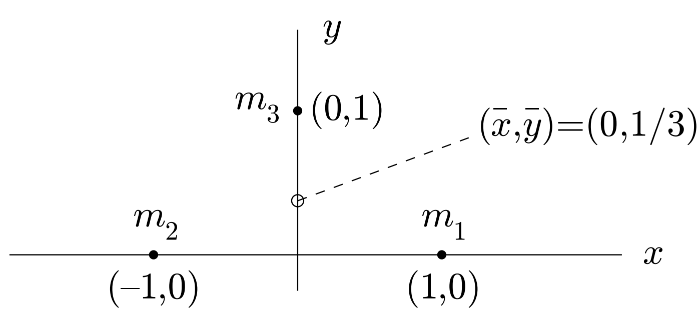
center of mass : $(\bar{x}, \bar{y}) = ?$

balance of moments : $\sum_{i=1}^{n} m_i(\bar{x} - x_i) = 0$, $\sum_{i=1}^{n} m_i(\bar{y} - y_i) = 0$

$\Rightarrow \bar{x} = \dfrac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \dfrac{M_y}{m}$, M_y : <u>moment about $y-$axis</u> $\Rightarrow M_y = m\bar{x}$

$\bar{y} = \dfrac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \dfrac{M_x}{m}$, M_x : <u>moment about $x-$axis</u> $\Rightarrow M_x = m\bar{y}$

<u>ex</u> : $m_1 = m_2 = m_3 = 1$

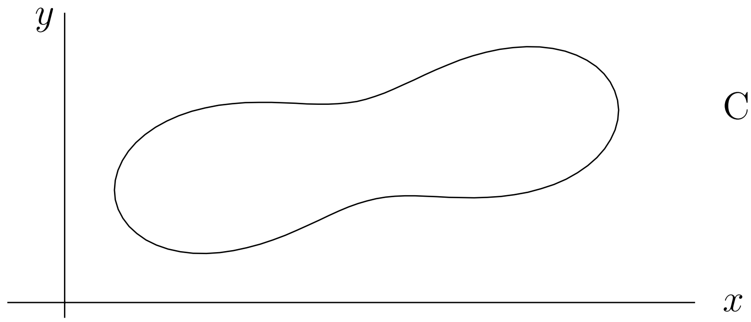


$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{-1 + 1 + 0}{3} = 0$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{0 + 0 + 1}{3} = \frac{1}{3}$$

<u>continuous mass distribution</u>

Consider a region of uniform density in the xy-plane.

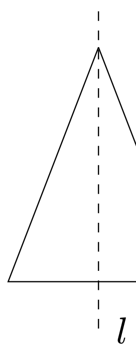


CM = $(\bar{x}, \bar{y})$ = ?

<u>symmetry principle</u>

If a region is symmetric about a line l, then CM lies on l.

<u>ex 1</u> : isoceles triangle , CM lies on l



<u>ex 2</u> : rectangle , CM lies on l_1 and l_2 $\Rightarrow$ CM is at center of rectangle

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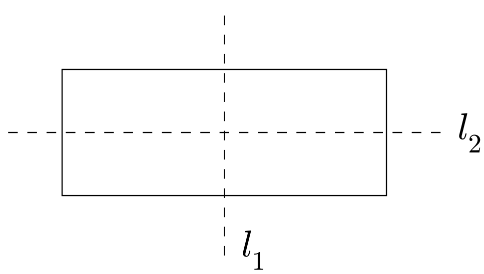
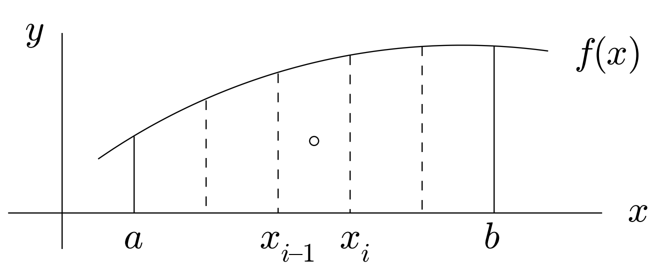


<u>case 1</u> : $R = \{(x, y) : a \le x \le b\,,\, 0 \le y \le f(x)\}$

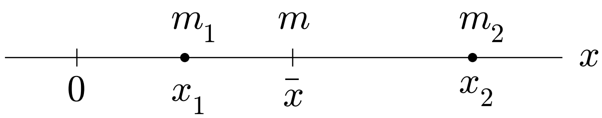


balance of moments $\Rightarrow$ $\bar{x} = \dfrac{M_y}{m}$, $\bar{y} = \dfrac{M_x}{m}$, but how do we find M_x, M_y?

$\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$

<u>moment principles</u>

1. The total moment of a mass distribution is the same as if all the mass is concentrated at the CM.

<u>ex</u> : 

$\Rightarrow \; M = m_1x_1 + m_2x_2 = m\bar{x}$

2. Let R_1, R_2 be two disjoint regions. Then $M(R_1 \cup R_2) = M(R_1) + M(R_2)$.

CM of ith rectangle $= (x_i^*, \frac{1}{2}f(x_i^*))$

mass of ith rectangle = density $\times$ area $= \rho f(x_i^*)\Delta x$

moment of ith rectangle about $y-$axis = mass $\times$ distance $= \rho f(x_i^*)\Delta x \cdot x_i^*$

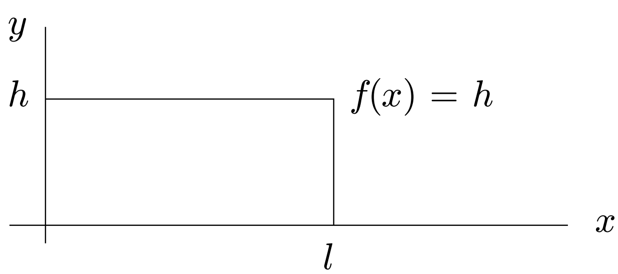
moment ....”........... $x-$axis = ...” .... $= \rho f(x_i^*)\Delta x \cdot \frac{1}{2}f(x_i^*)$

$$M_y = \lim_{n\to\infty}\sum_{i=1}^{n} \rho x_i^* f(x_i^*)\Delta x = \int_a^b \rho x f(x)dx = m\bar{x}$$

$$M_x = \lim_{n\to\infty}\sum_{i=1}^{n} \rho \tfrac{1}{2} f(x_i^*)^2\Delta x = \tfrac{1}{2}\int_a^b \rho f(x)^2 dx = m\bar{y}$$

$$\Rightarrow \; \bar{x} = \frac{M_y}{m} = \frac{\int_a^b \not\rho x f(x)dx}{\int_a^b \not\rho f(x)dx} \; , \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b \not\rho f(x)^2 dx}{\int_a^b \not\rho f(x)dx}$$

<u>ex</u> : rectangle ($\rho = 1$)



$$M_y = \int_a^b x f(x)dx = \int_0^l x \cdot h dx = h\frac{x^2}{2}\Big|_0^l = \frac{hl^2}{2}$$

$$M_x = \tfrac{1}{2}\int_a^b f(x)^2 dx = \tfrac{1}{2}\int_0^l h^2 dx = \frac{h^2 l}{2}$$

$$m = \int_a^b f(x)dx = \int_0^l h dx = hl$$

$$\bar{x} = \frac{M_y}{m} = \frac{hl^2}{2}\cdot\frac{1}{hl} = \frac{l}{2} \; , \quad \bar{y} = \frac{M_x}{m} = \frac{h^2 l}{2}\cdot\frac{1}{hl} = \frac{h}{2} \qquad \underline{\text{ok}}$$

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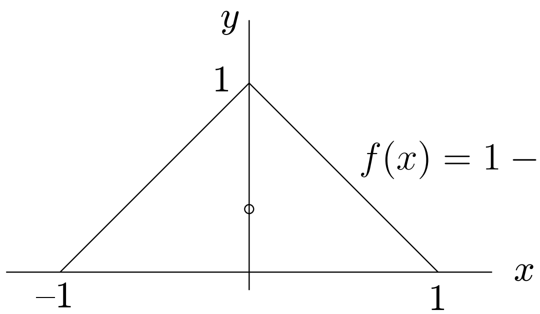
ex : triangular plate



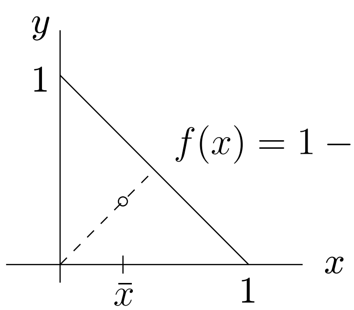
symmetry $\Rightarrow \bar{x} = 0$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b f(x)^2 dx}{\int_a^b f(x) dx} = \frac{\int_0^1 (1-x)^2 dx}{\frac{1}{2}\cdot 2 \cdot 1} = -\tfrac{1}{3}(1-x)^3\Big|_0^1 = -\tfrac{1}{3}(0-1) = \tfrac{1}{3}$$

$\Rightarrow$ CM $= (0, \frac{1}{3})$

---

ex : another triangular plate



symmetry $\Rightarrow$ CM lies on the line $y = x \Rightarrow \bar{y} = \bar{x}$

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{\int_0^1 x(1-x) dx}{\frac{1}{2}\cdot 1 \cdot 1} = \frac{\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1}{\frac{1}{2}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$\Rightarrow$ CM $= (\frac{1}{3}, \frac{1}{3})$

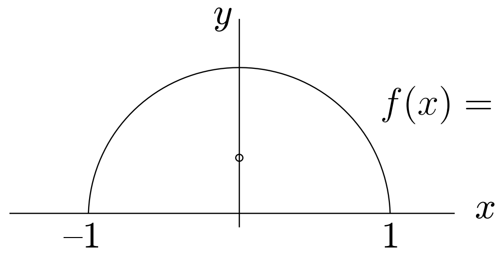
question : The line $x = \bar{x}$ divides the triangle into two parts. Which part has greater area?

area of left part $= \frac{1}{3} \cdot \frac{1}{2}(1 + \frac{2}{3}) = \frac{5}{18}$

area of right part $= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{18}$

So the left part has 25% greater area than the right part, even though the CM lies on the boundary of the two parts.

<u>ex</u> : half-disk



$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b f(x)^2 dx}{\int_a^b f(x) dx} = \frac{\int_0^1 (1-x^2) dx}{\pi/2} = \frac{\left(x - \frac{1}{3}x^3\right)\Big|_0^1}{\pi/2} = \frac{\frac{2}{3}}{\pi/2} = \frac{4}{3\pi} = 0.4244$$

<u>case 2</u> : $R = \{(x,y) : a \leq x \leq b\,,\ g(x) \leq y \leq f(x)\}$

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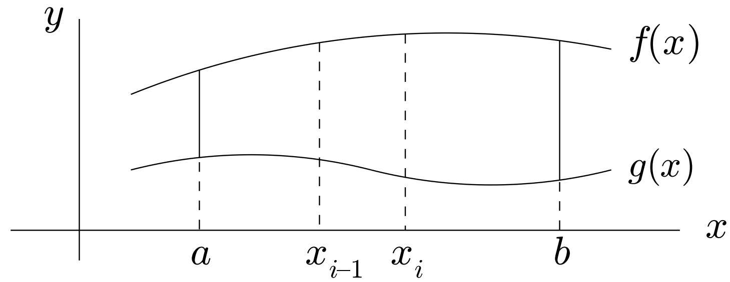
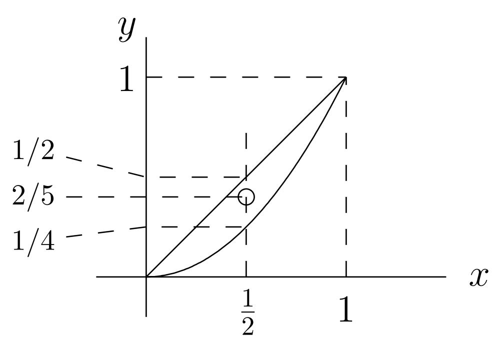


$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x(f(x) - g(x)) dx}{\int_a^b (f(x) - g(x)) dx} \ , \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b (f(x)^2 - g(x)^2) dx}{\int_a^b (f(x) - g(x)) dx}$$

<u>ex</u> : $f(x) = x$, $g(x) = x^2$, $0 \leq x \leq 1$



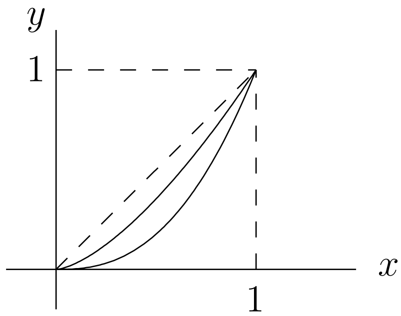
$$m = \int_a^b (f(x) - g(x)) dx = \int_0^1 (x - x^2) dx = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

$$M_x = \tfrac{1}{2}\int_a^b (f(x)^2 - g(x)^2) dx = \tfrac{1}{2}\int_0^1 (x^2 - x^4) dx = \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{1}{15}$$

$$M_y = \int_a^b x(f(x) - g(x)) dx = \int_0^1 (x^2 - x^3) dx = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2} \ , \ \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{15}}{\frac{1}{6}} = \frac{2}{5} \ \Rightarrow \text{CM is closer to top edge}$$

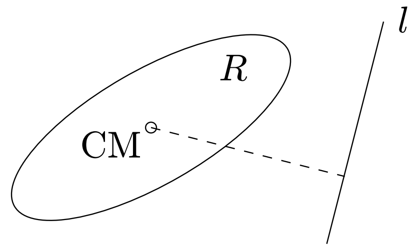
ex : $f(x) = x^m$, $g(x) = x^n$



For some choice of m, n, the CM lies outside the region. (hw7)

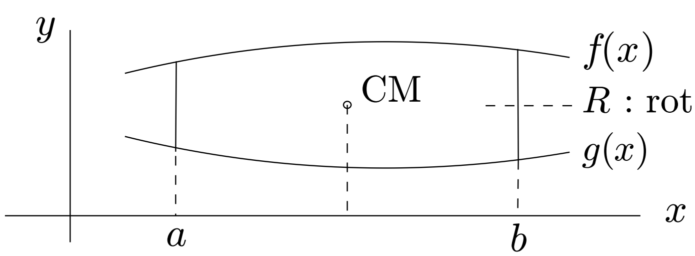
Theorem of Pappus

Let R be a region that lies on one side of a line l.



