1
Tues
9/5

preview : integrals, differential equations, series

part 1 : integrals

goal : prepare for FTC

1.1 sigma notation

def : $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$ : sum , series

$i$ : index , $a_i$ : terms , $1, n$ : limits

ex

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$$

note : A series can be written in different ways.

ex

$$\sum_{i=1}^{5} i = \sum_{j=0}^{4} (j+1) = 1 + 2 + 3 + 4 + 5 = 15$$

set $j = i - 1$

properties of series

1. $\sum_{i=1}^{n} ca_i = c \cdot \sum_{i=1}^{n} a_i$ , where $c$ is any constant

2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

3. $\sum_{i=1}^{n} a_i b_i \neq \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i$

pf

1. $\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c \cdot \sum_{i=1}^{n} a_i$ ok

2. hw

3. consider $n = 2$

$$\sum_{i=1}^{2} a_i b_i = a_1 b_1 + a_2 b_2$$

$$\sum_{i=1}^{2} a_i \cdot \sum_{i=1}^{2} b_i = (a_1 + a_2) \cdot (b_1 + b_2) = a_1 b_1 + \underline{a_1 b_2 + a_2 b_1} + a_2 b_2$$ ok

<u>thm</u>

1. $\sum_{i=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 = n$

2. $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ , e.g. $n = 5 \overset{\text{implies}}{\Rightarrow} S = 15$

3. $\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ , e.g. $n = 5 \Rightarrow S = 55$

<u>pf</u>

1. <u>ok</u>

2. $(i+1)^2 - i^2 = i^2 + 2i + 1 - i^2 = 2i + 1$

$$\sum_{i=1}^{n}((i+1)^2 - i^2) = \sum_{i=1}^{n}(2i+1) = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = \boxed{2S + n} \quad , \text{ where } S = \sum_{i=1}^{n} i$$

$$\sum_{i=1}^{n}((i+1)^2 - i^2) = (\cancel{2^2} - 1^2) + (\cancel{3^2} - \cancel{2^2}) + (\cancel{4^2} - \cancel{3^2}) + \cdots + ((n+1)^2 - \cancel{n^2})$$

$$= (n+1)^2 - 1^2 \text{ : example of a \underline{telescoping series}}$$

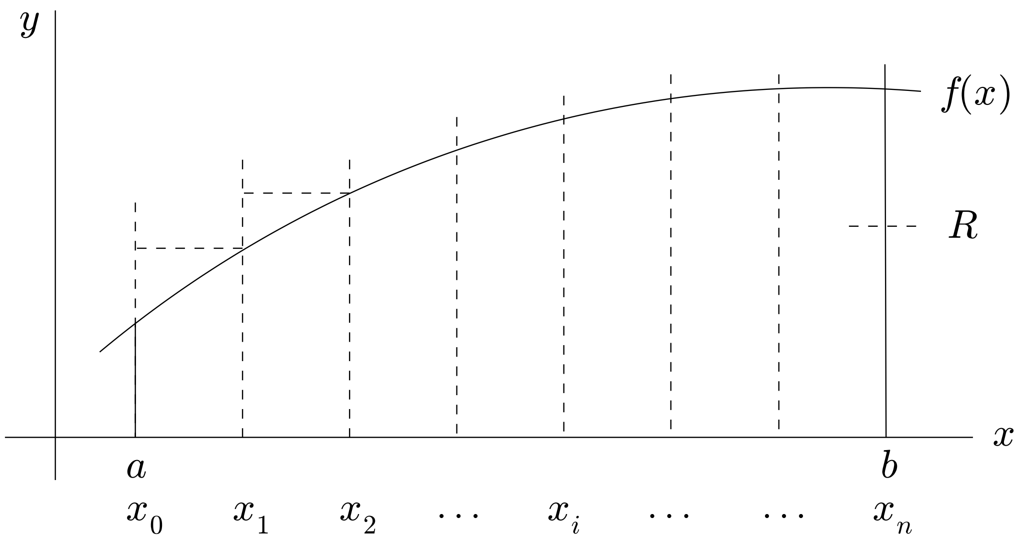
$$= \boxed{n^2 + 2n}$$

$$\Rightarrow 2S + n = n^2 + 2n \Rightarrow 2S = n^2 + n = n(n+1) \Rightarrow S = \frac{n(n+1)}{2} \quad \underline{\text{ok}}$$

3. $(i+1)^3 - i^3 = \cdots$ , hw

1.2 area

Given $f(x) \geq 0$, $a \leq x \leq b$.




$R$ = region in the $xy$-plane between $y = 0$ and $y = f(x)$ for $a \leq x \leq b$

$= \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$

problem : find the area of $R$

idea : approximate by rectangles

choose $n \geq 1$ , set $\Delta x = \dfrac{b-a}{n}$ , $x_i = a + i\Delta x$ , $i = 0, \ldots, n$ 2 Wed 9/6

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$\cdots$

$x_n = a + n\Delta x = a + b - a = b$

area of $i$th rectangle $= f(x_i)\Delta x$ for $i = 1, \ldots, n$

area of region $R \approx \sum_{i=1}^{n} f(x_i)\Delta x$

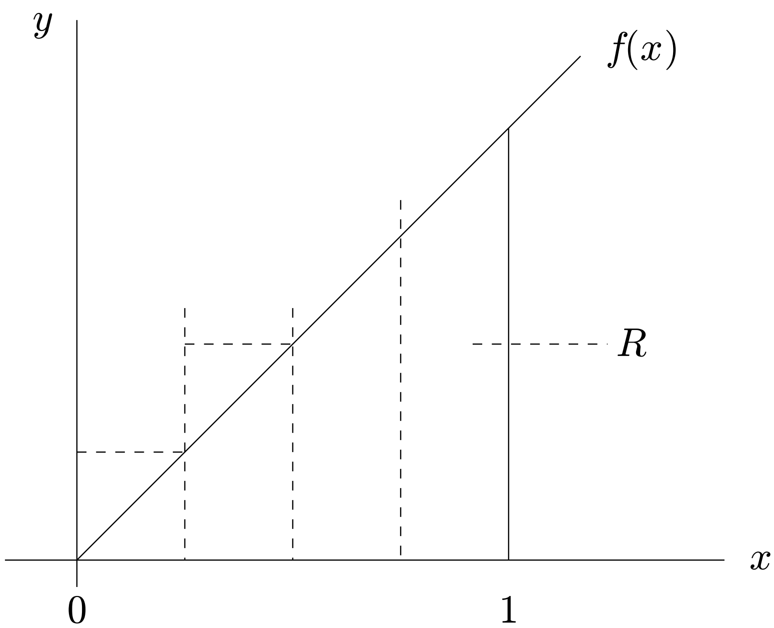
↑

approximately

area of region $R = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

<u>ex</u>

$f(x) = x \ , \ 0 \le x \le 1$




$R = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$

of course , area $= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2}$

$a = 0 \ , \ b = 1 \ , \ \Delta x = \frac{b-a}{n} = \frac{1}{n}$

$x_i = a + i\Delta x = \frac{i}{n}$

$f(x_i) = x_i = \frac{i}{n}$

$$\sum_{i=1}^{n} f(x_i)\Delta x \ = \ \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} \ = \ \frac{1}{n^2} \sum_{i=1}^{n} i \ = \ \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \ = \ \frac{n+1}{2n} \ = \ \frac{1}{2} + \frac{1}{2n}$$

area $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}$ <u>ok</u>

<u>1.3 definite integral</u>

As before, given $f(x)$ , $a \le x \le b$ , set $\Delta x = \dfrac{b-a}{n}$ , $x_i = a + i\Delta x$ , $i = 0, \ldots, n$.


Let $x_i^*$ be any point such that $x_{i-1} \le x_i^* \le x_i$.

Then $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ is a <u>Riemann sum</u>.

<u>ex</u>

$x_i^* = x_i$ : right-hand RS

$x_i^* = x_{i-1}$ : left-hand RS

$x_i^* = \dfrac{x_{i-1} + x_i}{2}$ : midpoint RS

<u>def</u> : $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$ : <u>definite integral</u> $= \begin{cases} \text{area, volume, } \ldots \\ \text{distance, work, } \ldots \\ \text{probability, } \ldots \end{cases}$

<u>ex</u>

$$\int_0^1 dx = \lim_{n\to\infty} \sum_{i=1}^{n} 1 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} 1 = \lim_{n\to\infty} \frac{1}{n} \cdot n = \lim_{n\to\infty} 1 = 1$$

$$a = 0 \ , \ b = 1 \ , \ \Delta x = \frac{b-a}{n} = \frac{1}{n} \ , \ x_i = a + i\Delta x = \frac{i}{n} \ , \ f(x) = 1$$

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$$\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \cdots = \frac{1}{2}$$

$$f(x) = x \ , \ f(x_i) = x_i = \frac{i}{n}$$

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$$\int_0^1 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \ldots = \frac{1}{3}$$

$$f(x) = x^2 \ , \ f(x_i) = x_i^2 = \left(\frac{i}{n}\right)^2$$

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$$\int_0^1 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(e^{1/n}\right)^i = \cdots = e - 1$$

$f(x) = e^x$ , $f(x_i) = e^{x_i} = e^{i/n}$

↑ geometric series (hw2)

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def : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ : right-hand RS , $\lim_{n\to\infty} R_n = \int_a^b f(x)\,dx$

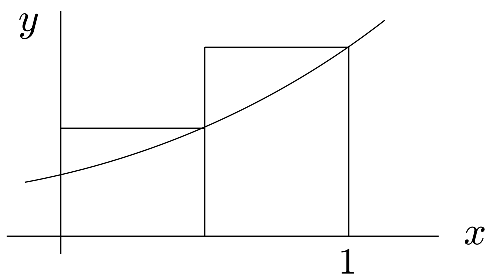
$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$ : midpoint RS , $\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx$

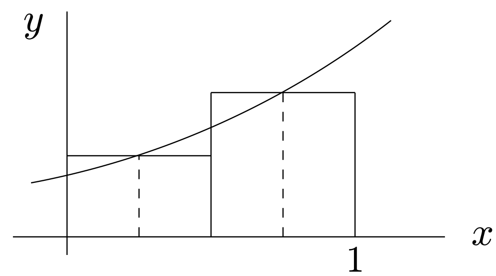
question : For a given value of $n$, which approximation is more accurate?

ex : $\int_0^1 e^x dx = e - 1 = 1.71828183 = I$

| $n$ | $\Delta x$ | $R_n$ | $\lvert I - R_n\rvert$ | $M_n$ | $\lvert I - M_n\rvert$ |
|---|---|---|---|---|---|
| 1 | 1 | 2.7183 | 1.0000 | 1.6487 | 0.0696 |
| 2 | 0.5 | 2.1835 | 0.4652 | 1.7005 | 0.0178 |
| 4 | 0.25 | 1.9420 | 0.2237 | 1.7138 | 0.0045 |

Hence the midpoint RS is more accurate than the right-hand RS. Why? Consider $n = 2$.




note : If $\Delta x$ decreases by a factor of 1/2,
then the error in $R_n$ decreases by a factor of approximately 1/2,
and $\cdots\cdots$ " $\cdots\cdots$ $M_n$ $\cdots\cdots\cdots\cdots\cdots\cdots$ " $\cdots\cdots\cdots\cdots\cdots\cdots$ 1/4.

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properties of the definite integral

1. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
2. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. If $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

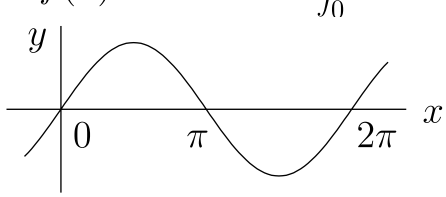
pf

1. $\int_a^b cf(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} cf(x_i)\Delta x = c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = c\int_a^b f(x)\,dx$ ok

2. hw2 , 3. , 4. omit

note : If $f(x)$ changes sign, then $\int_a^b f(x)\,dx$ = signed area.

ex : $f(x) = \sin x \Rightarrow \int_0^\pi \sin x\,dx > 0$ , $\int_\pi^{2\pi} \sin x\,dx < 0$ , $\int_0^{2\pi} \sin x\,dx = 0$




group work : $R = \{(x,y) : 1 \le x \le 2, 0 \le y \le x\}$ , sketch $R$ , find area by RS

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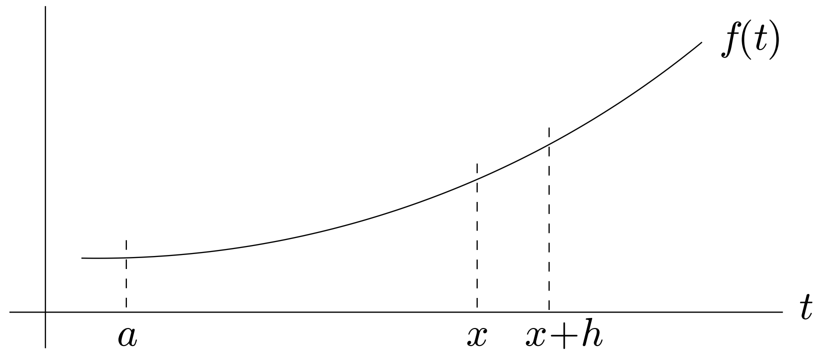
1.4 FTC

FTC, part 1

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

pf




define : $F(x) = \int_a^x f(t)\,dt$ , we need to show that $F'(x) = f(x)$

recall : $F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$

$$F(x+h) = \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt = F(x) + \int_x^{x+h} f(t)\,dt$$

$\Rightarrow F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt \approx f(t^*)\cdot h$ , where $x \le t^* \le x+h$

$$\Rightarrow \frac{F(x+h)-F(x)}{h} \approx f(t^*)$$

$\Rightarrow F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(t^*) = f(x)$ ok (if $f$ is continuous)

ex : $\frac{d}{dx}\int_0^x t\,dt = x$

check

$$\int_0^x t\,dt = \lim_{n\to\infty}\sum_{i=1}^n t_i\Delta t = \lim_{n\to\infty}\sum_{i=1}^n \frac{ix}{n}\cdot\frac{x}{n} = \lim_{n\to\infty}\frac{x^2}{n^2}\sum_{i=1}^n i = \lim_{n\to\infty}\frac{x^2}{n^2}\cdot\frac{n(n+1)}{2} = \frac{x^2}{2}$$

$a = 0$ , $b = x$ , $\Delta t = \frac{b-a}{n} = \frac{x}{n}$ , $t_i = a + i\Delta t = \frac{ix}{n}$

$\Rightarrow \frac{d}{dx}\int_0^x t\,dt = \frac{d}{dx}\left(\frac{x^2}{2}\right) = x$ ok

<u>FTC, part 2</u>

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

<u>pf</u>

$$\Delta x = \frac{b-a}{n}\ ,\ x_i = a + i\Delta x$$

$$\sum_{i=1}^n f'(x_i)\Delta x \approx \sum_{i=1}^n \left(\frac{f(x_i+\Delta x) - f(x_i)}{\cancel{\Delta x}}\right)\cancel{\Delta x}$$

$$= \sum_{i=1}^n (f(x_{i+1}) - f(x_i)) = f(x_{n+1}) - f(x_1) \quad : \quad \text{telescoping sum}$$

$$x_i + \Delta x = a + i\Delta x + \Delta x = a + (i+1)\Delta x = x_{i+1}$$

$$x_1 = a + \Delta x$$

$$x_{n+1} = a + (n+1)\Delta x = a + n\Delta x + \Delta x = a + b - a + \Delta x = b + \Delta x$$

$$\Rightarrow \int_a^b f'(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f'(x_i)\Delta x = \lim_{n\to\infty}(f(b+\Delta x) - f(a+\Delta x))$$

$$= f(b) - f(a) \quad \underline{\text{ok}} \ \ (\text{if } f \text{ is continuous})$$

<u>note</u>

1. We write $\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$.

2. To evaluate $\int_a^b f(x)\,dx$, if we can find a function $F(x)$ such that $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

We say that $F(x)$ is an <u>antiderivative</u> of $f(x)$ and we write $\int f(x)\,dx = F(x) + C$.