$A =$ area of R
$V =$ volume obtained by rotating R about l
$d =$ distance traveled by CM when R is rotated about l
$\Rightarrow V = A \cdot d$

pf : (special case)

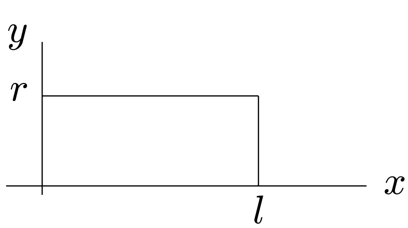


$$A = \int_a^b (f(x) - g(x))dx \quad , \quad V = \int_a^b \pi(f(x)^2 - g(x)^2)dx$$

$$d = 2\pi\bar{y} = 2\pi \cdot \frac{M_x}{m} = 2\pi \cdot \frac{\frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)dx}{\int_a^b (f(x) - g(x))dx} = \frac{V}{A} \quad \text{ok}$$

---

ex : cylinder



$$A = rl \ , \ d = 2\pi \cdot \frac{r}{2} = \pi r$$

$$\Rightarrow V = A \cdot d = rl \cdot \pi r = \pi r^2 l \quad \text{ok} \quad (\text{hw7 : torus})$$

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9.5 probability

X : random variable

ex

X = velocity of a gas molecule

X = waiting time in the supermarket check-out line

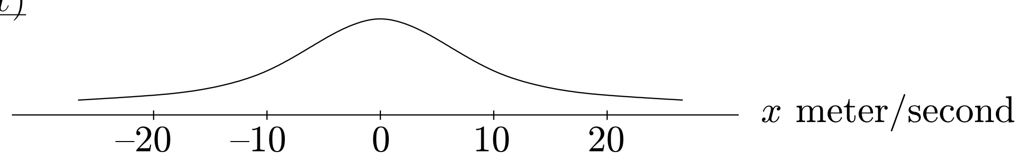
X = GPA of a college student

def

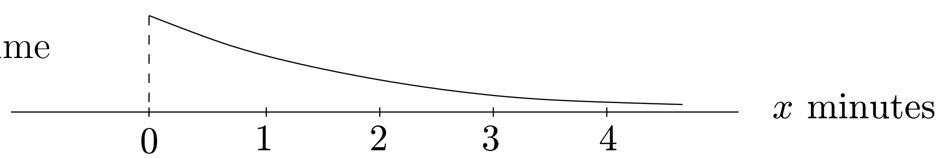
A random variable X has a probability density function $f(x)$ with the property that $\int_a^b f(x)dx$ = probability that X lies between a and b = prob$(a \le X \le b)$.

examples of $f(x)$

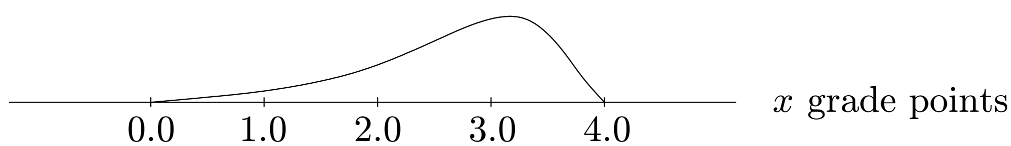
X = velocity



X = waiting time



X = GPA



note : large $f(x) \Rightarrow$ high probability that X is close to x

small $f(x) \Rightarrow$ low probability that X is close to x

In practice, the pdf $f(x)$ is obtained by empirical sampling or as the solution of a differential equation.

properties of a pdf

1. $f(x) \ge 0$
2. $\int_{-\infty}^{\infty} f(x)dx$ = prob$(-\infty < X < \infty) = 1$

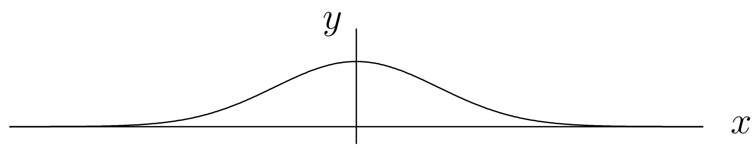
def : The mean value of a random variable X (average value, expected value) is denoted by $\mu = \mu(X)$ and is defined by the relation

$$\mu = \int_{-\infty}^{\infty} x f(x)dx \approx \sum_{i=1}^{n} x_i f(x_i)\Delta x \approx \sum_{i=1}^{n} x_i \cdot \text{prob}(x_{i-1} \le X \le x_i),$$

i.e. the values of x_i are weighted by the probability that X is close to x_i.

ex : Gaussian pdf

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$



1. $f(x) \geq 0$

2. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$ ok

note : $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (Math 255)

3. $\mu = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{\pi}} e^{-x^2} dx = 0$

def : The median value of a random variable X is denoted by $m = m(X)$ and is defined by the relation $\text{prob}(X \leq m) = \text{prob}(X \geq m) = \frac{1}{2}$.

note

1. This is equivalent to $\int_{-\infty}^{m} f(x)dx = \int_{m}^{\infty} f(x)dx = \frac{1}{2}$, i.e. half the area under the graph of $f(x)$ lies to the left of m and half lies to the right.

2. In general, $\mu \neq m$. (more later)

def

The standard deviation of a random variable X is denoted by $\sigma = \sigma(X)$ and is defined by the relation $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$.

note : small $\sigma \Rightarrow X$ is more likely to be close to μ

large $\sigma \Rightarrow X$ is less likely to be close to μ

ex

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$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \Rightarrow \mu = 0 , \ \sigma = ?$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{\pi}} e^{-x^2} dx \ , \quad u = x \ , \ dv = xe^{-x^2}dx$$

$$du = dx \ , \ v = \frac{e^{-x^2}}{-2}$$

$$= \frac{1}{\sqrt{\pi}} \left(\cancel{x \cdot \frac{e^{-x^2}}{-2} \Big|_{-\infty}^{\infty}} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx \right) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{1}{2} \Rightarrow \sigma = \frac{1}{\sqrt{2}} = 0.7071$$

<u>def</u> : Let μ and $\sigma > 0$ be given and define $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$. Then $f(x)$ is the pdf of a random variable X, called a <u>normal distribution</u>, with mean μ and standard deviation σ.

1. The Gaussian pdf corresponds to $\mu = 0$, $\sigma = \frac{1}{\sqrt{2}}$.
2. μ <u>shifts</u> the pdf along the x-axis and σ <u>scales</u> the height and width of the pdf

<u>check</u> (hw7)

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = 1$$

$$\int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} x\, \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \mu$$

$$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \sigma^2$$

<u>ex</u>

1. Find the probability that X is within 1 sd of μ.

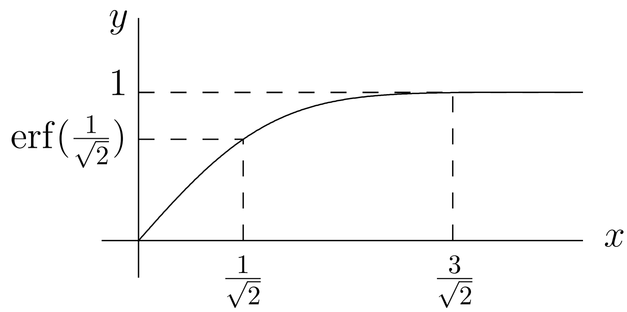
$$\text{prob}(\,\mu - \sigma \le X \le \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)dx$$

$$= \int_{\mu-\sigma}^{\mu+\sigma} \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-u^2} \cdot \sqrt{2}\sigma\, du = \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} e^{-u^2} du$$

$$\left(u = \frac{x-\mu}{\sqrt{2}\sigma}\ ,\ du = \frac{dx}{\sqrt{2}\sigma}\right)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} e^{-x^2} dx = \text{erf}(\tfrac{1}{\sqrt{2}}) = 0.6827$$

recall : $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2} dt$



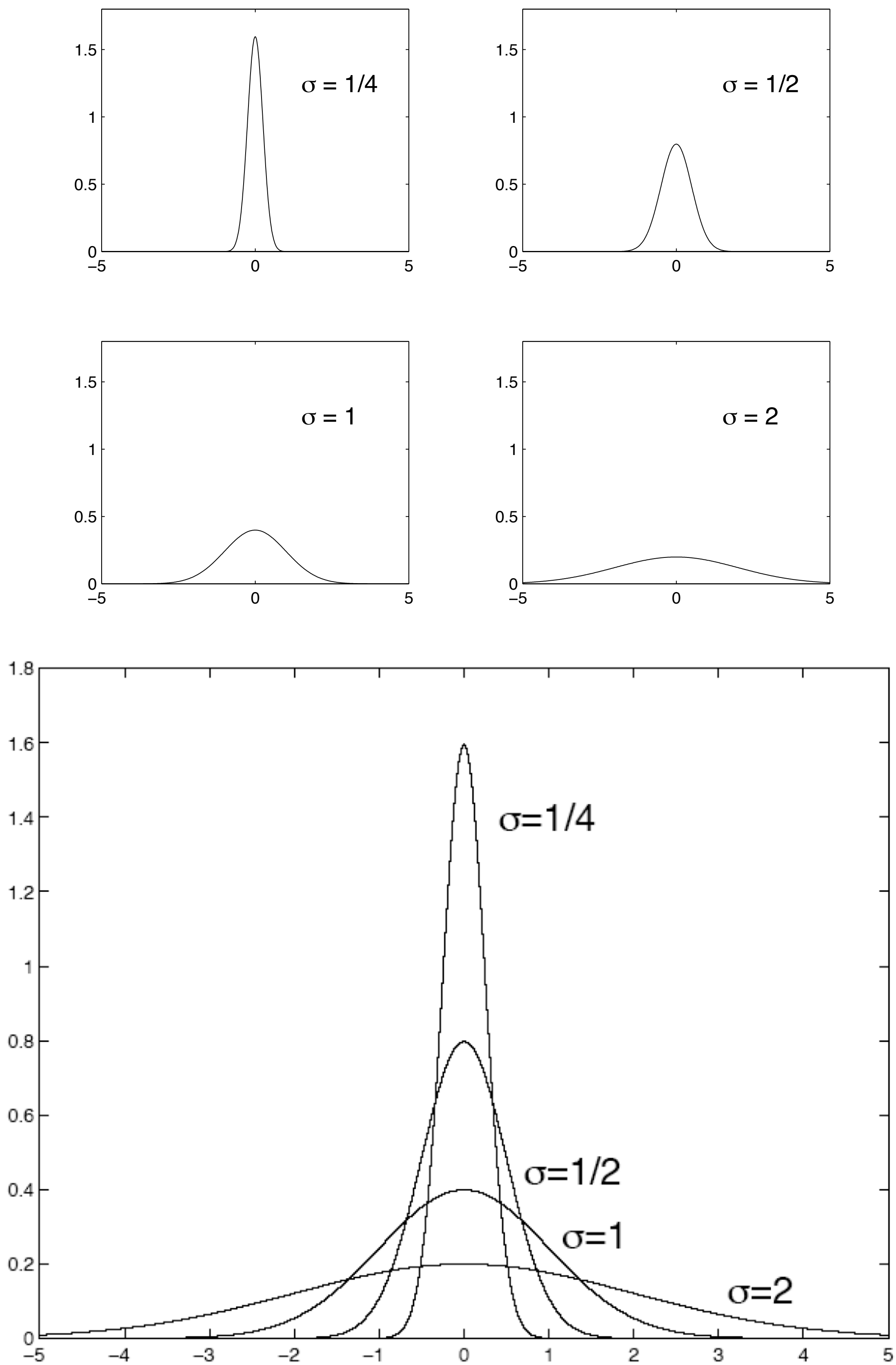
2. Find the probability that X is 3 sd or more greater than μ. 23 Mon 10/24

$$\text{prob}(X \ge \mu + 3\sigma) = \ldots = \frac{1}{\sqrt{\pi}} \int_{\frac{3}{\sqrt{2}}}^{\infty} e^{-x^2} dx = \tfrac{1}{2}(1 - \text{erf}(\tfrac{3}{\sqrt{2}})) = 0.001349$$

<u>ex</u> : annual rainfall in a certain state is normally distributed with $\mu = 25$in and $\sigma = 5$in.

1. 68% of years have rainfall between 20in and 30in (twice in 3 years)
2. 0.13% .... " .... greater than or equal to 40in (once in 740 years)

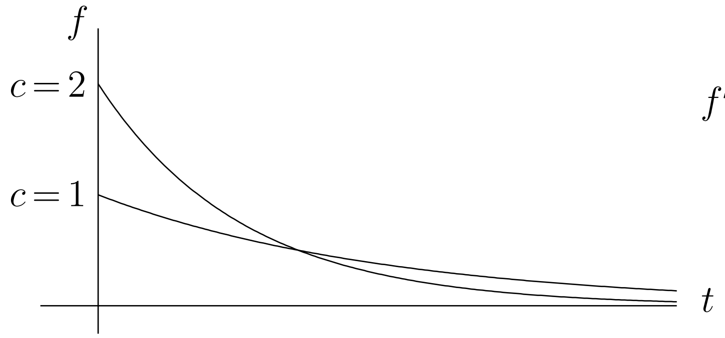
<u>pdf of a normal distribution</u> : $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $\mu = 0$



<u>exponential distribution</u>

T = waiting time in the supermarket check-out line (minutes)

$$f(t) = \begin{cases} c\,e^{-ct} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}, \quad \text{where } c > 0 \text{ is a constant}$$



$$f'(t) = -c^2 e^{-ct} \Rightarrow f'(0) = -c^2$$

check : $\int_{-\infty}^{\infty} f(t)dt = \int_0^{\infty} c\,e^{-ct}dt = c\,\frac{e^{-ct}}{-c}\Big|_0^{\infty} = 0 + 1 = 1$ <u>ok</u>

$$\text{average waiting time} = \mu = \int_{-\infty}^{\infty} t f(t)dt = \int_0^{\infty} t\,c\,e^{-ct}dt \quad (u = c\,t\,,\ du = c\,dt)$$

$$= \int_0^{\infty} u\,e^{-u}\frac{du}{c} = \frac{1}{c}$$

Assume the average waiting time is 5 minutes. Then $\mu = 5 \Rightarrow c = \frac{1}{5}$.

1. Find the probability that a shopper waits 1 minute or less.

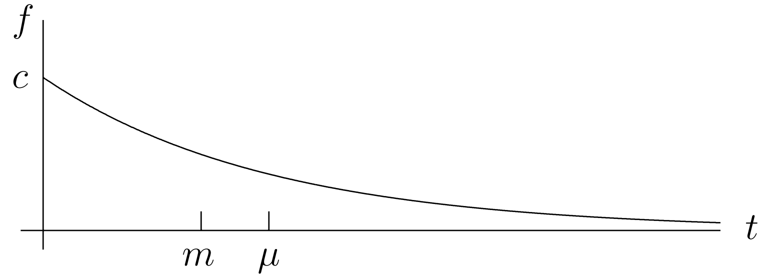
$$\text{prob}(0 \leq T \leq 1) = \int_0^1 f(t)\,dt = \int_0^1 c\,e^{-ct}\,dt = -e^{-ct}\Big|_0^1 = -e^{-c} + 1 = 1 - e^{-\frac{1}{5}} = 0.1813$$

Hence 18% of shoppers wait one minute or less.

2. Find the probability that a shopper waits 5 minutes or more.

$$\text{prob}(T \geq 5) = \int_5^{\infty} f(t)\,dt = \int_5^{\infty} c\,e^{-ct}\,dt = -e^{-ct}\Big|_5^{\infty} = e^{-1} = 0.3679$$

Even though the <u>average</u> waiting time is 5 minutes, only 37% of shoppers actually wait 5 minutes or more. In fact, the <u>median</u> waiting time is only 3.5 minutes (hw7), which implies that half the shoppers wait less than 3.5 minutes and half wait more.



$\Rightarrow$ "heavy tail" , "rare events"

The average waiting time ($\mu = 5$) is greater than the median waiting time ($m = 3.5$) because some of the shoppers who wait more than 3.5 minutes actually wait a lot longer (e.g. 10 minutes).

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10.1 differential equations

some famous differential equations

Newton's 2nd law : particle moving in a force field

Maxwell's eqs : electromagnetic waves

Schrodinger eq : quantum mechanics

def : A 1st order differential equation has the form $y' = f(y)$, where $y = y(t)$ is a function of time t and $y' = \frac{dy}{dt}$ is the 1st derivative.

ex

$y' = y$

$y' = -y$

$y' = 1 - y^2$

note

The function $y(t)$ represents a quantity that varies in time (e.g. population, temperature, value of an investment) and the differential equation $y' = f(y)$ controls the variation of $y(t)$. A function $y(t)$ is a solution of the differential equation if $y'(t) = f(y(t))$ for all $t \geq 0$; we also say that $y(t)$ satisfies the differential equation.

ex

Is the given $y(t)$ a solution of the differential equation $y' = y$?

$y(t) = e^t$: yes

$y(t) = e^t + 1$: no

$y(t) = 2e^t$: yes

$y(t) = e^{2t}$: no

$y(t) = -e^t$: yes

$y(t) = e^{-t}$: no

ex

The function $y(t) = \tanh t$ satisfies the differential equation $y' = 1 - y^2$.

pf : $y' = \frac{d}{dt}\tanh t = \frac{d}{dt}\frac{\sinh t}{\cosh t} = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = 1 - \tanh^2 t = 1 - y^2$ ok note

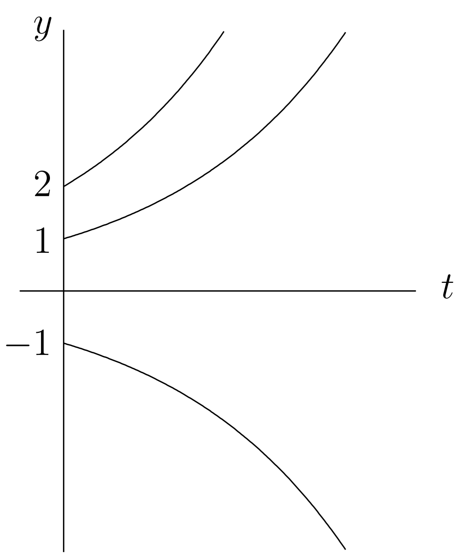
A differential equation $y' = f(y)$ has infinitely many solutions $y(t)$, but there is a unique solution satisfying the initial condition $y(0) = c$ for any value c.

ex

$y(t) = e^t$ is the solution of $y' = y$ satisfying $y(0) = 1$

$y(t) = 2e^t$ .... " .... $y(0) = 2$

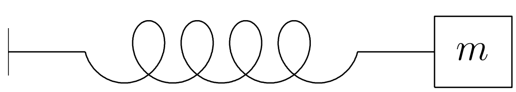
$y(t) = -e^t$ .... " .... $y(0) = -1$



note

We can also consider 2nd order differential equations, i.e. equations involving the 2nd derivative.

ex : mass-spring system



$y(t)$: displacement of spring from its natural length

$$\left.\begin{array}{l}\text{Newton's 2nd law : } F = m\,y'' \\ \text{Hooke's law : } F = -ky \;\; (k > 0\,,\text{restoring force})\end{array}\right\} \Rightarrow \; m\,y'' = -ky$$

$y(t) = \cos\omega t$ is a solution $\Leftrightarrow m \cdot -\omega^2 \cos\omega t = -k\cos\omega t \Leftrightarrow \omega = \sqrt{k/m}$

The system oscillates with frequency ω; this is called simple harmonic motion.

other examples : pendulum , Earth's orbit , heartbeat , ... (Math 256)

10.4 exponential growth/decay

Let $y(t)$ be a population size at time t and assume that the population changes at a rate proportional to its size. Then $y' = ky$, where k is a constant.

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$k > 0 \;\Rightarrow\; y' > 0 \;\Rightarrow$ population grows

$k < 0 \;\Rightarrow\; y' < 0 \;\Rightarrow$ population decays

$k = \dfrac{y'}{y}$: relative growth rate , units = time^{-1}

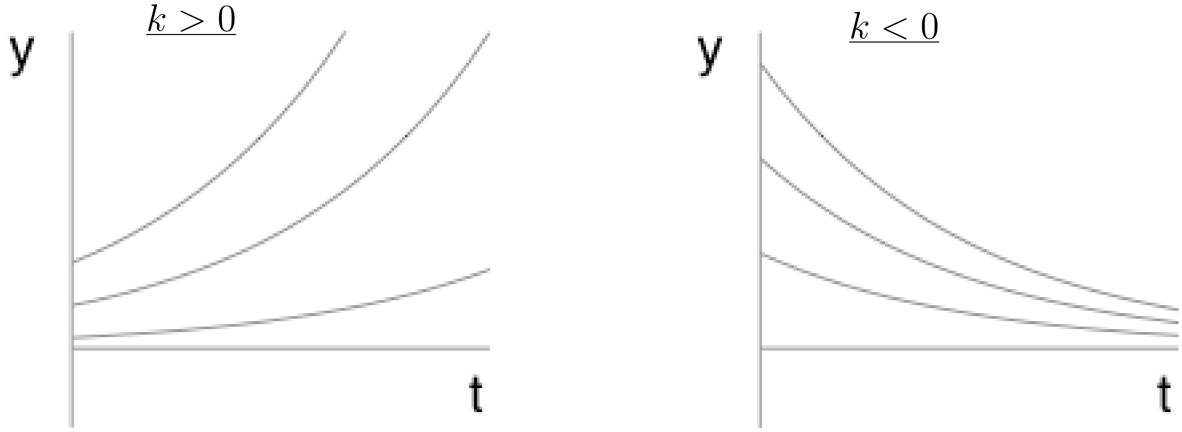
solution

$y' = ky \;\Rightarrow\; \dfrac{dy}{dt} = ky \;\Rightarrow\; \dfrac{dy}{y} = k\,dt$: separation of variables

$\Rightarrow\; \ln y = kt + c \;\Rightarrow\; y = e^{kt+c} = e^{kt} \cdot e^{c} = Ae^{kt}$

$t = 0 \;\Rightarrow\; y(0) = A \;\Rightarrow\; y(t) = y(0)e^{kt}$: exponential growth/decay , check ...

$k > 0$ $\qquad\qquad$ $k < 0$



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ex : A bacteria culture starts with 1000 cells and grows at a rate proportional to its size. If there are 2500 cells present after 2 hours, then how many cells are present after 4 hours?

$y(t)$: number of cells present after t hours

$y(0) = 1000$, $y(2) = 2500 \;\Rightarrow\; y(4) = 4000$: no

$y(t) = y(0)\,e^{kt} = 1000\,e^{kt}$

$y(2) = 1000\,e^{2k} = 2500 \;\Rightarrow\; e^{2k} = 2.5 \;\Rightarrow\; 2k = \ln 2.5 \;\Rightarrow\; k = \frac{1}{2}\ln 2.5 = 0.46$

$y(t) = 1000\,e^{0.46t}$

alternative : $e^{kt} = e^{(\frac{1}{2}\ln 2.5)t} = (e^{\ln 2.5})^{t/2} = (2.5)^{t/2}$, recall : $e^{ab} = (e^{a})^{b}$

$y(t) = 1000\,(2.5)^{t/2}$, check : $y(0) = 1000\,,\; y(2) = 2500$ ok

$\Rightarrow\; y(4) = 1000\,(2.5)^2 = 6250$ cells ex : A radioactive sample has mass 100 mg and decays at a rate proportional to its size with half-life 1600 years. (The half-life is the time required for the sample to lose half its mass.)

a) Find the mass remaining after t years.

$y(t)$: mass (mg) remaining after t years

$y(t) = y(0)\,e^{kt} = 100\,e^{kt}$

$y(1600) = 100\,e^{1600\,k} = 50 \;\Rightarrow\; e^{1600\,k} = \frac{1}{2} \;\Rightarrow\; 1600\,k = \ln\frac{1}{2} = -\ln 2$

$k = \frac{-\ln 2}{1600} = -0.00043 \;\Rightarrow\; y(t) = 100\,e^{-0.00043\,t}$

alternative : $y(t) = 100\,e^{(-\ln 2/1600)t} = 100\,(e^{\ln 2})^{-t/1600} = 100 \cdot 2^{-t/1600}$

check : $y(0) = 100 \cdot 2^0 = 100$, $y(1600) = 100 \cdot 2^{-1} = 50$ ok

b) How much mass is remaining after 800 years?

$y(800) = 100 \cdot 2^{-800/1600} = 100 \cdot 2^{-1/2} = \frac{100}{\sqrt{2}} = 71$ mg

c) When will the sample be reduced to 25 mg?

$y(t) = 100 \cdot 2^{-t/1600} = 25 \;\Rightarrow\; 2^{-t/1600} = \frac{1}{4} = 2^{-2} \;\Rightarrow\; \frac{-t}{1600} = -2$

$t = 3200$ years $= 2$ half-lives

---

ex : compound interest

x : initial investment (principal) , r : annual interest rate

compounded annually

after 1 year, the investment is worth $x + rx = (1+r)x$

..... 2 .... $(1+r)^2x$

..... t .... $(1+r)^tx$

compounded semi-annually : 2 times/year

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after $\frac{1}{2}$ year, the investment is worth $x + \frac{r}{2}x = (1+\frac{r}{2})x$

..... 1 .... $(1+\frac{r}{2})^2x$

..... t .... $(1+\frac{r}{2})^{2t}x$

note : $(1+\frac{r}{2})^2 = 1 + r + \frac{r^2}{4} > 1 + r$: more frequent compounding is beneficial

compounded daily : 365 times/year

after $\frac{1}{365}$ year, the investment is worth $x + \frac{r}{365}x = (1+\frac{r}{365})x$

..... $\frac{2}{365}$ .... $(1+\frac{r}{365})^2x$

..... 1 .... $(1+\frac{r}{365})^{365}x$

..... t .... $(1+\frac{r}{365})^{365t}x$

<u>continuous compounding</u>

after t years, the investment is worth

$$\lim_{n\to\infty}(1+\tfrac{r}{n})^{nt}x = \Big(\lim_{n\to\infty}(1+\tfrac{r}{n})^{n}\Big)^{t}x = (e^r)^t x = e^{rt}x \ : \ \text{hw7}$$

<u>ex</u> : $x = \$1000$, $r = 0.10\,(\text{i.e. } 10\%)$, $t = 10$ years

compounded annually : $(1+r)^t x = (1.1)^{10}\cdot 1000 = \$2593.74$

compounded continuously : $e^{rt}x = e^{(0.1\cdot 10)}\cdot 1000 = \$2718.28$

<u>note</u>

1. $e^{0.1} = 1.1052 = 1 + 0.1052 \Rightarrow$ the equivalent annual interest rate is 10.52%

$\Rightarrow$ 10% compounded continuously is the same as 10.52% compounded annually

2. continuous compounding $\Leftrightarrow$ $y(t) = y(0)e^{rt}$ $\Leftrightarrow$ $y' = ry$

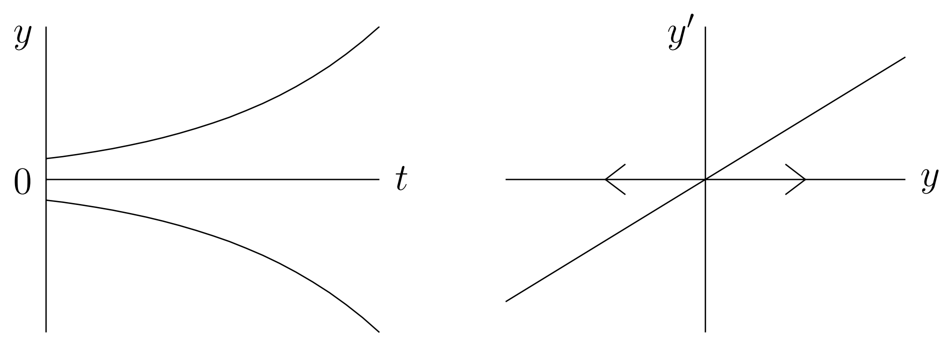
<u>def</u> : A differential equation $y' = f(y)$ has a <u>constant solution</u> $y(t) = c \Leftrightarrow f(c) = 0$.