<u>ex</u>

1. recall : $\displaystyle\int_0^1 x\,dx = \lim_{n\to\infty}\sum_{i=1}^n \frac{i}{n}\,\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^2}\,\frac{n(n+1)}{2} = \frac{1}{2}$

now we can use the FTC : $\displaystyle\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

2. $\displaystyle\int_{-1}^1 x^2\,dx = \frac{x^3}{3}\Big|_{-1}^1 = \frac{1}{3} - \left(\frac{-1}{3}\right) = \frac{2}{3}$

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| $\underline{f(x)}$ | $\underline{F(x)}$ |
|---|---|
| $x^n$ | $\dfrac{x^{n+1}}{n+1}$ , $n \neq -1$ |
| $x^{-1}$ | $\ln x$ |
| $\ln x$ | $x \ln x - x$ |
| $e^x$ | $e^x$ |
| $\sin x$ | $-\cos x$ |
| $\cos x$ | $\sin x$ |
| $\sinh x = \dfrac{e^x - e^{-x}}{2}$ | $\cosh x$ } more later |
| $\cosh x = \dfrac{e^x + e^{-x}}{2}$ | $\sinh x$ } more later |
| $\dfrac{1}{x^2+1}$ | $\tan^{-1} x$ |
| $e^{x^2}$ | $\int_a^x e^{t^2}\, dt$ : cannot be expressed in terms of elementary functions |

<u>final comment on FTC</u>

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$$

<u>pf</u>

Set $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$ and $F(g(x)) = \int_a^{g(x)} f(t)dt$.

$$\Rightarrow \frac{d}{dx}\int_a^{g(x)} f(t)dt = \frac{d}{dx}F(g(x)) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x) \quad \underline{\text{ok}}$$

<u>ex</u>

$$\frac{d}{dx}\int_a^{x^2} e^t dt = e^{x^2} \cdot 2x$$

<u>check</u>

$$\int_a^{x^2} e^t dt = e^t \Big|_a^{x^2} = e^{x^2} - e^a \Rightarrow \frac{d}{dx}\int_a^{x^2} e^t dt = \frac{d}{dx}\left(e^{x^2} - e^a\right) = e^{x^2} \cdot 2x \quad \underline{\text{ok}}$$

1.5 work

work = force × distance , force = mass × acceleration

| units | metric | British |
|---|---|---|
| mass | kilogram : kg | slug : ? |
| distance | meter : m | foot : ft |
| time | second : s | second : s |
| force | Newton : N = kg · m/s$^2$ | pound : lb |
| work | Joule : J = N · m | foot-pound : ft-lb |

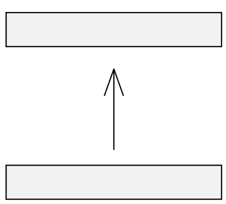
conversion

$1\,\text{m} = 3.28\,\text{ft}$ , $1\,\text{N} = 0.225\,\text{lb} \;\Rightarrow\; 1\,\text{J} = 0.738\,\text{ft-lb}$

$g = 9.8\ \text{m/s}^2 = 32.2\ \text{ft/s}^2$

ex

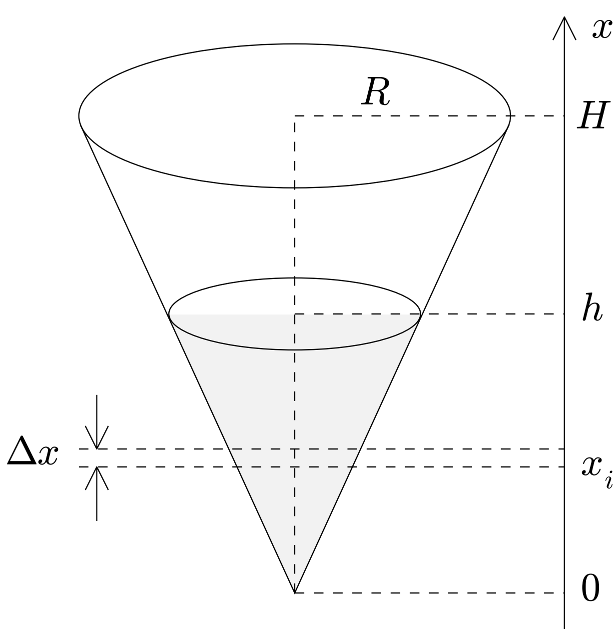
Find the work done in lifting a 1 kg book to a height of 1 m above a table.



$$W = \text{force} \times \text{distance} = mg \times d = 1\,\text{kg} \times 9.8\,\text{m/s}^2 \times 1\,\text{m} = 9.8\,\text{J}$$

ex

A water tank has the shape of an inverted cone.




$R$ : base radius of cone

$H$ : height of cone

$h$ : water level

$\rho$ : water density

problem : Find the work done in pumping the water to the top of the tank.
idea : Think of the water as a stack of books.

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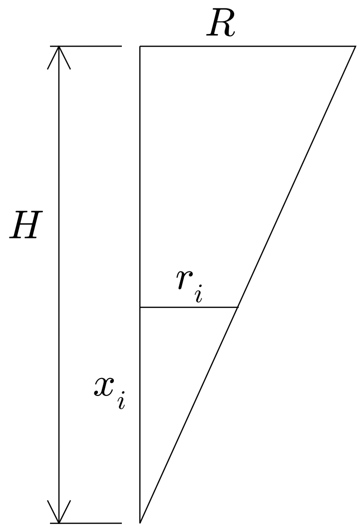
$\Delta x = \frac{h}{n}$ : width of a water layer

$x_i = i\Delta x$ : height of $i$th layer ($x_0 = 0$, $x_n = h$)

work = force × distance = mass × acceleration × distance

mass of water in $i$th layer = density × volume $\approx \rho \cdot \pi r_i^2 \Delta x$

$r_i$ : radius of $i$th layer




$$\frac{R}{H} = \frac{r_i}{x_i} \;\Rightarrow\; r_i = x_i \frac{R}{H}$$

check : $x_i = 0 \;\Rightarrow\; r_i = 0$

$x_i = H \;\Rightarrow\; r_i = R$ <u>ok</u>

force acting on $i$th layer $= \rho\pi r_i^2 \Delta x \cdot g$

work done in raising $i$th layer $= \rho g \pi r_i^2 \Delta x \cdot (H - x_i)$

work done in raising entire water volume

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \rho g \pi r_i^2 (H - x_i)\Delta x = \int_0^h \rho g \pi x^2 \frac{R^2}{H^2}(H - x)dx$$

$$\int_0^h x^2(H-x)dx = \int_0^h (x^2 H - x^3)dx = \left(\frac{x^3}{3}H - \frac{x^4}{4}\right)\Big|_0^h = \frac{h^3}{3}H - \frac{h^4}{4} = \frac{h^3}{12}(4H - 3h)$$

$W = \rho g \pi \frac{R^2}{H^2} \cdot \frac{h^3}{12}(4H - 3h)$ , check : $h = 0 \Rightarrow W = 0$ <u>ok</u>

plug in numbers

$R = 4\,\mathrm{m}$ , $H = 10\,\mathrm{m}$ , $h = 8\,\mathrm{m}$ , $\rho = 1000\,\mathrm{kg/m^3}$ , $g = 9.8\,\mathrm{m/s^2}$

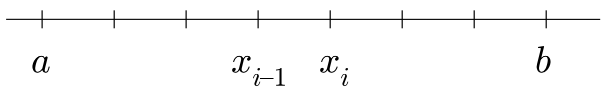
$$W = 10^3\,(9.8)\,(3.14)\,\frac{16}{10^2} \cdot \frac{8^3}{12}\,(40 - 24)\,\frac{\mathrm{kg}}{\cancel{\mathrm{m}}^3}\,\frac{\mathrm{m}}{\mathrm{s}^2}\,\frac{\cancel{\mathrm{m}}^2}{\cancel{\mathrm{m}}^2}\,\cancel{\mathrm{m}}^3 \cdot \mathrm{m}$$

$$\approx 10 \cdot 10 \cdot \cancel{3} \cdot \frac{4}{\cancel{3}} \cdot 5 \cdot 10^2 \cdot 16\,\mathrm{J} \;\approx\; 3.2 \cdot 10^6\,\mathrm{J} \;=\; 3.2\,\mathrm{MJ}$$

<u>note</u>

In these examples, the force (due to gravity) is assumed to be constant as the mass is raised (book, water), but in general the force may depend on the displacement of the mass.

<u>ex</u>: Compute the work done in moving an object from $x = a$ to $x = b$, subject to a variable force $f(x)$.



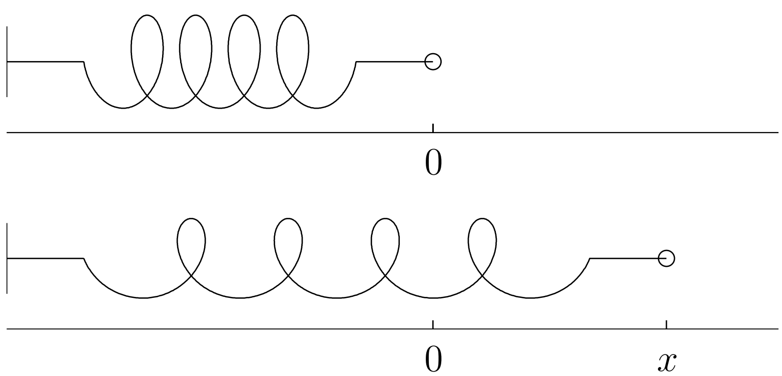

, $\Delta x = \dfrac{b-a}{n}$ , $x_i = a + i\Delta x$

work done in moving object from $x_{i-1}$ to $x_i \approx f(x_i)\Delta x$

................ " .............. $a$ to $b \approx \sum_{i=1}^{n} f(x_i)\Delta x$

$$\Rightarrow \; W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

<u>ex</u>: A spring is stretched $x$ units from its natural length.




<u>Hooke's law</u> : The force needed to stretch a spring is proportional to the displacement of the spring from its natural length.

$f(x) = kx$ , $k$ : <u>spring constant</u> , $x$ : displacement from natural length

<u>ex</u>: A 40 N force is needed to stretch a spring from its natural length of 10 cm to a length of 15 cm. 7 Fri 9/15

a) Find the work done in stretching the spring from length 10 cm to 15 cm.

$$f(x) = kx \;\Rightarrow\; 40\,\text{N} = k\cdot(15\,\text{cm} - 10\,\text{cm}) = k\cdot 5\,\text{cm} \;\Rightarrow\; k = 8\,\text{N/cm}$$

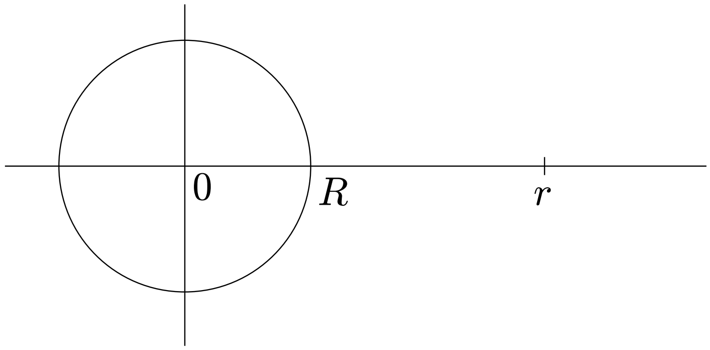
$$W = \int_a^b f(x)dx = \int_0^5 kxdx = k\frac{x^2}{2}\bigg|_0^5 = 8\left(\frac{25}{2} - 0\right) = 100\frac{\text{N}}{\text{cm}}\,\text{cm}^2 = 1\,\text{N m} = 1\,\text{J}$$

b) Find ................ " .................. from length 15 cm to 20 cm.

$$W = \int_a^b f(x)dx = \int_5^{10} kxdx = k\frac{x^2}{2}\bigg|_5^{10} = \cdots = 3\,\text{J}$$

<u>ex</u>

Find the work done in moving a particle from the Earth's surface to $\infty$.