<u>ex</u> : $y' = 1 - y^2$ has two constant solutions : $c = \pm 1$

<u>def</u> : A constant solution $y(t) = c$ is <u>stable</u> $\Leftrightarrow \lim_{t\to\infty} y(t) = c$ for all nearby solutions; otherwise, it is <u>unstable</u>.

<u>ex</u> : $y' = ky$ $\Rightarrow$ 0 is a constant solution. Is it stable or unstable?

case 1 : $k > 0 \Rightarrow 0$ is an unstable constant solution



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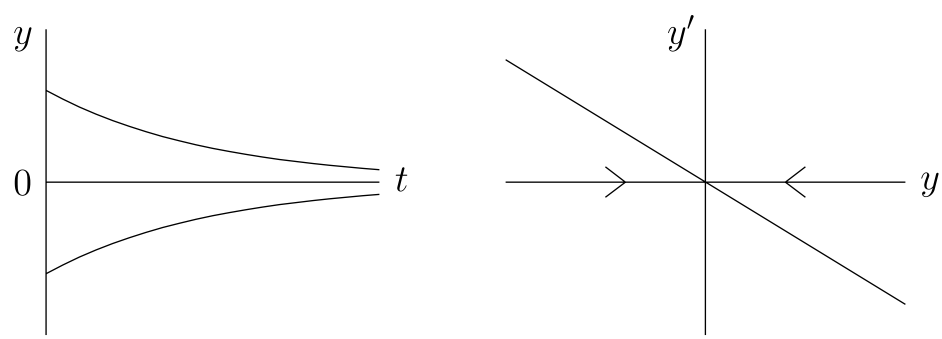
case 2 : $k < 0 \Rightarrow 0$ is a stable constant solution



<u>note</u> : The $y - y'$ plane is called the <u>phase plane</u>; it gives a concise description of the dynamical system.

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ex : Newton's law of cooling/heating

The rate of change of temperature of an object is proportional to the temperature difference between the object and its environment.

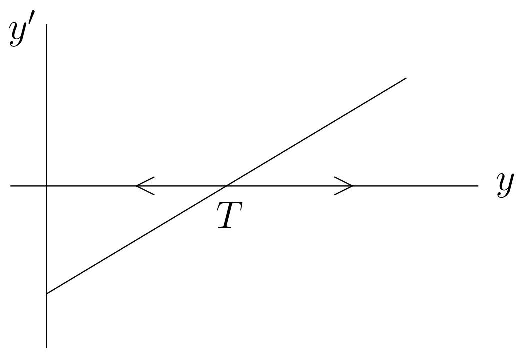
$y(t)$: temperature of object

T : temperature of environment

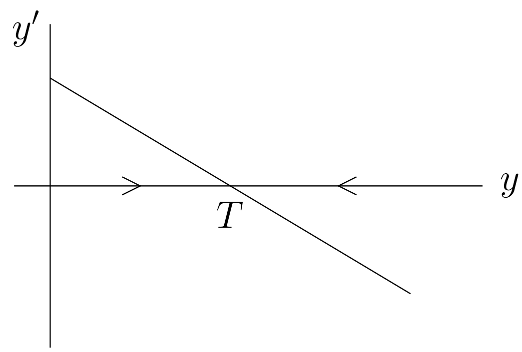
$\Rightarrow\ y' = k(y - T)$

note : There is one constant solution $y(t) = T$.

case 1 : $k > 0 \Rightarrow T$ is unstable



case 2 : $k < 0 \Rightarrow T$ is stable



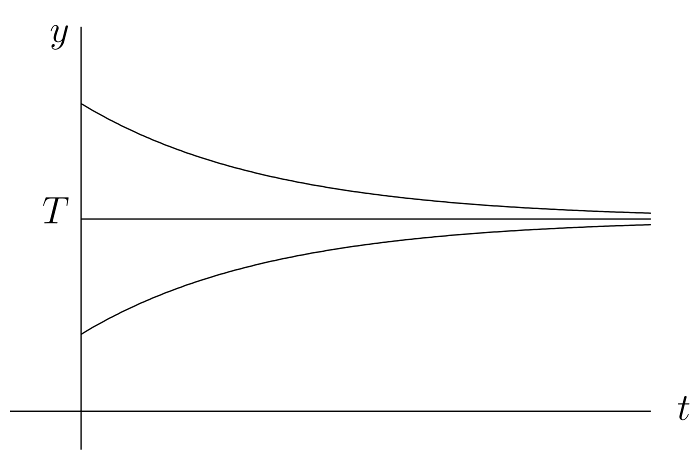
note : The case $k > 0$ is not physical, so we assume $k < 0$.

solution

$$\frac{dy}{dt} = k(y - T) \Rightarrow \frac{dy}{y - T} = kdt \Rightarrow \ln(y - T) = kt + C \Rightarrow y - T = e^{kt+C} = Ae^{kt}$$

$t = 0 \Rightarrow y_0 - T = A$, where $y_0 = y(0)$

$\Rightarrow y - T = (y_0 - T)e^{kt} \Rightarrow y(t) = T + (y_0 - T)e^{kt}$, check ...



$y_0 > T \Rightarrow$ object is cooling

$y_0 < T \Rightarrow$ object is heating

<u>ex</u>

A can of soda takes 1 hour to cool from 30°C to 20°C in a refrigerator at 10°C.

a) Find the soda temperature 30 minutes after it starts to cool.

$y(t) = T + (y_0 - T)e^{kt} = 10 + (30 - 10)e^{kt} = 10 + 20e^{kt}$

measure t in minutes

$y(60) = 10 + 20e^{60k} = 20 \Rightarrow 20e^{60k} = 10 \Rightarrow e^{60k} = \frac{1}{2} \Rightarrow e^k = \left(\frac{1}{2}\right)^{\frac{1}{60}} = 2^{-\frac{1}{60}}$

$\Rightarrow y(t) = 10 + 20 \cdot 2^{-t/60} \Rightarrow y(30) = 10 + 20 \cdot 2^{-1/2} = 10 + \frac{20}{\sqrt{2}} \approx 24°\text{C}$

b) How long does it take for the soda to cool an additional 6°C?

$y(t) = 10 + 20 \cdot 2^{-t/60} = 30 - 24 = 18°\text{C}$

$\Rightarrow 20 \cdot \left(\frac{1}{2}\right)^{t/60} = 8 \Rightarrow 0.5^{t/60} = 0.4 \Rightarrow t = 60 \cdot \frac{\ln 0.4}{\ln 0.5} \approx 80 \text{ min}$

It takes 50 minutes for the soda to cool an additional 6°C.

note : as $y(t) \to T$, the rate of cooling decreases

<u>ex</u>

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A 5000 L tank contains salt water with salt concentration 0.004 kg/L. Seawater with salt concentration 0.03 kg/L starts pouring into the tank at a rate of 25 L/min. The solution is kept well mixed and it drains from the tank at the same rate as it enters. How much salt is present in the tank after 30 mins?

$y(t)$: amount of salt (kg) in tank after t mins

$y(0) = 0.004 \frac{\text{kg}}{\text{L}} \cdot 5000 \text{ L} = 20 \text{ kg}$, $y(30) = ?$ (we expect $y(30) > 20$)

$y' =$ rate of change of amount of salt in tank $(\frac{\text{kg}}{\text{min}})$

= (rate coming in) - (rate going out)

rate coming in $= 0.03 \frac{\text{kg}}{\text{L}} \cdot 25 \frac{\text{L}}{\text{min}} = 0.75 \frac{\text{kg}}{\text{min}}$

rate going out $= \frac{y \text{ kg}}{5000 \text{ L}} \cdot 25 \frac{\text{L}}{\text{min}} = 0.005\, y \frac{\text{kg}}{\text{min}}$

$y' = 0.75 - 0.005\, y = -0.005(y - 150) = k(y - T)$

$y(t) = T + (y_0 - T)e^{kt} = 150 + (20 - 150)e^{-0.005t} = 150 - 130e^{-0.005t}$

$y(30) = 150 - 130e^{-0.005 \cdot 30} = 150 - 130e^{-0.15} = 38 \text{ kg}$

question : what does T represent in this problem? $\lim_{t \to \infty} y(t) = T = 150 \text{ kg}$

10.5 logistic equation

$y(t)$: population

M : maximum population (due to finite resources)

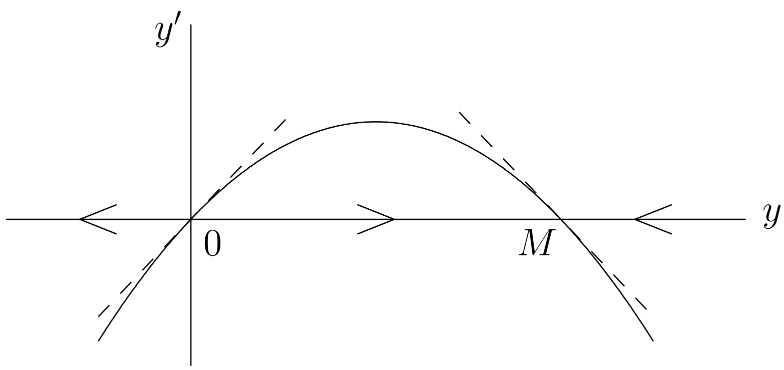
$y' = k(M - y)y$: logistic equation

note : textbook uses $P' = k\left(1 - \frac{P}{K}\right)P$

assume $k > 0$, $0 \le y \le M$

then $k(M - y)$ is a variable growth rate

There are two constant solutions : $y = 0$, $y = M$.

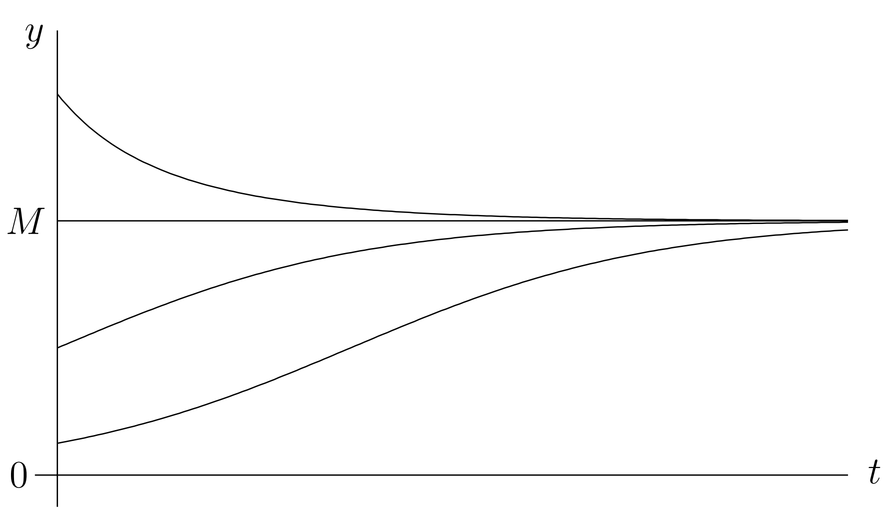


$y = 0$ is unstable

$y = M$ is stable

if $y \approx 0$, then $y' \approx kMy \Rightarrow$ exponential growth

if $y \approx M$, then $y' \approx kM(M - y) \Rightarrow$ cooling/heating



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<u>solution</u>

$$\frac{dy}{dt} = k(M-y)y \;\Rightarrow\; \frac{dy}{(M-y)y} = kdt$$

$$\frac{1}{(M-y)y} = \frac{a}{M-y} + \frac{b}{y} = \frac{ay + b(M-y)}{(M-y)y} = \frac{bM + y(a-b)}{(M-y)y}$$

$$\left.\begin{array}{ll} bM = 1 & \Rightarrow b = \dfrac{1}{M} \\ a - b = 0 & \Rightarrow a = \dfrac{1}{M} \end{array}\right\} \Rightarrow \frac{1}{(M-y)y} = \frac{1}{M(M-y)} + \frac{1}{My} \quad , \text{ check } \ldots$$

$$\int \frac{dy}{(M-y)y} = \int \frac{dy}{M(M-y)} + \int \frac{dy}{My} = -\frac{1}{M}\ln(M-y) + \frac{1}{M}\ln y$$

$$= \frac{1}{M}\ln\left(\frac{y}{M-y}\right) = \int kdt \;=\; kt + c$$

$$\ln\left(\frac{y}{M-y}\right) = kMt + cM \;\Rightarrow\; \frac{y}{M-y} = e^{kMt+cM} = De^{kMt}$$

$$t = 0 \;\Rightarrow\; \frac{y_0}{M-y_0} = D$$

$$\frac{y}{M-y} = \frac{y_0}{M-y_0}e^{kMt} \;\Rightarrow\; y = (M-y)\frac{y_0}{M-y_0}e^{kMt}$$

$$y\Big(1 + \frac{y_0}{M-y_0}e^{kMt}\Big) = \frac{My_0}{M-y_0}e^{kMt} \;\Rightarrow\; y\Big(M - y_0 + y_0e^{kMt}\Big) = My_0e^{kMt}$$

$$y\Big(\Big(M-y_0\Big)e^{-kMt} + y_0\Big) = My_0 \;\Rightarrow\; y(t) = \frac{My_0}{y_0 + (M-y_0)e^{-kMt}}$$

check : $y(0) = y_0$, ...

<u>note</u>

$$\left.\begin{array}{ll} y_0 = 0 & \Rightarrow y(t) = 0 \\ y_0 = M & \Rightarrow y(t) = M \end{array}\right\} : \text{ these are the 2 constant solutions}$$

if $0 < y_0 < M$, then $\lim_{t\to\infty} y(t) = M \;\Rightarrow\; \left\{\begin{array}{ll} M & \text{is stable} \\ 0 & \text{is unstable} \end{array}\right.$

This confirms what we knew from the phase plane analysis.

10.2 Euler's method

consider $y' = f(y)$, $y(0) = y_0$, goal : approximate $y(t)$

choose $\Delta t = h =$ time step , set $t_n = nh$, $n = 0, 1, 2, \ldots$

define $y_n = y(t_n)$: exact solution at time t_n

u_n : numerical solution at time t_n (approximation)

$\frac{u_{n+1} - u_n}{h} = f(u_n)$: Euler's method , finite-difference scheme

$\Rightarrow u_{n+1} = u_n + hf(u_n)$, $u_0 = y_0$

ex

$y' = y$, $y_0 = 1 \Rightarrow y(t) = e^t$

$u_{n+1} = u_n + hu_n = (1+h)u_n$

$u_0 = 1$

$u_1 = (1+h)u_0 = 1 + h$

$u_2 = (1+h)u_1 = (1+h)^2$

$\ldots$

$u_n = (1+h)u_{n-1} = (1+h)^n$

for example, suppose $t_n = nh = 1$, then $y_n = y(t_n) = y(1) = e = 2.7182818$

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h	u_n	$\lvert y_n - u_n\rvert$	$\lvert y_n - u_n\rvert/h$
0.1	2.5937425	0.1245	1.245
0.05	2.6532977	0.0650	1.300
0.025	2.6850638	0.0332	1.328
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	e	0	$e/2$, pf : later

note

1. $\lim_{h\to 0} u_n = \lim_{h\to 0}(1+h)^n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e \Rightarrow$ Euler's method converges

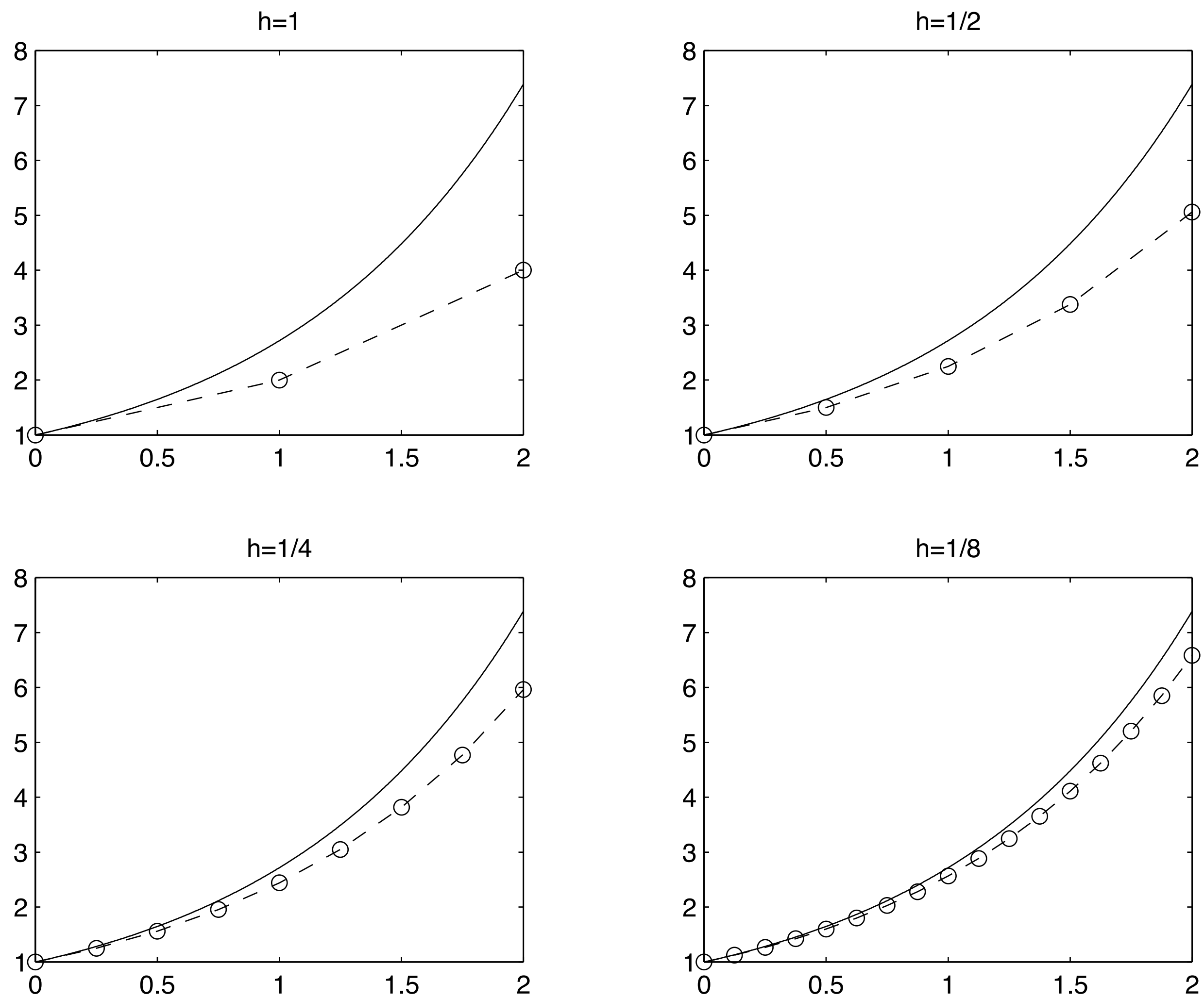
2. If the time step h decreases by a factor of $\frac{1}{2}$, then the error in the numerical solution given by Euler's method decreases by a factor of approximately $\frac{1}{2}$. Hence the error is proportional to h; we say "the error $= O(h)$".

3. There are higher order methods, i.e. methods for which the error $= O(h^p)$ for some $p \geq 2$. (Math 371, numerical methods)

Euler's method

differential equation : $y' = y\,,\ y(0) = 1 \ \Rightarrow\ y(t) = e^t$

numerical method : $u_{n+1} = u_n + hu_n\,,\ u_0 = 1 \ \Rightarrow\ u_n = (1+h)^n$



note

1. The solid line is the exact solution $y(t)$.

   The circles are the numerical solution u_n.

   The numerical solution values are connected by a dashed line.

2. For a fixed time t, the error decreases as $h \to 0$.

3. For a fixed time step h, the error increases as $t \to \infty$.

chapter 12. sequences and series

preview

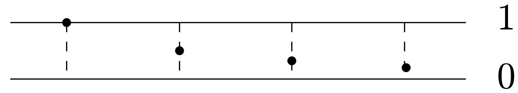
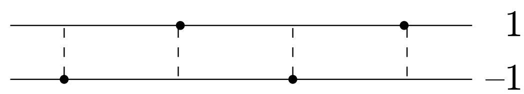
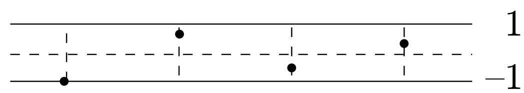
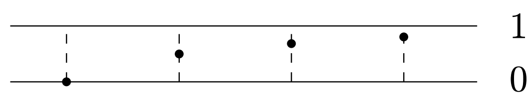
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ : \ \text{Taylor series}$$

questions : meaning? utility?

12.1 sequences

def : A sequence is a list of numbers $\{a_1, a_2, a_3, \ldots\} = \{a_n\}$. If the sequence converges to a limit L, we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$ as $n \to \infty$, and if the sequence does not converge, we say it diverges.

ex

	terms		converges?
$a_n = \frac{1}{n}$	$1\,,\ \frac{1}{2}\,,\ \frac{1}{3}\,,\ \frac{1}{4}\,,\ \ldots$		$L = 0$
$b_n = (-1)^n$	$-1\,,\ 1\,,\ -1\,,\ 1\,,\ \ldots$		diverges
$c_n = \frac{(-1)^n}{n}$	$-1\,,\ \frac{1}{2}\,,\ -\frac{1}{3}\,,\ \frac{1}{4}\,,\ \ldots$		$L = 0$
$d_n = 1 - \frac{1}{n}$	$0\,,\ \frac{1}{2}\,,\ \frac{2}{3}\,,\ \frac{3}{4}\,,\ \ldots$		$L = 1$

def : A sequence $\{a_n\}$ is ...

... bounded if there are constants m, M such that $m \le a_n \le M$ for all n. 31 Mon 11/7

... increasing if $a_{n+1} > a_n$ for all n.

... decreasing if $a_{n+1} < a_n$ for all n.

ex : bounded? : all the above , increasing? : $\{d_n\}$, decreasing? : $\{a_n\}$

thm

1. A convergent sequence is bounded.
2. A bounded increasing or decreasing sequence converges.

pf : Math 451 (advanced calculus), but it's clear from the examples

note : The converse is false.

1. A bounded sequence is not necessarily convergent.
2. A convergent sequence is not necessarily increasing or decreasing.

pf : hw9 (by example)

<u>comparison theorem</u>

If $a_n \le b_n \le c_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

<u>pf</u> : omit

<u>ex</u> : $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (used later)

<u>pf</u> :

n	1	2	3	4	$\cdots$
$\frac{n!}{n^n}$	$\frac{1}{1}$	$\frac{2\cdot 1}{2\cdot 2}$	$\frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3}$	$\frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 4\cdot 4\cdot 4}$	$\cdots$

$\Rightarrow 0 \le \frac{n!}{n^n} \le \frac{1}{n}$

then $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{n!}{n^n} = 0$: by comparison theorem <u>ok</u>

<u>12.2 series</u>

<u>def</u>

$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$: <u>series</u>

note : the starting index may be $n = 0$ or $n = 1$ or ...

<u>def</u>

$\{s_n\}$: sequence of <u>partial sums</u> of the series

$s_0 = a_0$

$s_1 = a_0 + a_1$

$s_2 = a_0 + a_1 + a_2$

$\vdots$

$s_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{i=0}^{n} a_i$

<u>ex</u>

$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = ?$

$s_0 = 1$

$s_1 = 1 + \frac{1}{2} = \frac{3}{2}$

$s_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

$s_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

$\vdots$

$\Rightarrow \lim_{n\to\infty} s_n = 2$

<u>pf</u> : soon

<u>def</u> : $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n} a_i = \lim_{n\to\infty} s_n$

If the limit exists, the series <u>converges</u>; otherwise, it <u>diverges</u>.