$R$ : radius of Earth

$r$ : distance from particle to center of Earth

$f(r) = \dfrac{GMm}{r^2}$ : force on particle due to gravity (Newton)

$G$ : gravitational constant

$M$ : Earth mass

$m$ : particle mass

$$W = \int_R^\infty f(r)dr = \int_R^\infty GMm\frac{dr}{r^2} = GMm \cdot \frac{-1}{r}\bigg|_R^\infty = GMm \cdot \left(0 - \frac{-1}{R}\right) = \frac{GMm}{R}$$

↑
<u>improper integral</u> , more later

note : We can compute the <u>escape velocity</u> of the particle.

work = kinetic energy

$$\frac{GMm}{R} = \frac{1}{2}mv_{\text{esc}}^2 \;\Rightarrow\; v_{\text{esc}} = \left(\frac{2GM}{R}\right)^{1/2}$$

$$v_{\text{esc}} = \left(2 \cdot \frac{6.67 \cdot 10^{-11}\text{N} \cdot \text{m}^2/\text{kg}^2 \times 5.97 \cdot 10^{24}\text{kg}}{6.37 \cdot 10^6\,\text{m}}\right)^{1/2}$$

$$\approx 11\,\frac{\text{km}}{\text{s}} = 33 \cdot \text{speed of sound in air at STP}$$

note : $R \to 0$ : black hole , $v_{\text{esc}} \to \infty$ : impossible (Einstein)

$\Rightarrow$ it is impossible to escape the gravitational field of a black hole

1.6 improper integrals

def

$$\int_a^b f(x)\,dx \text{ is a proper integral if } \begin{cases} (a,b) \text{ is a bounded interval} \\ \text{and} \\ f(x) \text{ is a bounded function for } a \le x \le b \end{cases}$$

$$\text{otherwise, } \int_a^b f(x)\ dx \text{ is an improper integral, i.e. if } \begin{cases} a = -\infty \text{ or } b = \infty \\ \text{or} \\ f(x) \to \pm\infty \text{ in } (a,b) \end{cases}$$

An improper integral is evaluated by taking a limit of proper integrals. If the limit is finite, the integral converges; otherwise, it diverges.

ex 1 : $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\Big(-\frac{1}{b}+1\Big) = 1$ : converges

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$\Rightarrow$ the area under the graph of $y = \dfrac{1}{x^2}$ from $x=1$ to $x=\infty$ is finite

short cut : $\displaystyle\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty}+1 = 1$

---

ex 2 : $\displaystyle\int_1^\infty \frac{dx}{x} = \ln x\Big|_1^\infty = \ln\infty - \ln 1 = \infty$ : diverges

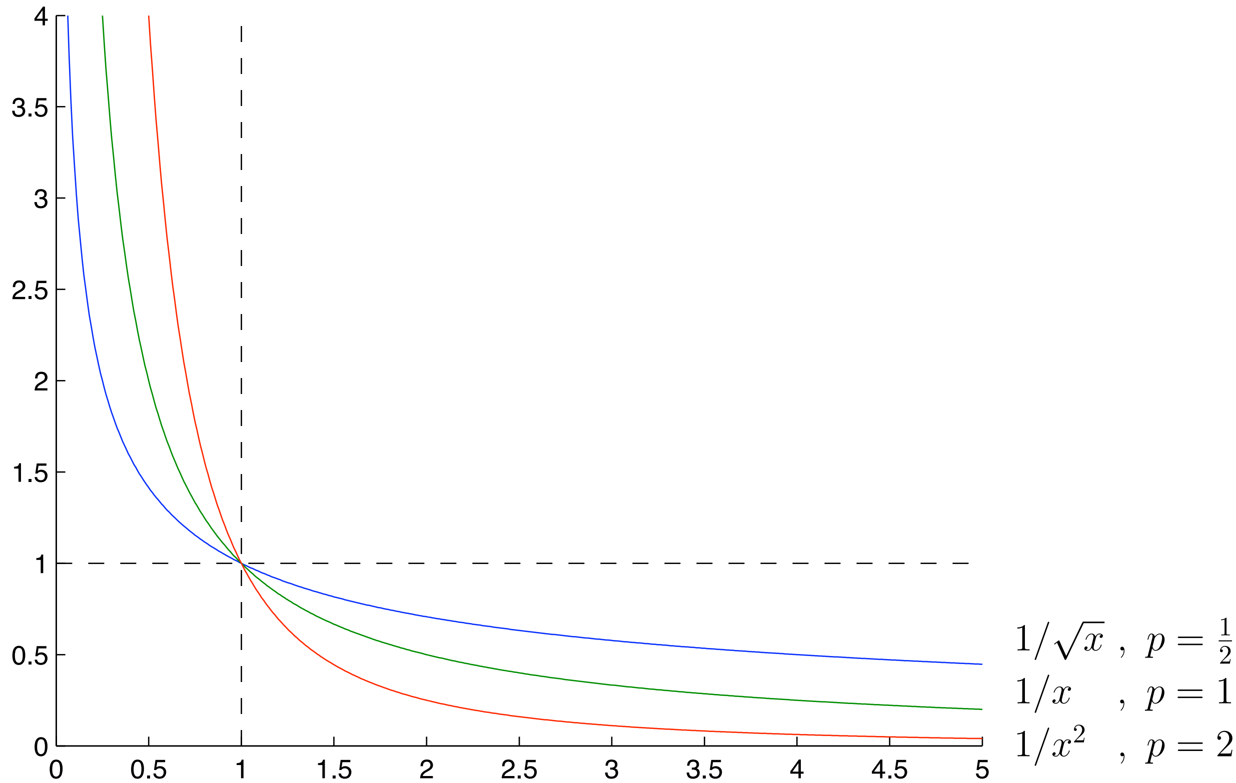
$\Rightarrow$ the area under the graph of $y = \dfrac{1}{x}$ from $x=1$ to $x=\infty$ is infinite

---

ex 3 : $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ : diverges

pf 1 : $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_1^\infty = 2\sqrt{\infty}-2 = \infty$ : diverges

pf 2 : $\displaystyle x \ge 1 \Rightarrow \frac{1}{x} \le \frac{1}{\sqrt{x}} \Rightarrow \int_1^\infty \frac{dx}{x} \le \int_1^\infty \frac{dx}{\sqrt{x}}$ : diverges (comparison test) comparison of $y = \frac{1}{x^2}$ , $\frac{1}{x}$ , $\frac{1}{\sqrt{x}}$ , general form : $\frac{1}{x^p}$

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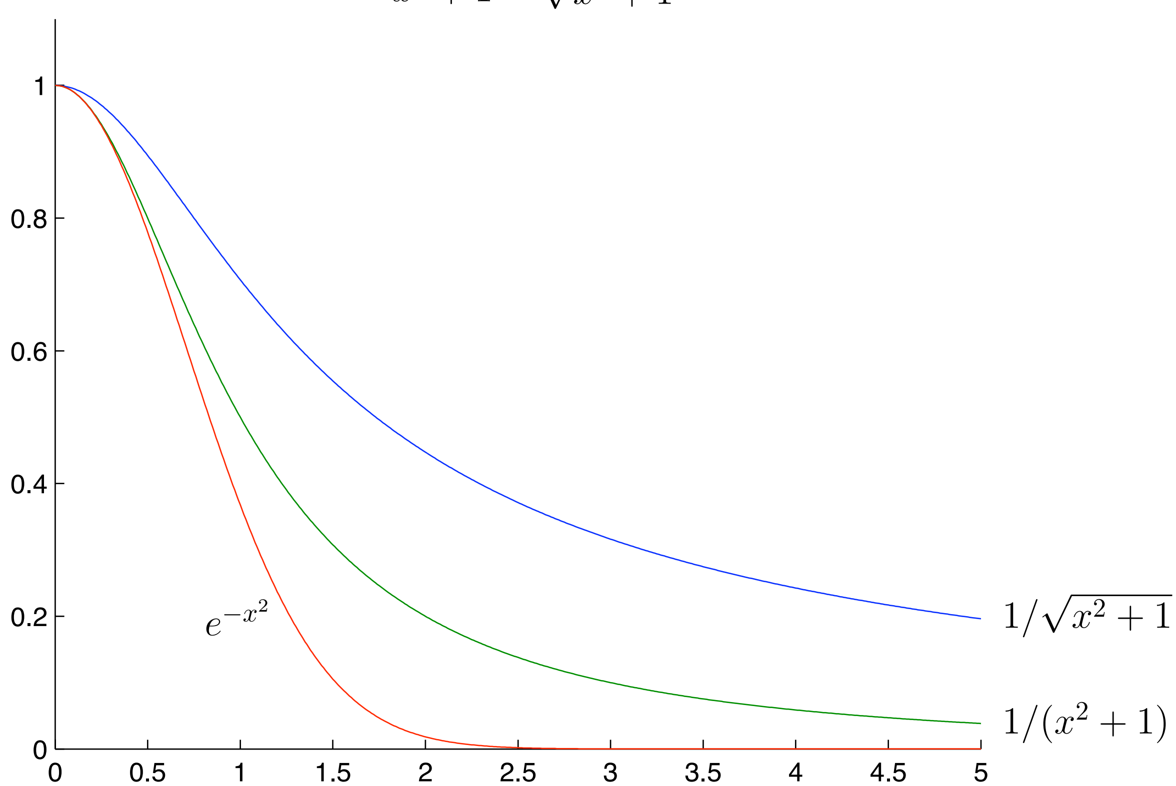
comparison of $y = e^{-x^2}$ , $\frac{1}{x^2+1}$ , $\frac{1}{\sqrt{x^2+1}}$



So far we considered $\int_1^\infty f(x)\,dx$, where $f(x) \to 0$ as $x \to \infty$.

Now consider $\int_0^1 f(x)\,dx$ , where $f(x) \to \infty$ as $x \to 0$.

$$\int_0^1 \frac{dx}{\sqrt{x}} \;=\; 2\sqrt{x}\Big|_0^1 \;=\; 2-0 \;=\; 2 \;:\; \text{converges}$$

$$\int_0^1 \frac{dx}{x} \;=\; \ln x\Big|_0^1 \;=\; \ln 1 - \ln 0 \;=\; 0-(-\infty) \;=\; \infty \;:\; \text{diverges}$$

$$\int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^1 = -\frac{1}{1} - \Big(-\frac{1}{0}\Big) = -1+\infty = \infty \;:\; \text{diverges}$$

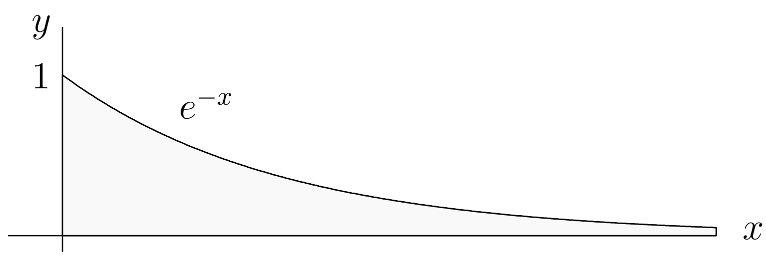
The comparison test can also be used in the last case.

<u>summary</u> ($p$-test)

$$\int_1^\infty \frac{dx}{x^p} : \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \le 1 \end{cases} , \quad \int_0^1 \frac{dx}{x^p} : \begin{cases} \text{converges if } p<1 \\ \text{diverges if } p \ge 1 \end{cases} , \quad \text{pf : omit}$$

---

<u>ex</u> : $\int_0^\infty e^{-x}dx \;=\; -e^{-x}\Big|_0^\infty = \; -e^{-\infty} - (-e^0) \;=\; 0-(-1) \;=\; 1 \;:\;$ converges

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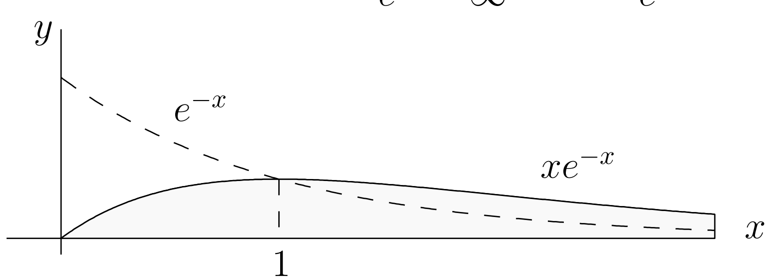
<u>ex</u> : $\int_0^\infty xe^{-x}dx \;=\; 1 \;:\;$ converges

$$\lim_{x\to\infty} xe^{-x} = \infty \cdot 0 = \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \quad \text{(l'Hôpital's rule, pf : later)}$$




$\int_0^\infty xe^{-x}dx \;=\; ?$ , integration by parts : $\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$ 9 Tues 9/19

choose : $u = x$ , $dv = e^{-x}dx \;\Rightarrow\; du = dx$ , $v = -e^{-x}$

$$\Rightarrow \int_0^\infty xe^{-x}dx \;=\; \cancel{-xe^{-x}}\Big|_0^\infty + \int_0^\infty e^{-x}dx \;=\; 0+1 \;=\; 1$$

question : What happens if we choose $u = e^{-x}$, $dv = xdx$? …

<u>integration by parts</u>

$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x)$

$(uv)' = uv' + u'v$

$\Rightarrow \int_a^b (uv)'dx = \int_a^b uv'dx + \int_a^b u'vdx$

$\Rightarrow \int_a^b uv'dx = \int_a^b (uv)'dx - \int_a^b u'vdx$

$v'dx = \frac{dv}{dx} \cdot dx = dv \ , \ u'dx = \frac{du}{dx} \cdot dx = du$

$\Rightarrow \int_a^b udv = uv\Big|_a^b - \int_a^b vdu$ <u>ok</u>

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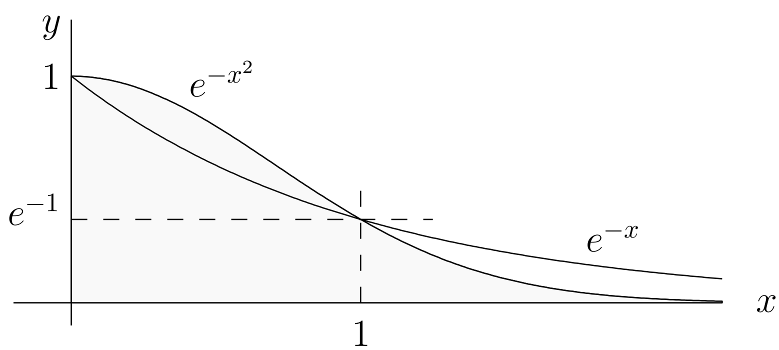
<u>ex</u>

$\int_0^\infty e^{-x^2}dx$ : converges




The antiderivative is not an elementary function, and integration by parts doesn't help (try it!), so we use a different approach.