<u>ex</u> : geometric series

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$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \begin{cases} \dfrac{1}{1-r} & , \text{ if } -1 < r < 1 \\ \text{diverges} , & \text{otherwise} \end{cases}$$

<u>pf</u>

recall : finite geometric series : $1 + r + r^2 + \cdots + r^n = \begin{cases} \dfrac{1-r^{n+1}}{1-r} & , \text{ if } r \neq 1 \\ n+1 & , \text{ if } r = 1 \end{cases}$

$$\lim_{n\to\infty} r^{n+1} = \begin{cases} 0 & , \text{ if } -1 < r < 1 \\ 1 & , \text{ if } r = 1 \\ \text{diverges} , & \text{otherwise} \end{cases}$$ <u>ok</u>

<u>ex</u>

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

$$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots = \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 2 \cdot \frac{1}{1-\frac{2}{3}} = 6$$

<u>ex</u>

A rubber ball is dropped from height 1, bounces back to height p $(0 < p < 1)$, and continues bouncing after that. Find the distance traveled by the ball.

initial height $= 1$

height after 1 bounce $= p$

$\cdots\cdots\cdots\cdots$ 2 bounces $= p^2$

$\cdots\cdots\cdots\cdots$ n bounces $= p^n$

distance $= 1 + 2p + 2p^2 + \cdots = 2(1 + p + p^2 + \cdots) - 1 = 2 \cdot \frac{1}{1-p} - 1$

$= \frac{2-(1-p)}{1-p} = \frac{1+p}{1-p}$

for example : $p = \frac{1}{2} \Rightarrow$ distance $= \frac{3/2}{1/2} = 3$

$p \to 1 \Rightarrow$ distance $\to \infty$

$p \to 0 \Rightarrow$ distance $\to 1$

<u>harmonic series</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \ : \ \text{diverges}$$

<u>pf</u>

$s_1 = 1$

$s_2 = s_1 + \frac{1}{2} = \frac{3}{2}$

$s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + 4 \cdot \frac{1}{8} = \frac{5}{2}$

$s_{16} = s_8 + \frac{1}{9} + \cdots + \frac{1}{16} > \frac{5}{2} + 8 \cdot \frac{1}{16} = 3$

$\ldots$

$\Rightarrow \lim_{n\to\infty} s_n = \infty$ <u>ok</u>

<u>thm</u>

If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

<u>pf</u>

set $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = s$

$s_n = a_0 + a_1 + \cdots + a_n$

$s_{n-1} = a_0 + a_1 + \cdots + a_{n-1}$

$s_n - s_{n-1} = a_n$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$ <u>ok</u>

<u>note</u>

1. The converse is false, i.e. if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ may converge or diverge.

$$\left.\begin{array}{l} \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} \ : \ \text{converges} \\ \\ \displaystyle\sum_{n=0}^{\infty} \frac{1}{n} \ : \ \text{diverges} \end{array}\right\} : \ \text{both satisfy } \lim_{n\to\infty} a_n = 0$$

2. However, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

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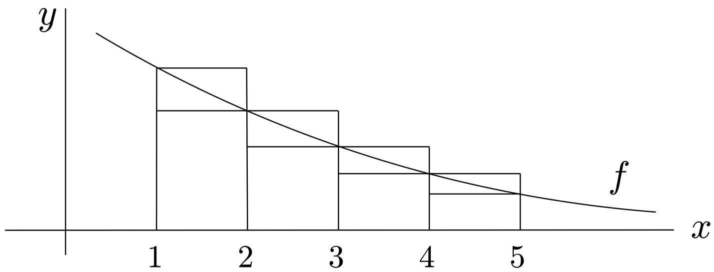
12.3 integral test for series

In this section assume $a_n = f(n)$, where $f(x)$ is positive and decreasing.

integral test : $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \int_1^{\infty} f(x)dx$ converges

pf



$\Rightarrow\ a_2 + a_3 + a_4 + a_5 \le \int_1^5 f(x)dx \le a_1 + a_2 + a_3 + a_4$ ok

---

ex : $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$: converges

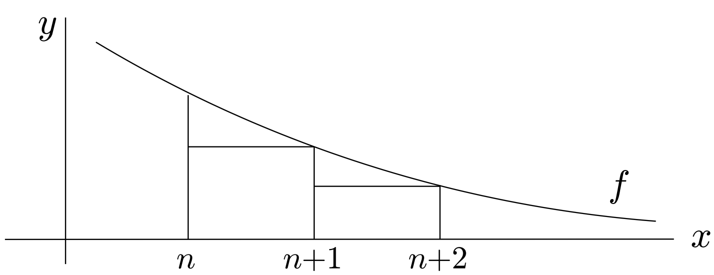
pf : $f(x) = \frac{1}{x^2} \Rightarrow f(n) = \frac{1}{n^2} \Rightarrow \int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{dx}{x^2}$: converges , $p = 2$ ok

note : We don't know the value of the sum s, but we can approximate it by s_n.

for example : $s_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{144+36+16+9}{144} = \frac{205}{144} = 1.42361111$

question : How large is the error?

$s - s_n = a_1 + \cdots + a_n + a_{n+1} + \cdots - (a_1 + \cdots + a_n) = a_{n+1} + \cdots \le \int_n^{\infty} f(x)dx$



$\Rightarrow 0 \le s - s_n \le \int_n^{\infty} f(x)dx$: error bound $\Rightarrow s_n \le s \le s_n + \int_n^{\infty} f(x)dx$

for example : $\int_4^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_4^{\infty} = \frac{1}{4} = 0.25 \Rightarrow 1.42361111 < s < 1.67361111$

question : How large must n be to ensure that the error is less than 10^{-3}?

$s - s_n \le \int_n^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_n^{\infty} = \frac{1}{n} = 10^{-3} \Rightarrow n = 1000$

$\Rightarrow s_{1000} < s < s_{1000} + 10^{-3} \Rightarrow 1.64393457 \le s \le 1.64493457$

in fact , $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.64493407$: discovered by Leonhard Euler in 1735

---

note : there is also a p-test for series , $\sum_{n=1}^{\infty} \frac{1}{n^p}$: $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$, pf : ...

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12.4 comparison test for series

Assume $0 \le a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

pf : $\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n \quad \ldots$ ok

ex : $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$: converges

pf : $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$: converges , $p = 2$ ok

in fact , $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$: hw10

12.5 alternating series

In sections 12.3, 12.4, we considered series with only positive terms.

def : An alternating series is a series whose terms alternate in sign.

ex : $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$: alternating harmonic series

general form : $\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots$, where $a_n > 0$

alternating series test

If a_n satisfies :

1. $a_n > 0$,

2. $a_{n+1} < a_n$ (i.e. $\{a_n\}$ is a decreasing sequence),

3. $\lim_{n\to\infty} a_n = 0$,

then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

<u>ex</u> : $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges

<u>pf</u>

$a_n = \frac{1}{n+1}$

$$\left.\begin{array}{l} 1.\ \frac{1}{n+1} > 0 \\ \\ 2.\ \frac{1}{n+2} < \frac{1}{n+1} \\ \\ 3.\ \lim_{n\to\infty} \frac{1}{n+1} = 0 \end{array}\right\} \Rightarrow \text{ series converges by AST}$$

recall : $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges

<u>pf</u> (AST)

$\sum_{n=0}^{\infty} (-1)^n a_n = \lim_{n\to\infty} s_n$: we need to show that the limit exists

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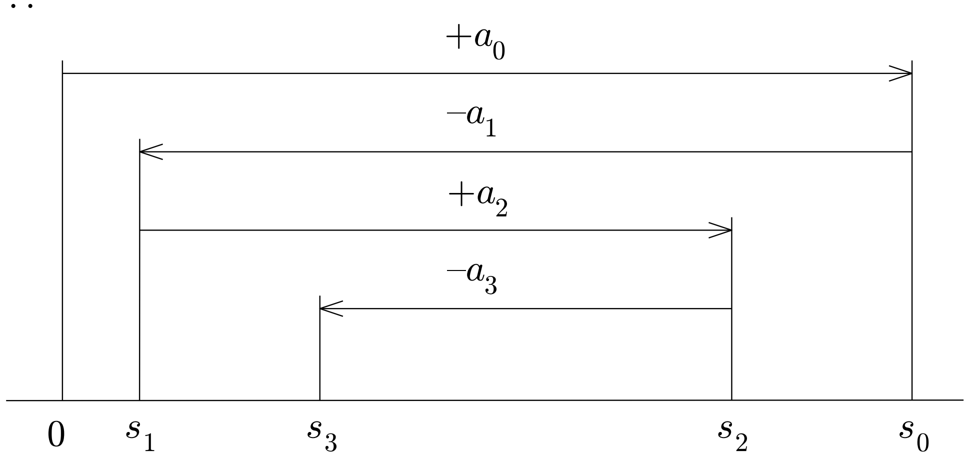
$s_0 = a_0$

$s_1 = a_0 - a_1$

$s_2 = a_0 - a_1 + a_2$

$s_3 = a_0 - a_1 + a_2 - a_3$

...



It follows that $\lim_{n\to\infty} s_n$ exists. <u>ok</u>

error bound for alternating series

Let $s = \sum_{n=0}^{\infty} (-1)^n a_n$, where a_n satisfies the conditions of the AST.

Then $|s - s_n| \le a_{n+1}$, i.e. the error is bounded by the first neglected term.

pf

$$s = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n + (-1)^{n+1} a_{n+1} + \cdots$$

$$s_n = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n$$

$$s - s_n = (-1)^{n+1} a_{n+1} + (-1)^{n+2} a_{n+2} + \cdots$$

$$= (-1)^{n+1}(a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots)$$

$$|s - s_n| = |(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots|$$

$$= (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots$$

$$= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots$$

$$\le a_{n+1} \quad \text{ok}$$

note : $|s - s_n| \le a_{n+1} \Leftrightarrow -a_{n+1} \le s - s_n \le a_{n+1} \Leftrightarrow s_n - a_{n+1} \le s \le s_n + a_{n+1}$

ex : $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = s$

check : $a_n = \frac{1}{n+1}$: satisfies the conditions of the AST

$$s_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{12 - 6 + 4 - 3}{12} = \frac{7}{12} = 0.5833$$

$$|s - s_3| \le a_4 = \frac{1}{5} = 0.2 \Rightarrow 0.3833 \le s \le 0.7833$$

in fact , $s = \ln 2 = 0.6931$ (pf : soon)

12.6 ratio test

Let a_n be positive or negative, and suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

note : If $L = 1$, the ratio test does not apply; the series may converge or diverge.

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<u>ex 1</u> : $\sum_{n=1}^{\infty} \frac{1}{2^n}$: converges , geometric series , $r = \frac{1}{2}$

$$a_n = \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} \right| = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

ratio test $\Rightarrow$ series converges

<u>ex 2</u> : $\sum_{n=1}^{\infty} \frac{1}{n}$: diverges , harmonic series

$$a_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot n = 1$$

ratio test does not apply

<u>ex 3</u> : $\sum_{n=1}^{\infty} \frac{1}{n^2}$: converges , p-test , $p = 2$

$$a_n = \frac{1}{n^2} \Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot n^2 = 1$$

ratio test does not apply

<u>ex 4</u> : $\sum_{n=0}^{\infty} \frac{1}{n!}$: converges

$$a_n = \frac{1}{n!} \Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot n! = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

ratio test $\Rightarrow$ series converges

<u>pf</u> (ratio test)

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| \approx L$ when n is sufficiently large

$$\Rightarrow |a_{n+1}| \approx L|a_n|$$

$$|a_{n+2}| \approx L|a_{n+1}| \approx L^2|a_n|$$

$$|a_{n+3}| \approx L|a_{n+2}| \approx L^3|a_n|$$

$$\cdots$$

$$\Rightarrow |a_n| + |a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \cdots$$

$$\sim |a_n| + L|a_n| + L^2|a_n| + L^3|a_n| + \cdots$$

$$= |a_n|(1 + L + L^2 + L^3 + \cdots)$$ <u>ok</u>

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12.8 power series

def

A power series has the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$.

c_n : coefficients , x : variable

note

For a given value of x, the power series may converge or diverge.

---

ex 1

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$: converges for $-1 < x < 1$

pf : geometric series , $r = x$

---

ex 2

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$: converges for $-\infty < x < \infty$

pf : $a_n = \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \text{ for all } x$$

note : This implies that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x.

---

ex 3 (more later)

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \text{ : converges for } -\infty < x < \infty$$

$$2^{2n}(n!)^2 = (2^n)^2(n!)^2 = (2 \cdot 2 \cdot 2 \cdots 2)^2(n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1)^2$$
$$= ((2n)(2n-2)(2n-4)\cdots 6 \cdot 4 \cdot 2)^2$$

pf : $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty} \frac{|x|^2}{2^2(n+1)^2} = 0 \text{ for all } x$$

def : A power series centered at $x = a$ has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

---

ex 1

$$\sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + \cdots \ : \text{ converges for } 0 < x < 2$$

pf : geometric series , $r = x - 1$, $-1 < r < 1 \Leftrightarrow -1 < x - 1 < 1 \Leftrightarrow 0 < x < 2$

---

ex 2

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n} = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots \ : \text{ converges for } 0 \le x < 2$$

pf : $a_n = \dfrac{(x-1)^n}{n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n}\right| = \lim_{n\to\infty} |x-1| \cdot \frac{n}{n+1} = |x-1| < 1$$

For $x = 0$ or $x = 2$, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$ so the ratio test does not apply.