$x \geq 1 \Rightarrow x^2 \geq x \Rightarrow -x^2 \leq -x \Rightarrow e^{-x^2} \leq e^{-x}$

(this is because $e^x$ is an <u>increasing</u> function, i.e. $a \leq b \Rightarrow e^a \leq e^b$)

$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$

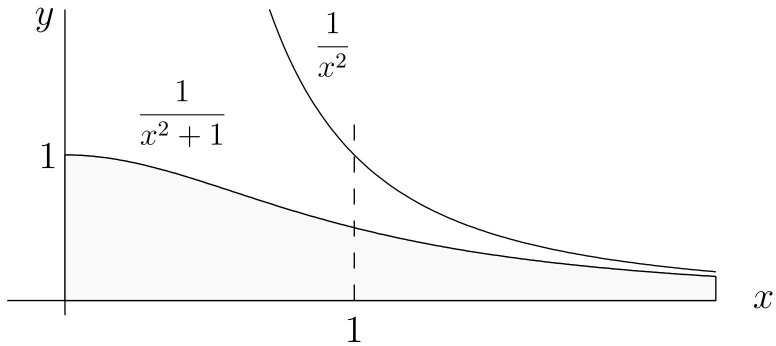
$\int_0^1 e^{-x^2}dx$ : converges (proper integral)

$\int_1^\infty e^{-x^2}dx \leq \int_1^\infty e^{-x}dx$ : converges (comparison test) <u>ok</u>

note : $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} = 0.8862$ , pf : multivariable calculus

<u>ex</u>

$$\int_0^\infty \frac{dx}{x^2+1} \ : \ \text{converges}$$




$$\frac{1}{x^2+1} \le \frac{1}{x^2} \Rightarrow \int_0^\infty \frac{dx}{x^2+1} \le \int_0^\infty \frac{dx}{x^2} = -\frac{1}{x}\bigg|_0^\infty = -\frac{1}{\infty} + \frac{1}{0} = \infty \ : \ \text{diverges}$$

Hence the comparison test yields no information.

$$\int_0^\infty \frac{dx}{x^2+1} = \int_0^1 \frac{dx}{x^2+1} + \int_1^\infty \frac{dx}{x^2+1}$$

$\uparrow$ proper $\quad$ $\uparrow$ converges because $\int_1^\infty \frac{dx}{x^2+1} \le \int_1^\infty \frac{dx}{x^2}$ : converges $\quad$ <u>ok</u>

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<u>alternative</u> : $\int \frac{dx}{x^2+1} = \tan^{-1} x = \arctan x$

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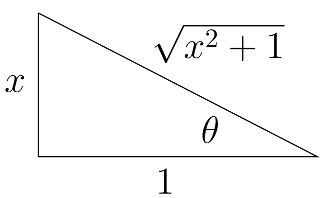


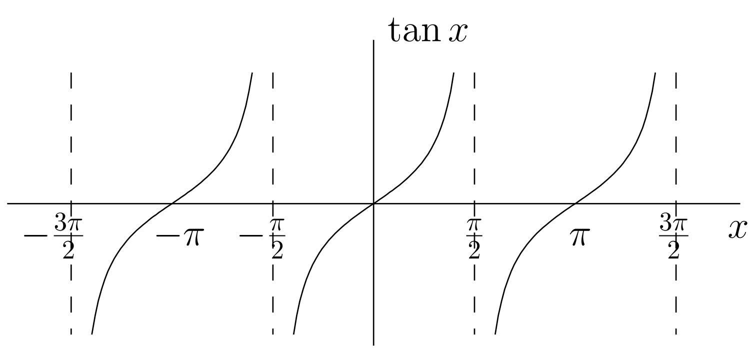

$\tan\theta = x$ : trigonometric substitution

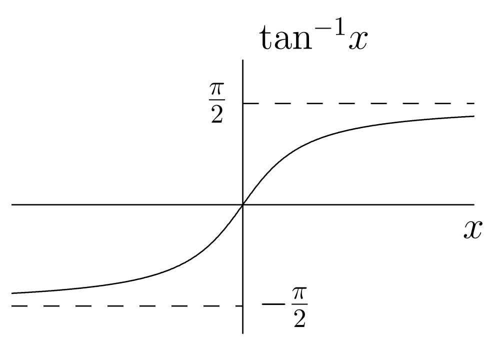
$\sec^2\theta \, d\theta = dx$

$\sec\theta = \sqrt{x^2+1}$

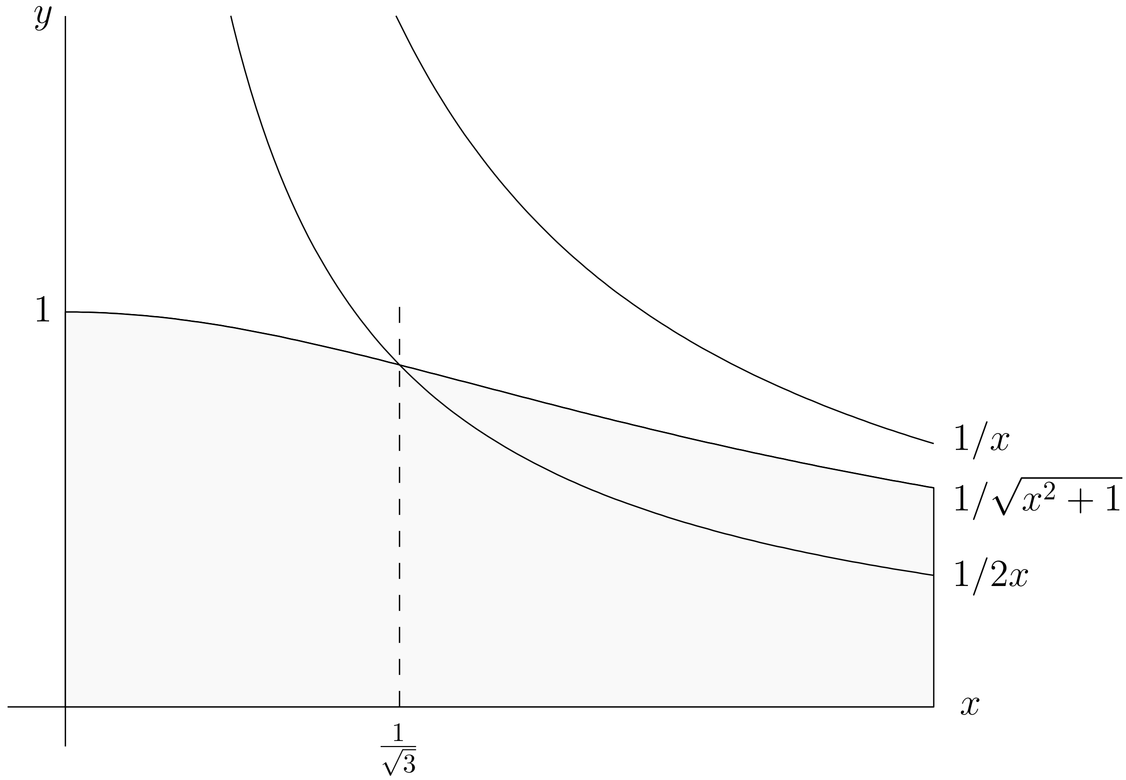
$$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta \, d\theta}{\sec^2\theta} = \int d\theta = \theta = \tan^{-1} x$$

$$\int_0^\infty \frac{dx}{x^2+1} = \tan^{-1} x \bigg|_0^\infty = \tan^{-1}\infty - \tan^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \underline{\text{ok}}$$



ex : $\int_0^\infty \frac{dx}{\sqrt{x^2+1}}$ : diverges




idea : $\frac{1}{\sqrt{x^2+1}} \sim \frac{1}{x}$ as $x \to \infty$ , so we expect that the integral diverges

↑ asymptotic

def: $f(x) \sim g(x)$ as $x \to \infty$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

To prove that the integral diverges, we can use the comparison test.

$$\frac{1}{\sqrt{x^2+1}} \leq \frac{1}{x} \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^2+1}} \leq \int_0^\infty \frac{dx}{x} \ : \ \text{diverges}$$

This yields no information; instead we need a reverse inequality.

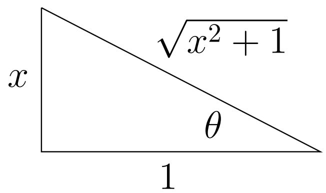
$$\frac{1}{\sqrt{x^2+1}} \geq \frac{1}{2x} \Leftrightarrow 2x \geq \sqrt{x^2+1} \Leftrightarrow 4x^2 \geq x^2+1 \Leftrightarrow 3x^2 \geq 1 \Rightarrow x \geq \frac{1}{\sqrt{3}}$$

↑ if and only if

$$\int_0^\infty \frac{dx}{\sqrt{x^2+1}} = \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{x^2+1}} + \int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{\sqrt{x^2+1}}$$

1st integral : proper

2nd integral : $\int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{\sqrt{x^2+1}} \geq \int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{2x}$ : diverges ok alternative




$$\tan\theta = x$$
$$\sec^2\theta \, d\theta = dx$$
$$\sec\theta = \sqrt{x^2+1}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\sec^2\theta}{\sec\theta} d\theta = \int \sec\theta d\theta = \ ?$$

$$\int \sec\theta d\theta = \int \frac{1}{\cos\theta} d\theta = \int \frac{\cos\theta}{\cos^2\theta} d\theta = \int \frac{\cos\theta}{1-\sin^2\theta} d\theta = \int \frac{du}{1-u^2} = \ ?$$

$$u = \sin\theta \ \Rightarrow \ du = \cos\theta d\theta$$

partial fractions

idea : $\dfrac{3}{10} = \dfrac{3}{2\cdot 5} = \dfrac{5-2}{2\cdot 5} = \dfrac{1}{2} - \dfrac{1}{5}$

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{a}{1+u} + \frac{b}{1-u} = \frac{a(1-u)+b(1+u)}{(1+u)(1-u)}$$

$$\Rightarrow \ a(1-u) + b(1+u) = 1 \ \Rightarrow \ (a+b) + u(-a+b) = 1$$

$$\Rightarrow \ a + b = 1 \, , \ -a + b = 0 \ \Rightarrow \ a = b = \frac{1}{2}$$

$$\Rightarrow \ \frac{1}{1-u^2} = \frac{1/2}{1+u} + \frac{1/2}{1-u} \quad , \ \text{check} \ \ldots$$

$$\int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{1-u} = \frac{1}{2}\ln(1+u) - \frac{1}{2}\ln(1-u) = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$

recall : $\ln a + \ln b = \ln(ab)$ , $\ln a - \ln b = \ln(a/b)$ , $a \ln b = \ln(b^a)$

$$\int \sec\theta d\theta = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\right) = \frac{1}{2}\ln\left(\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right)$$ 11 Mon 9/25

$$= \frac{1}{2}\ln\left(\frac{(1+\sin\theta)^2}{\cos^2\theta}\right) = \ln\left(\frac{1+\sin\theta}{\cos\theta}\right) = \ln(\sec\theta + \tan\theta)$$

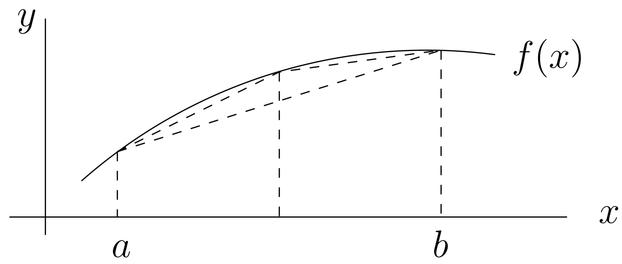
$$\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta) \quad , \quad \text{check} \ \ldots$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln(\sqrt{x^2+1} + x) \quad , \quad \text{check} \ \ldots$$

$$\int_0^\infty \frac{dx}{\sqrt{x^2+1}} = \ln(\sqrt{x^2+1} + x)\Big|_0^\infty = \ln\infty - \ln 1 = \infty : \ \text{diverges}$$ ok

1.7 arclength

problem : find the length of the graph of a function




1st approximation : $\sqrt{(b-a)^2+(f(b)-f(a))^2}$

2nd approximation : set $x_0 = a$ , $x_1 = \dfrac{a+b}{2}$ , $x_2 = b$

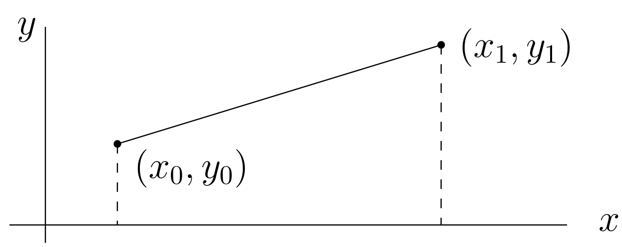
$$\sqrt{(x_1-x_0)^2+(f(x_1)-f(x_0))^2}+\sqrt{(x_2-x_1)^2+(f(x_2)-f(x_1))^2}$$

$n$th approximation : $\Delta x = \dfrac{b-a}{n}$ , $x_i = a + i\Delta x$ , $x_i - x_{i-1} = \Delta x$

$$\sum_{i=1}^{n}\sqrt{(x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))^2} = \sum_{i=1}^{n}\sqrt{(x_i-x_{i-1})^2\left[1+\left(\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right)^2\right]}$$

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+(f'(x_i))^2}\cdot\Delta x = \int_a^b\sqrt{1+(f'(x))^2}\,dx$$

ex : straight line

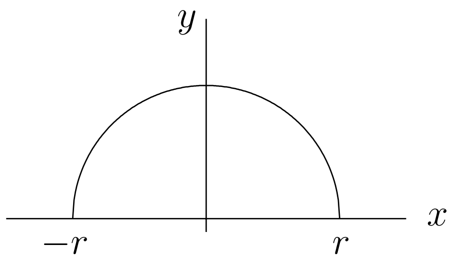



$$y = f(x) = mx+b \;\Rightarrow\; f'(x) = m = \frac{y_1-y_0}{x_1-x_0}$$

$$L = \int_a^b\sqrt{1+(f'(x))^2}\,dx = \int_{x_0}^{x_1}\sqrt{1+\left(\frac{y_1-y_0}{x_1-x_0}\right)^2}dx = \sqrt{1+\left(\frac{y_1-y_0}{x_1-x_0}\right)^2}\cdot\int_{x_0}^{x_1}dx$$

$$= \sqrt{1+\left(\frac{y_1-y_0}{x_1-x_0}\right)^2}\cdot(x_1-x_0) = \sqrt{(x_1-x_0)^2+(y_1-y_0)^2} \quad \underline{\text{ok}}$$

ex : circumference of a circle of radius $r$ , $L = 2\pi r$




$$L = 2\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$x^2+y^2=r^2 \;\Rightarrow\; f(x)=(r^2-x^2)^{1/2}\ ,\ f'(x)=\frac{1}{2}(r^2-x^2)^{-1/2}\cdot(-2x)=\frac{-x}{\sqrt{r^2-x^2}}$$

$$1+(f'(x))^2 = 1+\frac{x^2}{r^2-x^2} = \frac{r^2-x^2+x^2}{r^2-x^2} = \frac{r^2}{r^2-x^2}$$

$$L \;=\; 2\int_a^b \sqrt{1+(f'(x))^2}\,dx \;=\; 2\int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\,dx \;:\; \text{improper}$$

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Tues
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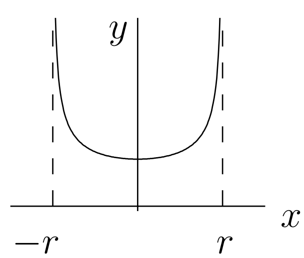
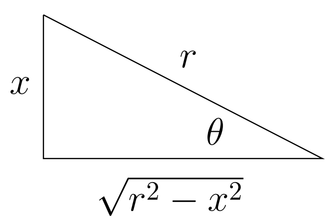
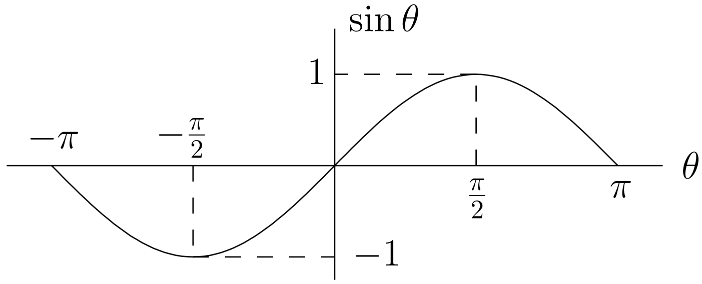



$$\sin\theta = \frac{x}{r} \;\Rightarrow \cos\theta\,d\theta = \frac{dx}{r}\ ,\ \ \sec\theta = \frac{r}{\sqrt{r^2-x^2}}$$



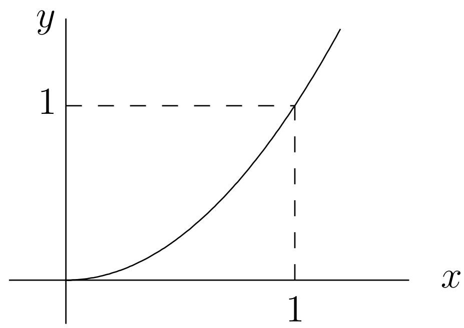

$$x = r \Rightarrow \sin\theta = 1 \Rightarrow \theta = \tfrac{\pi}{2}$$

$$x = -r \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\tfrac{\pi}{2}$$

$$L = 2\int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\,dx = 2\int_{-\pi/2}^{\pi/2} \sec\theta \cdot r\cos\theta\,d\theta$$

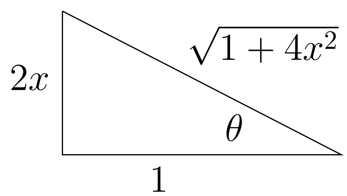
$$= 2r\int_{-\pi/2}^{\pi/2} d\theta = 2r\cdot\theta\Big|_{-\pi/2}^{\pi/2} = 2r\cdot\left(\frac{\pi}{2}-\Big(-\frac{\pi}{2}\Big)\right) = 2\pi r \qquad \underline{\text{ok}}$$

<u>ex</u> : parabola , $y = x^2$ , $0 \le x \le 1$




note : $L > \sqrt{2} = 1.4142$

$$f(x) = x^2 \;\Rightarrow\; f'(x) = 2x \;\Rightarrow\; L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \;=\; \int_0^1 \sqrt{1 + 4x^2}\, dx$$




$$\tan\theta = 2x \;\Rightarrow\; \sec^2\theta d\theta = 2dx$$

$$\sec\theta = \sqrt{1 + 4x^2}$$

$$\int \sqrt{1 + 4x^2}\, dx \;=\; \int \sec\theta \cdot \tfrac{1}{2} \sec^2\theta d\theta \;=\; \tfrac{1}{2} \int \sec^3\theta d\theta$$

$$\int \sec^3\theta d\theta \;=\; \int \frac{d\theta}{\cos^3\theta} \;=\; \int \frac{\cos\theta}{\cos^4\theta} d\theta \;=\; \int \frac{\cos\theta d\theta}{(1 - \sin^2\theta)^2} \;=\; \int \frac{du}{(1 - u^2)^2}$$

$$u = \sin\theta \;\Rightarrow\; du = \cos\theta d\theta$$

$$(1 - u^2)^2 = ((1 + u)(1 - u))^2 = (1 + u)^2 (1 - u)^2$$

$$\frac{1}{(1 - u^2)^2} = \frac{a}{1 + u} + \frac{b}{(1 + u)^2} + \frac{c}{1 - u} + \frac{d}{(1 - u)^2} = \cdots$$

<u>alternative</u>

$$\int \sec^3\theta d\theta \;=\; \int \sec\theta \cdot \sec^2\theta d\theta$$

$$u = \sec\theta\,,\; dv = \sec^2\theta d\theta \;\Rightarrow\; du = \sec\theta \tan\theta d\theta\,,\; v = \tan\theta$$

$$\int \sec^3\theta d\theta = \sec\theta \tan\theta - \int \sec\theta \tan^2\theta d\theta = \sec\theta \tan\theta - \int \sec\theta \cdot \frac{\sin^2\theta}{\cos^2\theta} d\theta$$

$$= \sec\theta \tan\theta - \int \sec^3\theta (1 - \cos^2\theta) d\theta$$

$$2 \int \sec^3\theta d\theta = \sec\theta \tan\theta + \int \sec\theta d\theta$$

$$\int \sec^3\theta d\theta = \tfrac{1}{2} \left(\sec\theta \tan\theta + \ln(\sec\theta + \tan\theta)\right)$$

$$\int_0^1 \sqrt{1 + 4x^2}\, dx = \tfrac{1}{4} \left( 2x\sqrt{1 + 4x^2} + \ln\left(2x + \sqrt{1 + 4x^2}\right)\right)\Big|_0^1$$

$$= \tfrac{1}{4} (2\sqrt{5} + \ln(2 + \sqrt{5}\,)) = 1.4789$$

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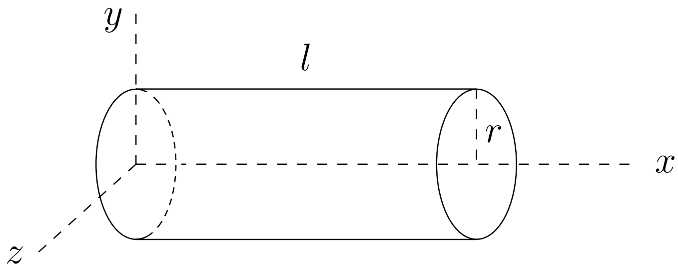
<u>1.8 surface area</u>

Consider a surface formed by rotating a curve about an axis.

<u>ex</u> : cylinder , $S = 2\pi rl$

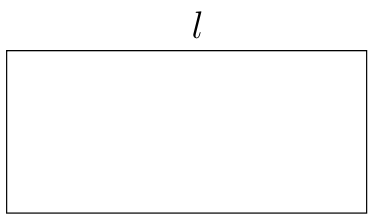



$l$ : length

$r$ : radius

$S$ : surface area

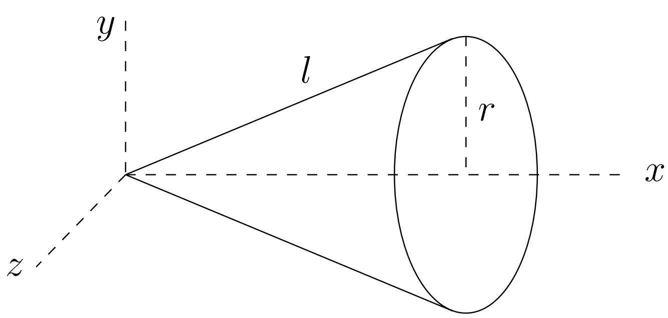
To find $S$, cut the cylinder and spread it flat to form a rectangle.




$2\pi r \;\Rightarrow\; S = 2\pi rl$ <u>ok</u>

---

<u>ex</u> : cone , $S = \pi rl$

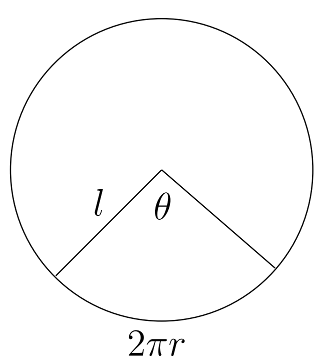



$l$ : <u>slant length</u>

$r$ : radius

$S$ : surface area

To find $S$, cut the cone and spread it flat to form a circular sector.




$l$ : radius of circle

$\theta$ : sector angle

$2\pi r$ : length of sector edge

$S$ : area of sector = area of cone

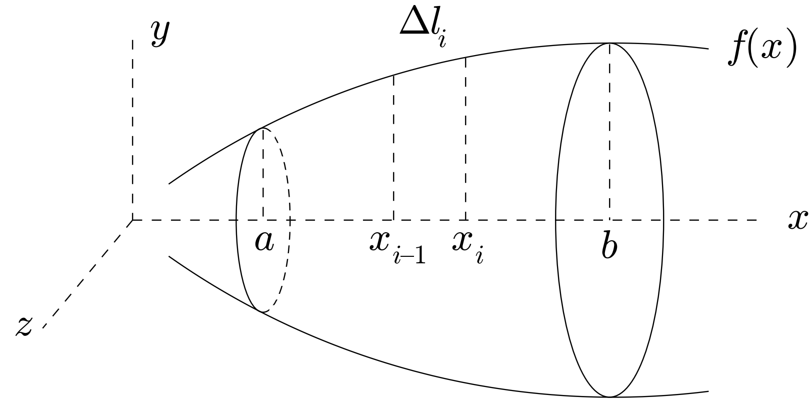
1. $2\pi r = l\theta$ , pf : $\dfrac{2\pi r}{2\pi l} = \dfrac{\theta}{2\pi} \Rightarrow 2\pi r = \cancel{2\pi} l \cdot \dfrac{\theta}{\cancel{2\pi}} = l\theta$

2. $S = \frac{1}{2}l^2\theta$ , pf : $\dfrac{S}{\pi l^2} = \dfrac{\theta}{2\pi} \Rightarrow S = \cancel{\pi} l^2 \cdot \dfrac{\theta}{2\cancel{\pi}} = \frac{1}{2}l^2\theta$

$\Rightarrow\; S = \frac{1}{2}l \cdot l\theta = \frac{1}{2}l \cdot 2\pi r = \pi rl$ <u>ok</u>

note : another proof on hw5 uses $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$ , $A = \int_a^b f(x)dx$

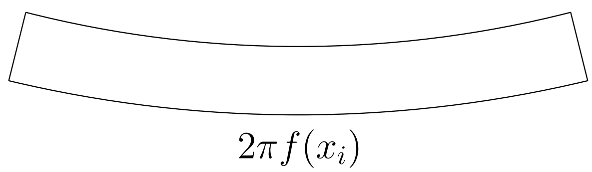
<u>general case</u>




$\Delta x = \dfrac{b-a}{n}$ , $x_i = a + i\Delta x$

$\Delta l_i$ : arclength of $i$th slice

$i$th slice :



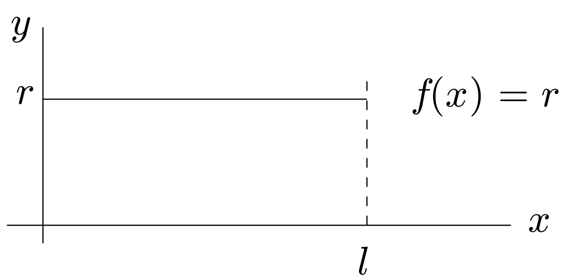

$\Delta l_i \;\Rightarrow$ surface area $\approx 2\pi f(x_i) \cdot \Delta l_i$

$$\Delta l_i \;\approx\; \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \;\approx\; \sqrt{1 + (f'(x_i))^2} \cdot \Delta x$$

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}\,\Delta x \;= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

---

check 1 : cylinder , $S = 2\pi r l$

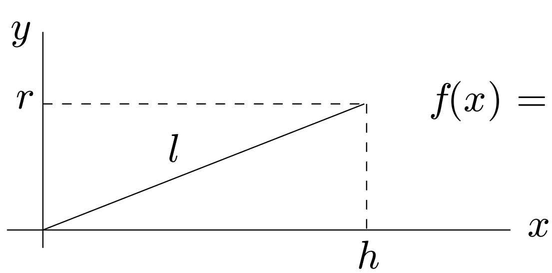



$f(x) = r$

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx \;= \int_0^l 2\pi r dx \;=\; 2\pi r \int_0^l dx \;=\; 2\pi r l$$ <u>ok</u>