$x = 0 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$: converges by AST

$x = 2 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$: diverges , $p = 1$

---

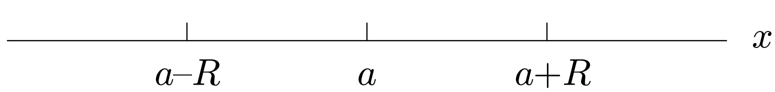
thm (convergence of power series) 38 Tues 11/22

Given a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, there exists $R \ge 0$ such that

1. if $0 < R < \infty$, then the series $\begin{cases} \text{converges for } |x-a| < R \\ \text{diverges for } |x-a| > R, \end{cases}$
2. if $R = \infty$, then the series converges for all x,
3. if $R = 0$, then the series converges for $x = a$ and diverges for $x \neq a$.

note : R is called the radius of convergence of the power series

$|x-a| < R \Leftrightarrow -R < x - a < R \Leftrightarrow a - R < x < a + R$



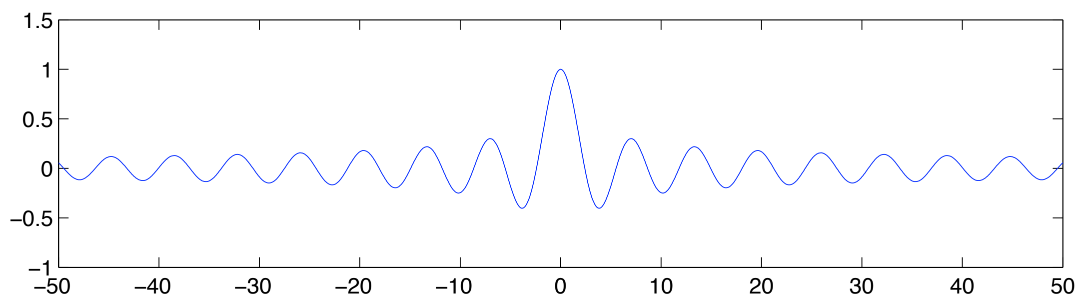
interval of convergence : the set of all points x for which the series converges

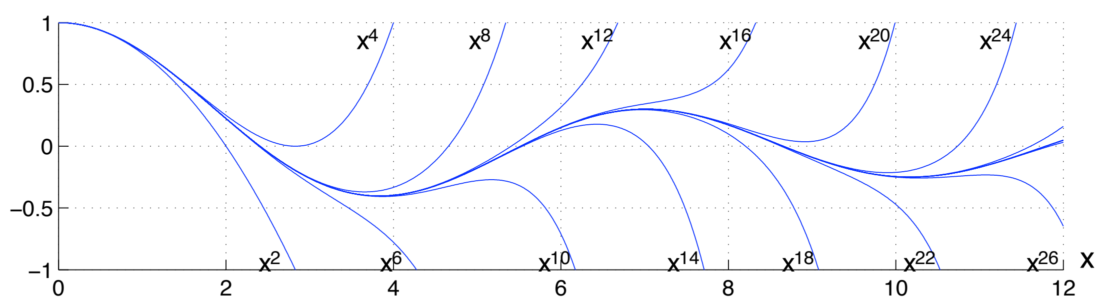
pf : omit

series	center	roc	ioc	sum
$\sum_{n=0}^{\infty} x^n$	$a=0$	$R=1$	$-1<x<1$	$\frac{1}{1-x}$
$\sum_{n=0}^{\infty} (x-1)^n$	$a=1$	$R=1$	$0<x<2$	$\frac{1}{1-(x-1)} = \frac{1}{2-x}$
$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$	$a=1$	$R=1$	$0 \le x < 2$	$-\ln(2-x)$: soon
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$a=0$	$R=\infty$	$-\infty<x<\infty$	e^x : soon
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$a=0$	$R=\infty$	$-\infty<x<\infty$	$J_0(x)$

Bessel function of order zero

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$





vibrations of a drum , ripples on a lake , Math 454 (partial differential equations)

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12.9 power series representation of a function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

problem : given $f(x)$, find c_n

ex

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \text{ for } -1 < x < 1$$

note : Power series can be added, subtracted, multiplied, and divided, and the result is another power series.

ex

$$\frac{1}{(1-x)^2} = \left(\sum_{n=0}^{\infty} x^n\right)^2 = (1 + x + x^2 + x^3 + \cdots)^2$$

$$= (1 + x + x^2 + x^3 + \cdots) \cdot (1 + x + x^2 + x^3 + \cdots)$$

$$= 1 + x(1+1) + x^2(1+1+1) + x^3(1+1+1+1) + \cdots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1) x^n \text{ for } -1 < x < 1$$

term-by-term differentiation and integration of power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\int f(x)dx = c_0 x + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1}$$

thm

1. The power series for $f'(x)$ and $\int f(x)dx$ have the same radius of convergence as the power series for $f(x)$, but the endpoints of the ioc must be checked in each case.

2. Similar results hold for power series centered at $x = a$.

pf : omit , but we will see examples

<u>ex 1</u>

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1$$

<u>differentiate</u>

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n \text{ for } -1 < x < 1$$

note : This agrees with the previous result obtained by multiplication.

<u>integrate</u>

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } -1 \le x < 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } -1 \le x < 1$$

$$x = 1 \Rightarrow \ln 0 = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots \; : \; \text{diverges}$$

$$x = -1 \Rightarrow \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \; : \; \text{converges}$$

$$x = \frac{1}{2} \Rightarrow \ln \frac{1}{2} = -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} - \frac{1}{64} - \cdots = -\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n} = -\ln 2$$

$$\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n2^n} = 0.693147\ldots \; : \; \#1$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 0.693147\ldots \; : \; \#2$$

question : Which series converges faster?

n	s_n , #1	s_n , #2
1	0.5000	1.0000
2	0.6250	0.5000
3	0.6667	0.8333
4	0.6823	0.5833

answer : #1 converges faster

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ex 2 : evaluate $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$

method 1 : FTC

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} = \left(\frac{\sqrt{2}}{8} \ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} \right) + \frac{\sqrt{2}}{4} (\tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1)) \right) \Bigg|_0^{\frac{1}{2}}$$

---

method 2 : Riemann sums

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} \approx \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{2n}\right)^4} \cdot \frac{1}{2n}$$

$a = 0$, $b = \frac{1}{2}$, $\Delta x = \frac{b-a}{n} = \frac{1}{2n}$, $x_i = a + i\Delta x = \frac{i}{2n}$

---

method 3 : series

$$\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = 1 - x^4 + x^8 - x^{12} + \cdots \text{ for } -1 < x < 1$$

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} = \int_0^{\frac{1}{2}} \left(1 - x^4 + x^8 - x^{12} + \cdots\right) dx = \left(x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots \right) \Bigg|_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{5 \cdot 2^5} + \frac{1}{9 \cdot 2^9} - \frac{1}{13 \cdot 2^{13}} + \cdots$$

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} \approx \frac{1}{2} - \frac{1}{5 \cdot 2^5} = \frac{1}{2} - \frac{1}{160} = \frac{79}{160} = 0.49375$$

$$\left| \int_0^{\frac{1}{2}} \frac{dx}{1+x^4} - 0.49375 \right| \le \frac{1}{9 \cdot 2^9} = \frac{1}{9 \cdot 512} = \frac{1}{4608} < \frac{1}{4000} = 0.00025$$

$$\Rightarrow 0.49350 < \int_0^{\frac{1}{2}} \frac{dx}{1+x^4} < 0.49400$$

---

method 4 : Maple

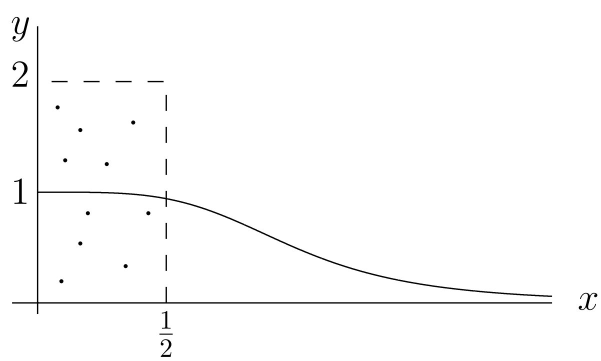
> Int(1/(1+x^4), x=0..1/2); $\quad \int_0^{\frac{1}{2}} \left(\frac{1}{1+x^4} \right) dx$

> evalf(%); $\quad$ 0.4939580511

question : which method does Maple use?

---

method 5 : Monte Carlo integration



$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} \approx \frac{\#\text{ points below graph}}{\#\text{ points total in box}}$$

<u>ex 3</u>

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \;:\; \text{converges for } -\infty < x < \infty$$

question : Can you identify $f(x)$?

$$f'(x) = 1 + x + \frac{x^2}{2} + \frac{x^2}{3!} + \frac{x^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow f'(x) = f(x) \;,\; f(0) = 1 \Rightarrow f(x) = e^x$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for } -\infty < x < \infty$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \text{ for } -\infty < x < \infty$$

$$x = 1 \Rightarrow e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots = 2.71828$$

$$e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \cdots = 0.36788$$

<u>12.10 Taylor series</u>

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$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \;,\; c_n = \,?$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1$$

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2}$$

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots$$

$$f'''(a) = 2\cdot 3c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!}$$

$$\cdots$$

$$f^{(n)}(a) = 2\cdot 3\cdot 4 \cdots n \cdot c_n \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

<u>def</u> : The <u>Taylor series</u> of $f(x)$ at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots .$$

<u>ex 1</u> : $f(x) = \dfrac{1}{1-x}$, $a = 0$: find the Taylor series

$f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$

$f'(x) = (1-x)^{-2} \Rightarrow f'(0) = 1$

$f''(x) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$

$f'''(x) = 3 \cdot 2(1-x)^{-4} \Rightarrow f'''(0) = 3!$

$\dots$

$f^{(n)}(0) = n!$

$$\text{TS} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } -1 < x < 1$$

<u>note</u> : Hence the geometric series is a special case of a Taylor series.

<u>ex 2</u> : $f(x) = \sin x$, $a = 0$: find the TS

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

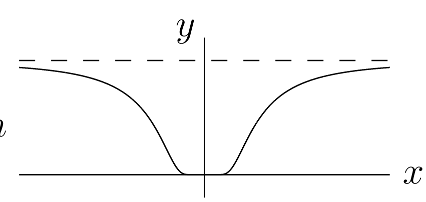
$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$

$$\begin{aligned}\text{TS} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \cancel{f(0)} + f'(0)x + \cancel{\frac{f''(0)}{2}}x^2 + \frac{f'''(0)}{3!}x^3 + \cancel{\frac{f^{(4)}(0)}{4!}}x^4 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \ : \text{ converges for all } x \text{ (ratio test)}\end{aligned}$$

<u>note</u> : We still need to show that the TS converges to the given $f(x)$; in some cases it does, and in some cases it doesn't.

does : $\dfrac{1}{1-x}$, e^x , $\sin x$, $\cos x$, $J_0(x)$, ...

doesn't : $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \Rightarrow f^{(n)}(0) = 0$ for all n



<u>error bound for Taylor approximation</u>

Recall the FTC and related formulas derived on hw.

$$f(x) = f(a) + \int_a^x f'(t)dt$$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)dt$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{(x-t)^2}{2} f'''(t)dt$$

$\cdots$

in general : $f(x) = T_n(x) + R_n(x)$

$T_n(x)$: <u>Taylor polynomial</u> of degree n of $f(x)$ about $x = a$ (partial sum of TS)

$$T_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$R_n(x)$: <u>remainder</u> , <u>error</u> 42 Fri 12/2

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{n+1}(t)dt$$

note : $n = 0 \Rightarrow f(x) = f(a) + \int_a^x f'(t)dt$

$|f(x) - T_n(x)| = |R_n(x)| \le \frac{1}{(n+1)!} M_{n+1} |x-a|^{n+1}$: error bound

where $M_{n+1} = \max |f^{(n+1)}(t)|$

<u>note</u> : The error bound resembles the first neglected term in the TS.

<u>thm</u>

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ for all } x$$

<u>pf</u> : consider $\sin x$, $\cos x$ is similar

we must show that $\lim_{n\to\infty} R_n(x) = 0$ for all x

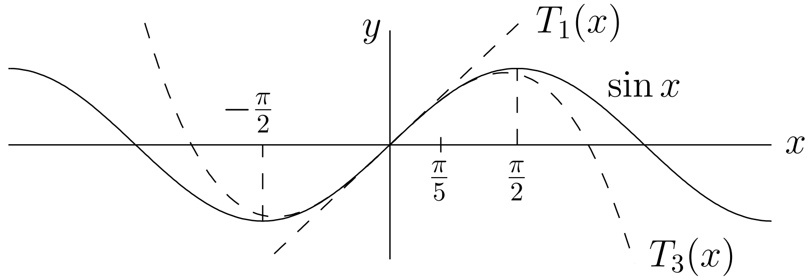
$|R_n(x)| \le \frac{1}{(n+1)!} M_{n+1} |x-a|^{n+1}$

$M_{n+1} = \max |f^{n+1}(t)| = 1 \Rightarrow |R_n(x)| \le \dfrac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$ <u>ok</u>

<u>summary</u>

1. We've shown that the TS for $\sin x$ about $x = 0$ converges to $\sin x$ for all x.
2. The Taylor polynomial $T_n(x)$ is an approximation to $f(x)$; the approximation becomes more accurate as n increases and $x \to a$.

ex 1 : Taylor approximation for $\sin\frac{\pi}{5}$



$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin\frac{\pi}{5} \approx \frac{\pi}{5} = 0.6283 \Rightarrow |\sin\frac{\pi}{5} - \frac{\pi}{5}| \le \frac{1}{3!}\left(\frac{\pi}{5}\right)^3 = 0.0413$$

$$\sin\frac{\pi}{5} \approx \frac{\pi}{5} - \frac{1}{3!}\left(\frac{\pi}{5}\right)^3 = 0.5870 \Rightarrow \left|\sin\frac{\pi}{5} - \left[\frac{\pi}{5} - \frac{1}{3!}\left(\frac{\pi}{5}\right)^3\right]\right| \le \frac{1}{5!}\left(\frac{\pi}{5}\right)^5 = 0.0008161$$

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ex 2 : water waves

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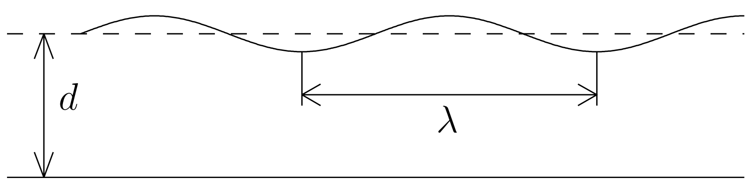
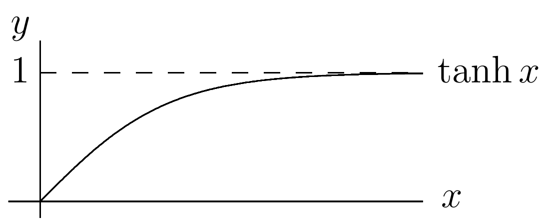


d : depth
λ : wavelength
v : wave speed
g : acceleration due to gravity

given : $v^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi d}{\lambda}$, two important cases : $\begin{cases} d/\lambda \to \infty & : \text{deep water} \\ d/\lambda \to 0 & : \text{shallow water} \end{cases}$



$$f(x) = f(0) + f'(0)x + \cdots \Rightarrow \tanh x \approx \begin{cases} x & \text{for } x \to 0 \\ 1 & \text{for } x \to \infty \end{cases}$$

$$f(x) = \tanh x \Rightarrow f(0) = 0$$

$$f'(x) = \text{sech}^2 x \Rightarrow f'(0) = 1$$

deep water

$$d/\lambda \to \infty \Rightarrow v^2 \approx \frac{g\lambda}{2\pi}\cdot 1 \Rightarrow v \approx \left(\frac{g\lambda}{2\pi}\right)^{1/2}$$

$\Rightarrow$ In deep water, long waves travel faster than short waves.

shallow water

$$d/\lambda \to 0 \Rightarrow v^2 \approx \frac{g\lambda}{2\pi}\cdot\frac{2\pi d}{\lambda} = gd \Rightarrow v \approx (gd)^{1/2}$$

$\Rightarrow$ In shallow water, the wave speed is independent of the wavelength.