---

check 2 : cone , $S = \pi r l$




$f(x) = \dfrac{r}{h}\,x$

$$S = \int_0^h 2\pi \frac{r}{h} x \sqrt{1 + \frac{r^2}{h^2}}\,dx \;=\; 2\pi \frac{r}{h}\sqrt{1 + \frac{r^2}{h^2}} \cdot \frac{h^2}{2} \;=\; \pi r \sqrt{h^2 + r^2} \;=\; \pi r l$$ <u>ok</u>

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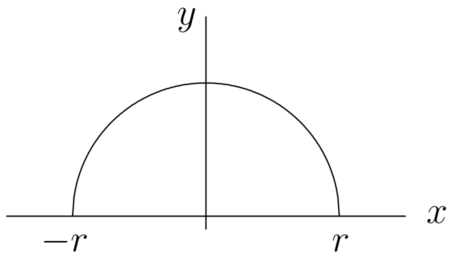
ex: sphere , $S = 4\pi r^2$




$$f(x) = (r^2 - x^2)^{1/2} \Rightarrow f'(x) = \tfrac{1}{2}(r^2 - x^2)^{-1/2} \cdot -2x = \frac{-x}{\sqrt{r^2 - x^2}}$$

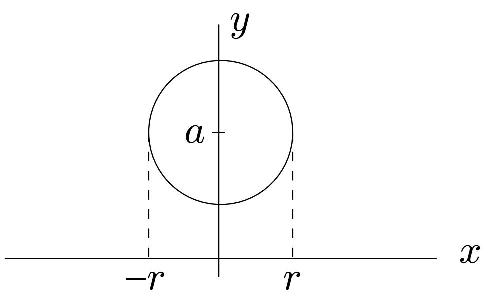
$$1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx = 2\pi \int_{-r}^{r} \cancel{\sqrt{r^2 - x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2 - x^2}}}\, dx$$

$$= 2\pi r \int_{-r}^{r} dx = 2\pi r \cdot 2r = 4\pi r^2 \quad \underline{\text{ok}}$$

---

ex : torus , $S = 4\pi^2 ar$




assume $a > r$

equation of circle : $x^2 + (y - a)^2 = r^2 \Rightarrow y = a \pm \sqrt{r^2 - x^2}$

upper semicircle : $f_+(x) = a + \sqrt{r^2 - x^2}$

lower semicircle : $f_-(x) = a - \sqrt{r^2 - x^2}$

$$S = S_+ + S_- = \int_{-r}^{r} 2\pi f_+(x)\sqrt{1 + (f_+'(x))^2}\, dx + \int_{-r}^{r} 2\pi f_-(x)\sqrt{1 + (f_-'(x))^2}\, dx$$

$$1 + (f_+'(x))^2 = \frac{r^2}{r^2 - x^2} = 1 + (f_-'(x))^2 \text{ , because } f_-'(x) = -f_+'(x)$$

$$S = 2\pi \int_{-r}^{r} (a + \cancel{\sqrt{r^2 - x^2}} + a - \cancel{\sqrt{r^2 - x^2}}) \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 2\pi \cdot 2a \cdot \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx = 2\pi \cdot 2a \cdot \underset{\substack{\uparrow \\ \text{recall}}}{\pi r} = 4\pi^2 ar \quad \underline{\text{ok}}$$

<u>note</u>

1. $S_{\text{torus}} = 4\pi^2 ar = 2\pi a \cdot 2\pi r$ , in fact a torus is the product of two circles

2. For the cylinder and cone, we can find $S$ by cutting the surface and spreading it flat, but this does not work for the sphere and torus. (differential geometry - Math 433)

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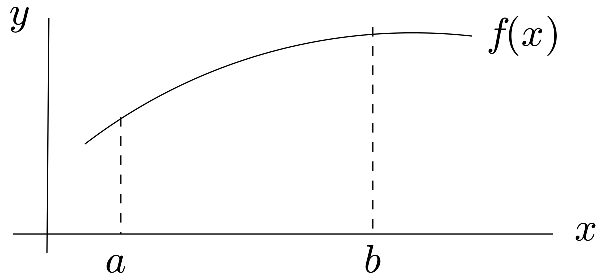
<u>summary</u>




area under graph of $y = f(x)$ for $a \le x \le b$ : $A = \int_a^b f(x)dx$

arclength ································· : $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

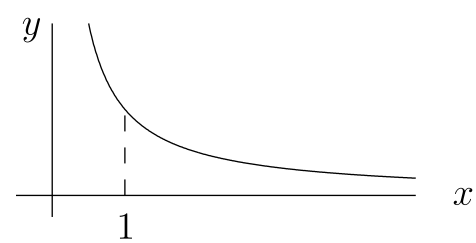
surface area ····························· : $S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$

volume ···································· : $V = \int_a^b \pi f(x)^2 dx$

<u>ex</u>

$f(x) = \dfrac{1}{x}$ , $1 \le x < \infty$

$A = \int_1^\infty \dfrac{dx}{x}$ : diverges, $p = 1$




$L = \int_1^\infty \sqrt{1 + \dfrac{1}{x^4}}\, dx$ : diverges , comparison test , $\sqrt{1 + \frac{1}{x^4}} \ge 1$

$S = \int_1^\infty 2\pi \dfrac{1}{x}\sqrt{1 + \dfrac{1}{x^4}}\, dx$ : diverges , comparison test , $\frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \ge \frac{1}{x}$

$V = \int_1^\infty \dfrac{\pi}{x^2} dx = -\dfrac{\pi}{x}\Big|_1^\infty = \pi$ : converges , $p = 2$

This shape is called <u>Gabriel's horn</u>; it has finite volume and infinite surface area.

1.9 center of mass

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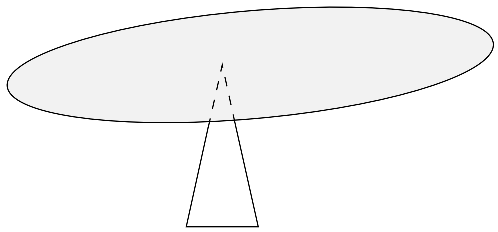
problem : Find the point at which a thin plate balances.



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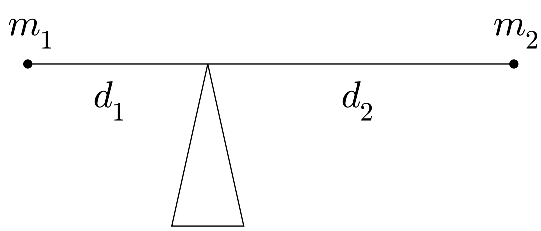
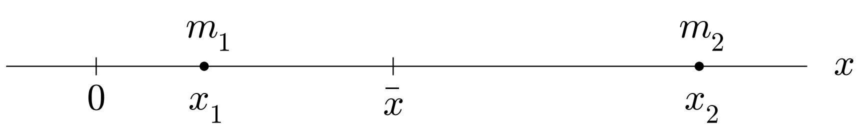
ex : 2 masses $m_1, m_2$ connected by a rod of negligible mass




balance of moments (prevents tipping) : $m_1 d_1 = m_2 d_2$




$x_1$ , $x_2$ , $\bar{x}$ : coordinates of $m_1$ , $m_2$ , CM

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \Rightarrow (m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2 \Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$m_i x_i$ : moment of mass $i$ about $x = 0$ , units are mass $\times$ distance

The balance of moments can also be written as $m_1(\bar{x} - x_1) + m_2(\bar{x} - x_2) = 0$.

---

ex : $n$ masses $m_1, \ldots, m_n$ connected by a rod of negligible mass

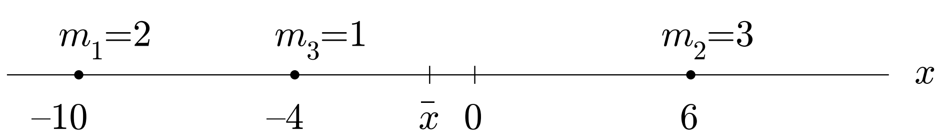
balance of moments $\Rightarrow \sum_{i=1}^{n} m_i(\bar{x} - x_i) = 0 \Rightarrow \sum_{i=1}^{n} m_i \bar{x} = \sum_{i=1}^{n} m_i x_i$

$$\Rightarrow \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M}{m} \ , \ M = \sum_{i=1}^{n} m_i x_i : \text{total moment} \ , \ m = \sum_{i=1}^{n} m_i : \text{total mass}$$

If $n = 2$, this agrees with the previous formula for $\bar{x}$.

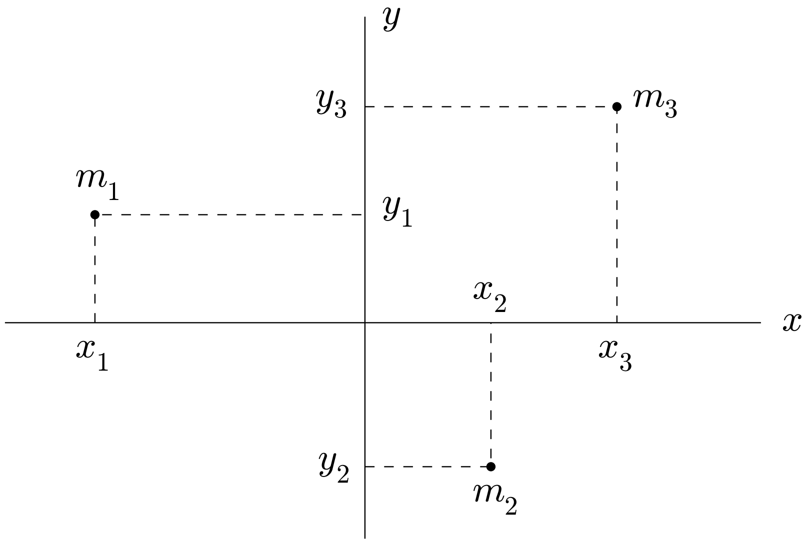
---

ex : $m_1 = 2$ , $m_2 = 3$ , $m_3 = 1$ , $x_1 = -10$ , $x_2 = 6$ , $x_3 = -4$ : find CM




$$\left.\begin{array}{l} M = m_1 x_1 + m_2 x_2 + m_3 x_3 = -20 + 18 - 4 = -6 \\ m = m_1 + m_2 + m_3 = 2 + 3 + 1 = 6 \end{array}\right\} \Rightarrow \bar{x} = \frac{M}{m} = \frac{-6}{6} = -1$$

<u>two-dimensional case</u>




CM : $(\bar{x}, \bar{y}) = ?$

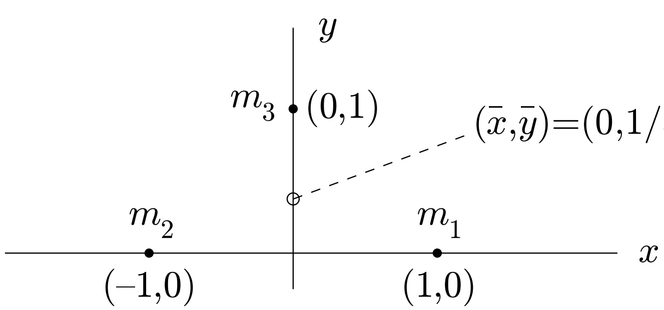
balance of moments : $\sum_{i=1}^{n} m_i(\bar{x} - x_i) = 0$ , $\sum_{i=1}^{n} m_i(\bar{y} - y_i) = 0$

$$\Rightarrow \ \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M_y}{m} \ , \ M_y \ : \ \underline{\text{total moment about } y\text{−axis}} \ , \ M_y = m\bar{x}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{M_x}{m} \ , \ M_x \ : \ \underline{\text{total moment about } x\text{−axis}} \ , \ M_x = m\bar{y}$$

---

<u>ex</u> : $m_1 = m_2 = m_3 = 1$




$$\bar{x} = \frac{M_y}{m} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{1 - 1 + 0}{3} = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{0 + 0 + 1}{3} = \frac{1}{3}$$

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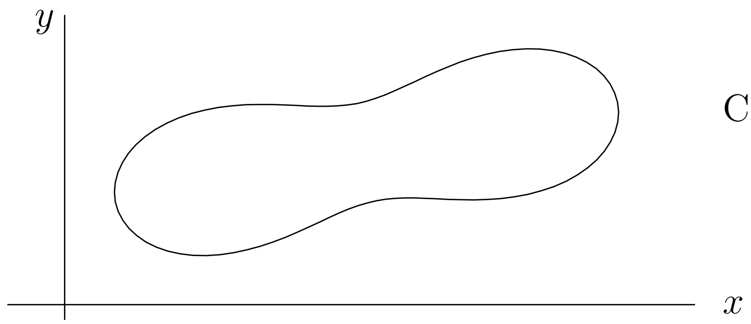
continuous mass distribution

Consider a region of uniform density $\rho$ in the $xy$-plane.

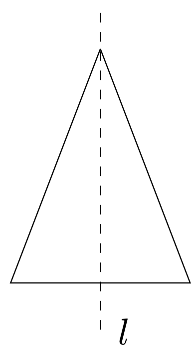



$\text{CM} = (\bar{x}, \bar{y}) = \,?$

symmetry principle

If a region is symmetric about a line $l$, then CM lies on $l$.

ex 1 : isoceles triangle , CM lies on $l$

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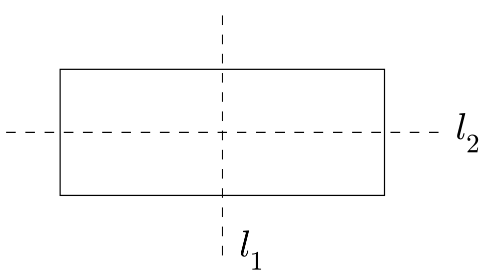
ex 2 : rectangle , CM lies on $l_1$ and $l_2 \Rightarrow$ CM is at center of rectangle

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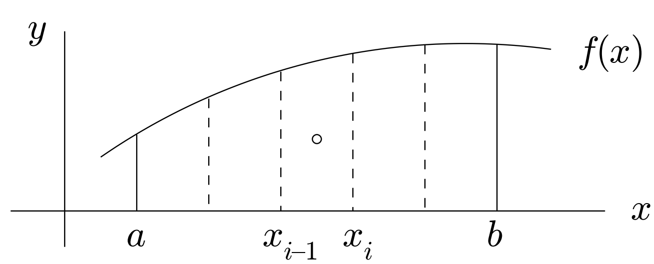
case 1 : $R = \{(x, y) : a \le x \le b\,,\, 0 \le y \le f(x)\}$




balance of moments $\Rightarrow \bar{x} = \dfrac{M_y}{m}$ , $\bar{y} = \dfrac{M_x}{m}$, but how do we find $M_x$, $M_y$?

$\Delta x = \dfrac{b-a}{n}$ , $x_i = a + i\Delta x$ , $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$

$R_i$ : $i$th rectangle , $i = 1, \ldots, n$

mass of $R_i$ = density × area = $\rho \cdot f(x_i^*)\Delta x$

CM of $R_i = (x_i^*, \frac{1}{2} f(x_i^*))$

moment of $R_i$ about $y-$axis = mass × distance = $\rho f(x_i^*)\Delta x \cdot x_i^*$

moment ..... ”..... $x-$axis = .......”....... $= \rho f(x_i^*)\Delta x \cdot \frac{1}{2} f(x_i^*)$

<u>moment principle</u>

The moment of a union of rectangles is the sum of the moments of each rectangle.

$$M(R_1 \cup R_2 \cdots \cup R_n) = \sum_{i=1}^{n} M(R_i)$$

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho x_i^* f(x_i^*)\Delta x = \int_a^b \rho x f(x) dx$$

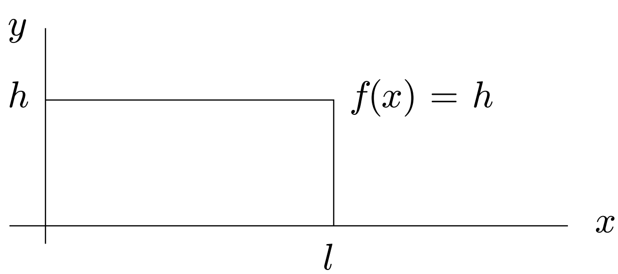
$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \tfrac{1}{2} f(x_i^*)^2 \Delta x = \tfrac{1}{2} \int_a^b \rho f(x)^2 dx$$

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b \not{\rho} x f(x) dx}{\int_a^b \not{\rho} f(x) dx} \quad , \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b \not{\rho} f(x)^2 dx}{\int_a^b \not{\rho} f(x) dx}$$

assume $\rho = 1$ from now on

---

<u>ex</u> : rectangle



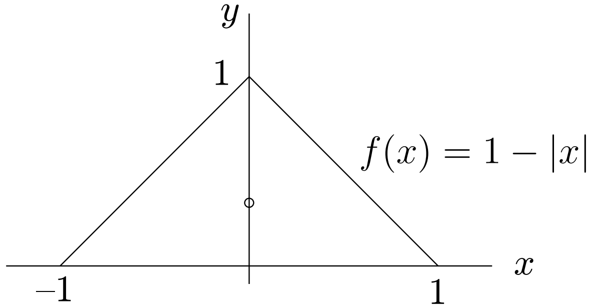

$$M_y = \int_a^b x f(x) dx = \int_0^l x \cdot h dx = h \frac{x^2}{2}\Big|_0^l = \frac{hl^2}{2}$$

$$M_x = \tfrac{1}{2}\int_a^b f(x)^2 dx = \tfrac{1}{2}\int_0^l h^2 dx = \frac{h^2 l}{2}$$

$$m = \int_a^b f(x) dx = \int_0^l h dx = hl$$

$$\bar{x} = \frac{M_y}{m} = \frac{hl^2}{2} \cdot \frac{1}{hl} = \frac{l}{2} \ , \ \bar{y} = \frac{M_x}{m} = \frac{h^2 l}{2} \cdot \frac{1}{hl} = \frac{h}{2} \qquad \underline{\text{ok}}$$

ex : triangular plate




symmetry $\Rightarrow\ \bar{x} = 0$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b f(x)^2 dx}{\int_a^b f(x) dx} = \frac{\frac{1}{2}\int_{-1}^0 (1+x)^2 dx \ + \ \frac{1}{2}\int_0^1 (1-x)^2 dx}{\frac{1}{2}\cdot 2\cdot 1}$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{3}(1+x)^3\Big|_{-1}^{0} + \tfrac{1}{2}\cdot -\tfrac{1}{3}(1-x)^3\Big|_0^1 = \tfrac{1}{6}(1-0) - \tfrac{1}{6}(0-1) = \tfrac{1}{3} \ \Rightarrow \ \text{CM} = (0, \tfrac{1}{3})$$

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ex : another triangular plate

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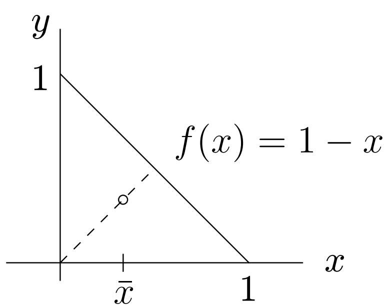



symmetry $\Rightarrow$ CM lies on the line $y = x \ \Rightarrow \ \bar{y} = \bar{x}$

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{\int_0^1 x(1-x) dx}{\frac{1}{2}\cdot 1\cdot 1} = \frac{\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1}{\frac{1}{2}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$\Rightarrow$ CM $= (\frac{1}{3}, \frac{1}{3})$

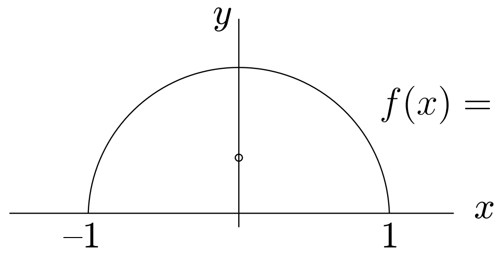
question : The line $x = \bar{x}$ divides the triangle into 2 parts; which part has larger area?

area of left part $= \frac{1}{3}\cdot\frac{1}{2}(1 + \frac{2}{3}) = \frac{5}{18}$

area of right part $= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{4}{18}$

So the left part has larger area than the right part, even though the CM lies on the boundary of the 2 parts.

ex : half-disk




$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b f(x)^2 dx}{\int_a^b f(x)dx} = \frac{\int_0^1 (1-x^2)dx}{\pi/2} = \frac{\left(x - \frac{1}{3}x^3\right)\Big|_0^1}{\pi/2} = \frac{\frac{2}{3}}{\pi/2} = \frac{4}{3\pi} = 0.4244$$

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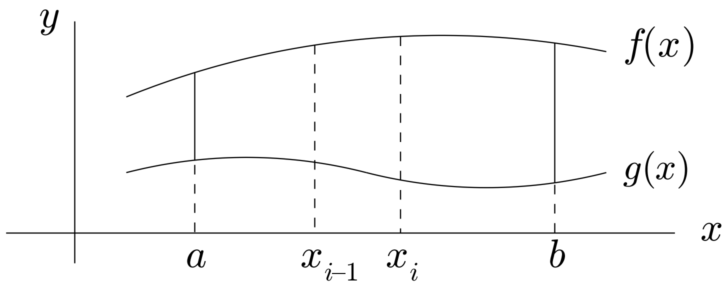
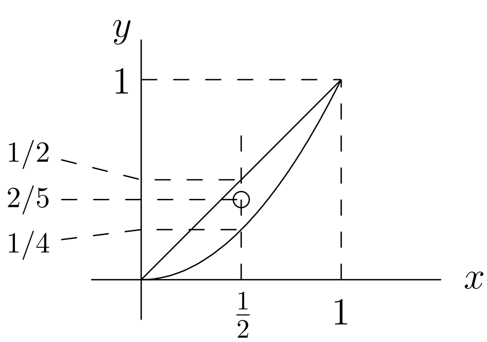
case 2 : $R = \{(x,y) : a \le x \le b\,,\ g(x) \le y \le f(x)\}$




$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x(f(x)-g(x))dx}{\int_a^b (f(x)-g(x))dx} \quad , \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)dx}{\int_a^b (f(x)-g(x))dx}$$

---

ex : $f(x) = x\ ,\ g(x) = x^2\ ,\ 0 \le x \le 1$



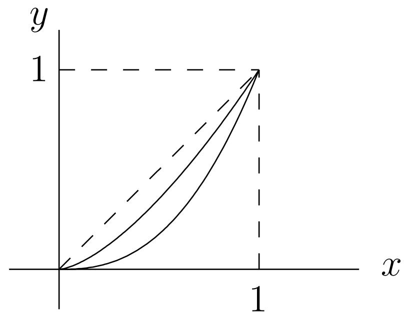

$$m = \int_a^b (f(x)-g(x))dx = \int_0^1 (x-x^2)dx = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

$$M_x = \tfrac{1}{2}\int_a^b (f(x)^2 - g(x)^2)dx = \tfrac{1}{2}\int_0^1 (x^2 - x^4)dx = \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{1}{15}$$

$$M_y = \int_a^b x(f(x)-g(x))dx = \int_0^1 (x^2 - x^3)dx = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{1}{12}}{\frac{1}{6}} = \tfrac{1}{2}\ ,\ \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{15}}{\frac{1}{6}} = \tfrac{2}{5} \Rightarrow \text{CM is closer to top edge}$$

ex : $f(x) = x^m$ , $g(x) = x^n$




For some choice of $m$ and $n$, the CM lies outside the region. (hw7)

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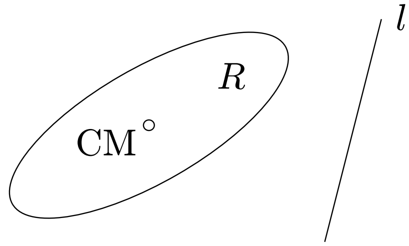
Theorem of Pappus

18 Tues 10/10

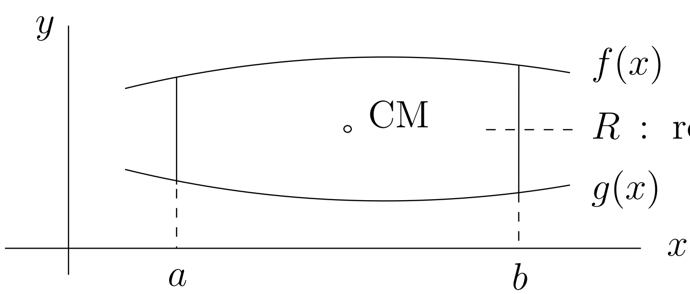
Let $R$ be a region that lies on one side of a line $l$.




$A =$ area of $R$
$V =$ volume obtained by rotating $R$ about $l$
$d =$ distance traveled by CM when $R$ is rotated about $l$

$\Rightarrow V = A \cdot d$

pf : (special case)

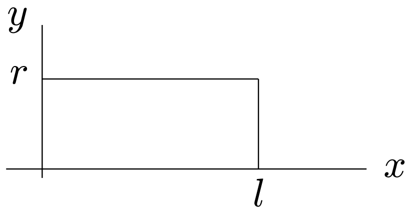



$$A = \int_a^b (f(x) - g(x))dx \quad , \quad V = \int_a^b \pi(f(x)^2 - g(x)^2)dx$$

$$d = 2\pi\bar{y} = 2\pi \cdot \frac{M_x}{m} = 2\pi \cdot \frac{\frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)dx}{\int_a^b (f(x) - g(x))dx} = \frac{V}{A} \quad \underline{\text{ok}}$$

---

ex : volume of a cylinder




$A = rl$ , $d = 2\pi \cdot \dfrac{r}{2} = \pi r$

$\Rightarrow V = A \cdot d = rl \cdot \pi r = \pi r^2 l$ ok

hw7 : volume of a torus

## 1.10 probability

$X$ : random variable

ex

$X$ = velocity of a gas molecule

$X$ = waiting time in the supermarket checkout line

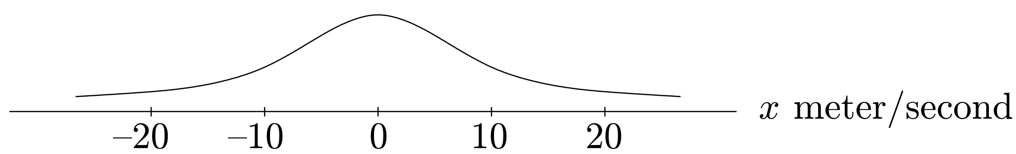
$X$ = GPA of a college student

def

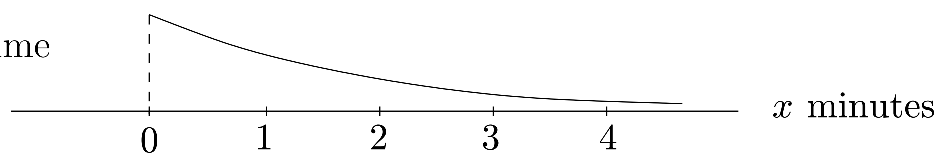
A random variable $X$ has a probability density function $f(x)$ with the property that $\int_a^b f(x)dx$ = probability that $X$ lies between $a$ and $b$ = prob($a \le X \le b$).

examples of $f(x)$

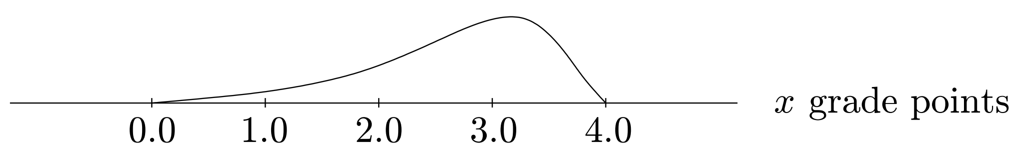
$X$ = velocity




$X$ = waiting time




$X$ = GPA




note : large $f(x) \Leftrightarrow$ high probability that $X$ is close to $x$

small $f(x) \Leftrightarrow$ low probability that $X$ is close to $x$

properties of a pdf

1. $f(x) \ge 0$
2. $\int_{-\infty}^{\infty} f(x)dx = \text{prob}(-\infty < X < \infty) = 1$

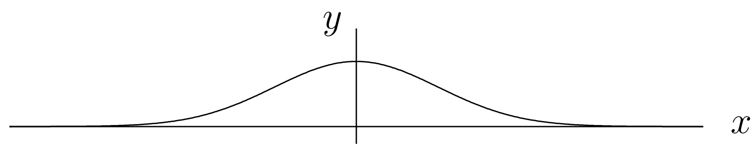
def : The mean value of a random variable $X$ (average value, expected value) is denoted by $\mu = \mu(X)$ and it is defined by

$$\mu = \int_{-\infty}^{\infty} xf(x)dx \approx \sum_{i=1}^{n} x_i f(x_i)\Delta x \approx \sum_{i=1}^{n} x_i \cdot \text{prob}(x_{i-1} \le X \le x_i),$$

i.e. the values of $x_i$ are weighted by the probability that $X$ is close to $x_i$.

ex : Gaussian pdf

$$f(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2}$$




1. $f(x) \geq 0$

2. $\int_{-\infty}^{\infty} f(x)dx \;=\; \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\, e^{-x^2} dx \;=\; \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx \;=\; \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \;=\; 1$ ok

note : $\int_{-\infty}^{\infty} e^{-x^2} dx \;=\; \sqrt{\pi}$ (Math 215/255)

3. $\mu = \int_{-\infty}^{\infty} x f(x)dx \;=\; \int_{-\infty}^{\infty} x \,\frac{1}{\sqrt{\pi}}\, e^{-x^2} dx \;=\; 0$ 19 Wed 10/11

def : The median value of a random variable $X$ is denoted by $m \;=\; m(X)$ and it is defined by prob$(X \leq m) \;=\;$ prob$(X \geq m) \;=\; \frac{1}{2}$.

note

1. This is equivalent to $\int_{-\infty}^{m} f(x)dx \;= \int_{m}^{\infty} f(x)dx \;=\; \frac{1}{2}$, i.e. half the area under the graph of $f(x)$ lies to the left of $m$ and half lies to the right.