<u>ex 3</u> : l'Hopital's rule

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{0}{0}$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

<u>pf</u> : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{\cancel{f(a)} + f'(a)\cancel{(x-a)} + \frac{1}{2}f''(a)(x-a)^{\cancel{2}} + \cdots}{\cancel{g(a)} + g'(a)\cancel{(x-a)} + \frac{1}{2}g''(a)(x-a)^{\cancel{2}} + \cdots} = \dfrac{f'(a)}{g'(a)}$ <u>ok</u>

<u>ex 4</u> : $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = \lim\limits_{x\to 0}\dfrac{x - \frac{1}{3!}x^3 + \cdots}{x} = \lim\limits_{x\to 0}(1 - \frac{1}{3!}x^2 + \cdots) = 1$

<u>12.11 binomial series</u>

$f(x) = (1+x)^k$, $a = 0$

$\text{TS} = f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots$

$f(x) = (1+x)^k \Rightarrow f(0) = 1$

$f'(x) = k(1+x)^{k-1} \Rightarrow f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2} \Rightarrow f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \Rightarrow f'''(0) = k(k-1)(k-2)$

$\ldots$

$$f^{(n)}(0) = \begin{cases} k(k-1)(k-2)\cdots(k-(n-1)) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

<u>thm</u>

$(1+x)^k = 1 + kx + \dfrac{k(k-1)}{2}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \cdots$ for $-1 < x < 1$

For $x = \pm 1$, the series may converge or diverge depending on the value of k.

<u>pf</u> : ratio test , error bound ...

<u>ex</u>

$$(1+x)^{-1} = 1 + (-1)x + \tfrac{(-1)(-2)}{2}x^2 + \tfrac{(-1)(-2)(-3)}{3!}x^3 + \cdots$$
$$= 1 - x + x^2 - x^3 + \cdots$$

$$(1+x)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)x + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots$$
$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

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replace x by $-x^2$

$$(1-x^2)^{-1/2} = 1 - \tfrac{1}{2}(-x^2) + \tfrac{3}{8}(-x^2)^2 - \tfrac{5}{16}(-x^2)^3 + \cdots$$
$$= 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots$$

ex 1 : Einstein's theory of special relativity

m_0 : mass of an object at rest

v : velocity of the object

m : mass of the object when moving at velocity v

c : velocity of light $= 3 \cdot 10^8\,\mathrm{m/s}$

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \Rightarrow \begin{cases} \text{if } v \to 0 \text{ , then } m \to m_0 \\ \text{if } v \to c \text{ , then } m \to \infty \end{cases}$$

kinetic energy = (total energy at velocity v) – (total energy at rest)

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)$$

assume $v << c$, set $x = \dfrac{v}{c} \Rightarrow \left(1 - \dfrac{v^2}{c^2}\right)^{-1/2} = \left(1 - x^2\right)^{-1/2} = 1 + \frac{1}{2}x^2 + \cdots$

$$K = m_0c^2\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots - 1\right) \approx m_0c^2 \cdot \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}m_0v^2$$

$\Rightarrow$ special relativity reduces to Newtonian mechanics for $v << c$.

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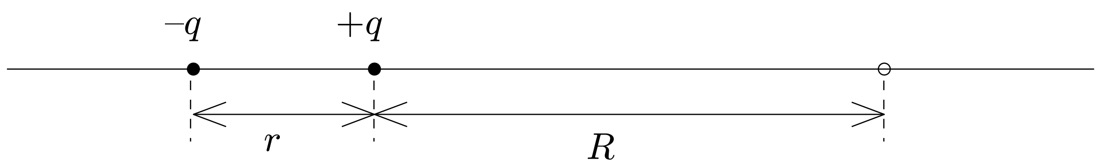
ex 2 : Two ions with electric charge $+q$ and $-q$ form an electric dipole.



The induced electric field at the observation point is $E = \dfrac{q}{R^2} + \dfrac{-q}{(R+r)^2}$.

This expression can be simplified for $R >> r$.

$$E = \frac{q}{R^2} - \frac{q}{(R+r)^2} = \frac{q}{R^2}\left(1 - \frac{1}{(1 + r/R)^2}\right) = \frac{q}{R^2}\left(1 - \left(1 + \frac{r}{R}\right)^{-2}\right)$$

set $k = -2$, $x = \dfrac{r}{R} \Rightarrow x << 1$

$$(1+x)^{-2} = 1 + (-2)x + \tfrac{(-2)(-3)}{2}x^2 + \cdots = 1 - 2x + 3x^2 - \cdots$$

$$E = \frac{q}{R^2}\left(1 - \left(1 - 2\frac{r}{R} + 3\frac{r^2}{R^2} - \cdots\right)\right) = \frac{q}{R^2}\left(2\frac{r}{R} - 3\frac{r^2}{R^2} + \cdots\right)$$

$\Rightarrow E \approx \dfrac{2qr}{R^3}$: far-field approximation for the electric field of a dipole

<u>recall</u>

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-(n-1))}{n!}x^n$$

<u>special case</u>

For the remainder of this section, let k be a positive integer (i.e. $k = 1,\ 2,\ 3,\ \ldots$). Then $f(x) = (1+x)^k$ is a polynomial of degree k and the series terminates at index $n = k$.

<u>ex</u>

$k = 2 \ \Rightarrow\ (1+x)^2 = 1 + 2x + x^2$ <u>ok</u>

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<u>alternative notation</u>

$$\frac{k(k-1)\cdots(k-(n-1))}{n!} \cdot \frac{(k-n)(k-(n+1))\cdots 3\cdot 2\cdot 1}{(k-n)(k-(n+1))\cdots 3\cdot 2\cdot 1}$$
$$= \frac{k!}{n!(k-n)!} = \binom{k}{n} \ : \ \underline{\text{binomial coefficient}}\ ,\ \binom{k}{0} = \frac{k!}{0!k!} = 1$$

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n = \sum_{n=0}^{k} \frac{k!}{n!\,(k-n)!}\, x^n \ \text{ for } \ k = 1, 2, 3, \ldots$$

<u>ex</u>

$k = 4\ ,\ (1+x)^4$

$$\binom{4}{0} = \frac{4!}{0!\,4!} = 1\ ,\ \binom{4}{1} = \frac{4!}{1!\,3!} = 4\ ,\ \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4\ ,\ \binom{4}{4} = \frac{4!}{4!\,0!} = 1$$

$$(1+x)^4 = \sum_{n=0}^{4} \binom{4}{n} x^n = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4$$
$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$
$$= (1+x)(1+x)(1+x)(1+x) \quad \underline{\text{ok}}$$

<u>interpretation</u>

$$\binom{k}{n} = \frac{k!}{n!\,(k-n)!} = \frac{k(k-1)\cdots(k-n+1)}{n!} = \underline{k \text{ choose } n}$$

$=$ number of ways of choosing n objects from a set of k objects, disregarding the order in which they are chosen

<u>ex</u> : $k = 4$, $n = 2$

4 objects : $\{A, B, C, D\}$

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \cdot 3}{2} = 6$$

$\Rightarrow$ there are 6 ways of choosing 2 objects from a set of 4 objects

$\{AB, AC, AD, BC, BD, CD\}$ note : BA is the same as AB

<u>pf</u> (general k , n)

$(1+x)^k = (1+x)(1+x)\cdots(1+x)$: k factors

$$= \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{n}x^n + \cdots + \binom{k}{k}x^k$$

$\binom{k}{n}$ = coefficient of x^n

= number of ways of choosing n factors from k factors <u>ok</u>

<u>thm</u>

$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$: <u>binomial formula</u>

<u>pf</u>

$$(a+b)^k = a^k\left(1 + \frac{b}{a}\right)^k = a^k \sum_{n=0}^{k} \binom{k}{n} \left(\frac{b}{a}\right)^n = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n \quad \underline{\text{ok}}$$

<u>ex</u>

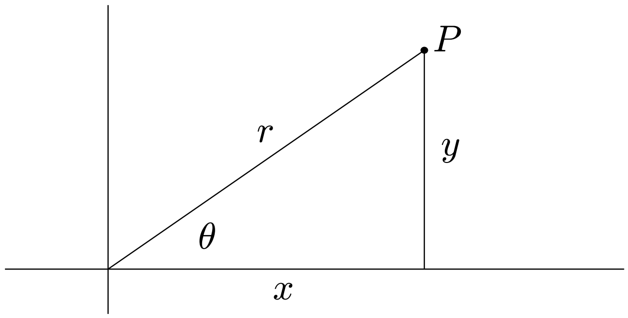
$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \text{ , check} \ldots$$

11.3 polar coordinates

(x,y) : Cartesian coordinates of a point P

(r,θ) : polar coordinates of P



$$r=\sqrt{x^2+y^2}\ ,\ \theta=\tan^{-1}\frac{y}{x}$$
$$x=r\cos\theta\ ,\ y=r\sin\theta$$

Appendix G. complex numbers

$$ax^2+bx+c=0\ \Rightarrow\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\ :\ \text{quadratic formula}$$

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pf

$$ax^2+bx+c\ =\ a\Big(x^2+\frac{b}{a}x+\frac{c}{a}\Big)\ =\ a\Big(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\Big)$$
$$=\ a\Big(\Big(x+\frac{b}{2a}\Big)^2-\frac{b^2}{4a^2}+\frac{c}{a}\Big)\ =\ 0$$

$$\Rightarrow\ \Big(x+\frac{b}{2a}\Big)^2\ =\ \frac{b^2}{4a^2}-\frac{c}{a}\ =\ \frac{b^2-4ac}{4a^2}\ \Rightarrow\ x+\frac{b}{2a}\ =\ \pm\frac{\sqrt{b^2-4ac}}{2a}\qquad \text{ok}$$

ex : $x^2-2x+2=0$

$x^2-2x+2=(x-1)^2+1>0\ \Rightarrow$ no real solutions

$$x\ =\ \frac{2\pm\sqrt{4-4\cdot 2}}{2}\ =1\pm\frac{\sqrt{-4}}{2}\ =1\pm\frac{\sqrt{-1}\sqrt{4}}{2}\ =1\pm i$$

def

A complex number has the form $z=x+iy$ where x,y are real and $i=\sqrt{-1}$.

x : real part , y : imaginary part , $\overline{z}=x-iy$: complex conjugate of z

arithmetic : $z_1+z_2\ ,\ z_1-z_2\ ,\ z_1z_2\ ,\ \dfrac{z_1}{z_2}$

ex

$$(1+i)+(1-i)\ =\ 2$$
$$(1+i)-(1-i)\ =\ 2i$$
$$(1+i)(1-i)\ =\ 1-i+i-i^2=2$$
$$\frac{1+i}{1-i}\ =\ \frac{1+i}{1-i}\cdot\frac{1+i}{1+i}\ =\ \frac{1+i+i+i^2}{2}\ =\ \frac{2i}{2}\ =\ i$$

<u>Euler's formula</u> : $e^{i\theta} = \cos\theta + i\sin\theta$

<u>pf</u> : $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!}\cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) = \cos\theta + i\sin\theta \quad \underline{\text{ok}}$$

<u>ex</u>

$e^{\pi i} = \cos\pi + i\sin\pi = -1$

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$

$e^{\pi i/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

<u>note</u>

1. A complex number $z = x + iy$ can be plotted as a point in the xy-plane using (x, y) as Cartesian coordinates.

2. A complex number can be written using polar coordinates, $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

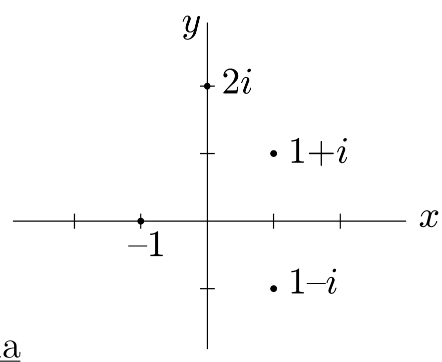
<u>ex</u>

$z = 2i = 2e^{\pi i/2}$

$z = -1 = e^{\pi i}$

$z = 1 + i = \sqrt{2}\, e^{\pi i/4}$

$z = 1 - i = \sqrt{2}\, e^{-\pi i/4}$



<u>applications of Euler's formula</u>

1. double-angle formulas

$$\left(e^{i\theta}\right)^2 = e^{2i\theta} = \cos 2\theta + i\sin 2\theta$$

$$\left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$$

$$\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta \,,\ \sin 2\theta = 2\sin\theta\cos\theta$$

2. addition formulas for sine and cosine

$$e^{ia}\cdot e^{ib} = e^{ia+ib} = e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$

$$e^{ia}\cdot e^{ib} = (\cos a + i\sin a)\cdot(\cos b + i\sin b)$$

$$= (\cos a\,\cos b - \sin a\,\sin b) + i\,(\cos a\,\sin b + \sin a\,\cos b)$$

$$\Rightarrow \begin{cases} \cos(a+b) = \cos a\,\cos b - \sin a\,\sin b \\ \sin(a+b) = \sin a\,\cos b + \cos a\,\sin b \quad \underline{\text{ok}} \end{cases}$$