2. For the Gaussian pdf we have $\mu = m$, but in general $\mu \neq m$. (more later)

def

The standard deviation of a random variable $X$ is denoted by $\sigma = \sigma(X)$ and is defined by $\sigma^2 \;=\; \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$.

note : small $\sigma \;\Leftrightarrow\; X$ is more likely to be close to $\mu$

large $\sigma \;\Leftrightarrow\; X$ is less likely to be close to $\mu$

ex

$$f(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2} \;\Rightarrow\; \mu = 0 \;,\; \sigma = \,?$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \;=\; \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{\pi}}\, e^{-x^2} dx \quad , \quad u = x \;,\; dv = xe^{-x^2} dx$$

$$du = dx \;,\; v = \frac{e^{-x^2}}{-2}$$

$$= \frac{1}{\sqrt{\pi}} \left( \cancel{x \cdot \frac{e^{-x^2}}{-2} \Big|_{-\infty}^{\infty}} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx \right) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{1}{2} \;\Rightarrow\; \sigma = \frac{1}{\sqrt{2}} \;=\; 0.7071$$

def : Let $\mu$ and $\sigma > 0$ be given and define $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$. Then $f(x)$ is the pdf of a random variable $X$, called a normal distribution, with mean $\mu$ and standard deviation $\sigma$.

1. The Gaussian pdf corresponds to $\mu = 0\,,\ \sigma = \frac{1}{\sqrt{2}}$.
2. $\mu$ shifts the pdf along the $x$-axis and $\sigma$ scales the height and width of the pdf

check (hw7)

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$$\int_{-\infty}^{\infty} f(x)dx \ = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \ = \ 1$$

$$\int_{-\infty}^{\infty} x f(x)dx \ = \int_{-\infty}^{\infty} x\, \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \ = \ \mu$$

$$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \ = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \ = \ \sigma^2$$

ex

1. Find the probability that $X$ is within 1 sd of $\mu$.

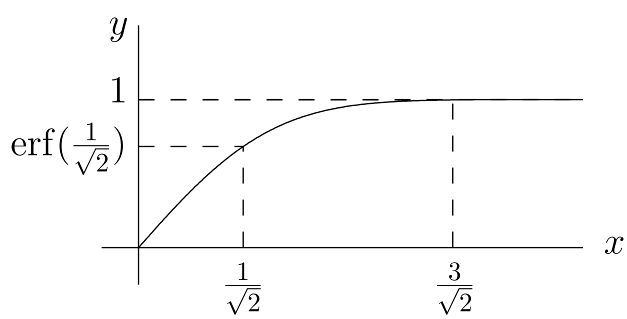
$$\text{prob}(\,\mu - \sigma \le X \le \mu + \sigma\,) \ = \int_{\mu-\sigma}^{\mu+\sigma} f(x)dx$$

$$= \int_{\mu-\sigma}^{\mu+\sigma} \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \ = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-u^2} \cdot \sqrt{2}\sigma\, du \ = \ \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} e^{-u^2} du$$

$$\left(u = \frac{x-\mu}{\sqrt{2}\sigma}\ ,\ du = \frac{dx}{\sqrt{2}\sigma}\right)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} e^{-t^2} dt \ = \ \text{erf}(\tfrac{1}{\sqrt{2}}) = 0.6827$$

recall : $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2} dt$



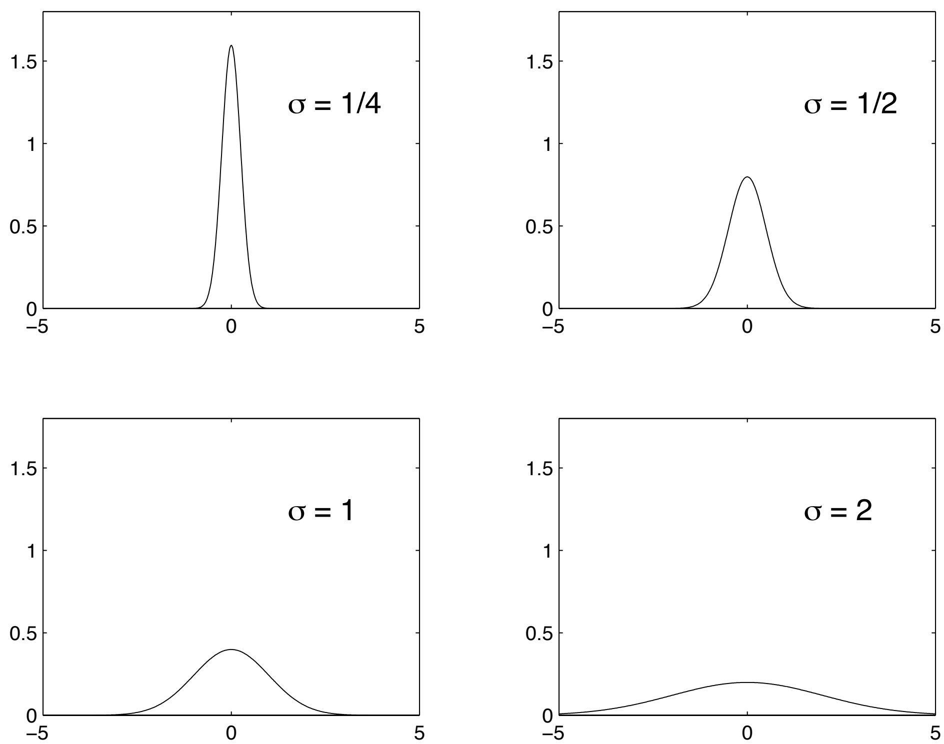

2. Find the probability that $X$ is 3 sd or more greater than $\mu$.

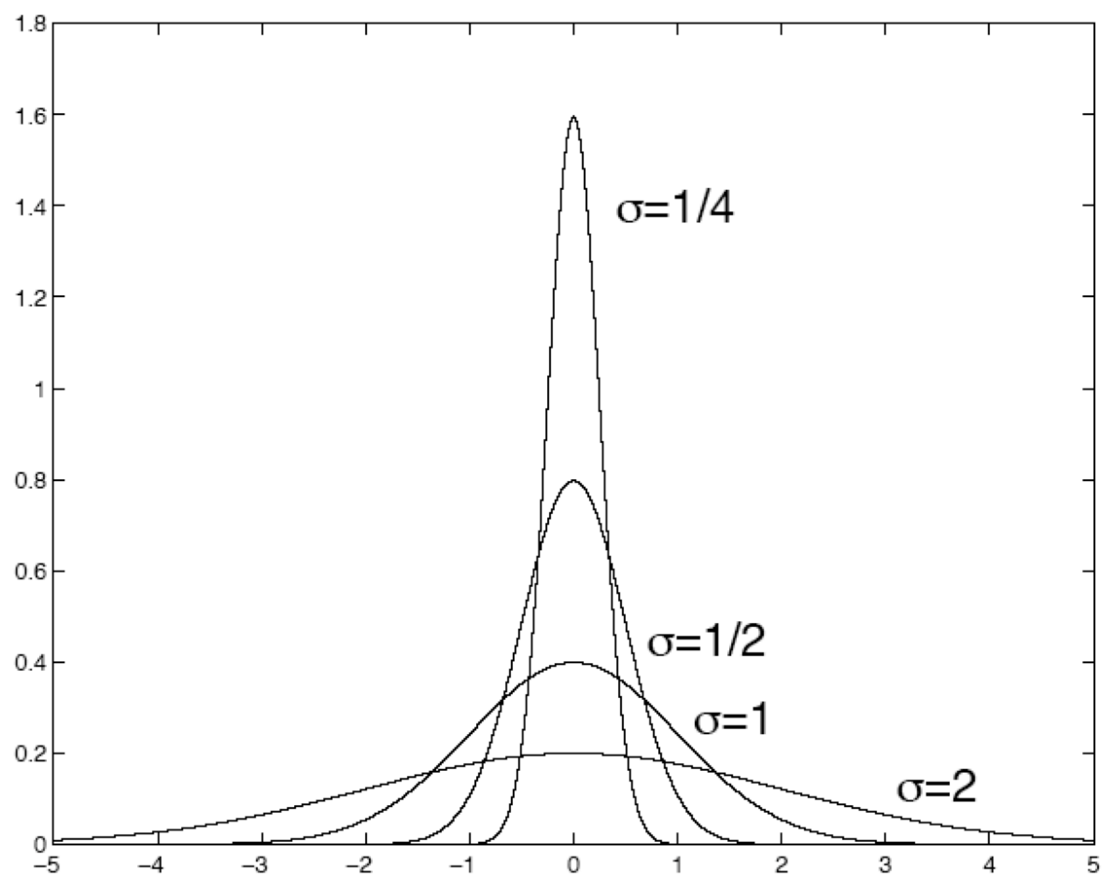
$$\text{prob}(X \ge \mu + 3\sigma) \ = \ \ldots \ = \ \frac{1}{\sqrt{\pi}} \int_{\frac{3}{\sqrt{2}}}^{\infty} e^{-x^2} dx \ = \ \tfrac{1}{2}(1 - \text{erf}(\tfrac{3}{\sqrt{2}})) = 0.001349$$

ex : Annual rainfall in a certain state is normally distributed with $\mu = 25$ in and $\sigma = 5$ in.

1. 68% of years have rainfall between 20 in and 30 in (twice in 3 years)
2. 0.13% . . . . . . . . . " . . . . . . . . greater than or equal to 40 in (once in 740 years)

$$\underline{\text{pdf of a normal distribution}} \ : \ f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \ , \ \mu = 0$$

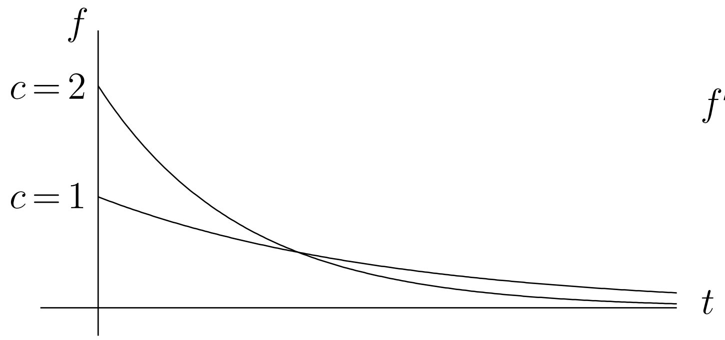


<u>exponential distribution</u>

$T$ = waiting time in the supermarket checkout line (minutes)

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ c\,e^{-ct} & \text{if } t \geq 0 \end{cases}, \quad \text{where } c > 0 \text{ is a constant}$$




$$f'(t) = -c^2 e^{-ct} \;\Rightarrow\; f'(0) = -c^2$$

check : $\int_{-\infty}^{\infty} f(t)dt = \int_0^{\infty} c\,e^{-ct}dt = c\,\dfrac{e^{-ct}}{-c}\Big|_0^{\infty} = 0 + 1 = 1$ <u>ok</u>

average waiting time $= \mu = \int_{-\infty}^{\infty} t f(t)dt = \int_0^{\infty} t\,c\,e^{-ct}dt \quad (u = c\,t\,,\ du = c\,dt)$

$$= \int_0^{\infty} u\,e^{-u}\frac{du}{c} = \frac{1}{c}$$

<u>ex</u>: Assume the average waiting time is 5 minutes. Then $\mu = 5 \Rightarrow c = \frac{1}{5}$.

1. Find the probability that a shopper waits 1 minute or less.

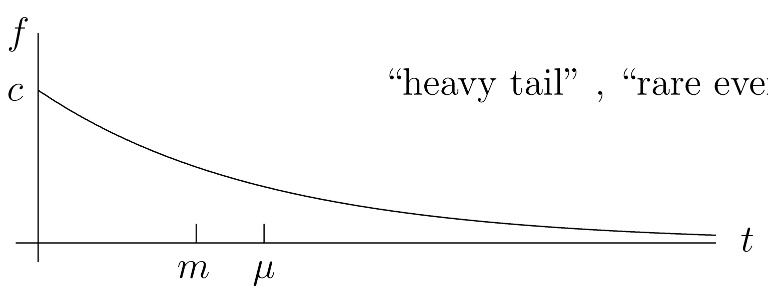
$\text{prob}(0 \leq T \leq 1) = \int_0^1 f(t)\,dt = \int_0^1 c\,e^{-ct}\,dt = -e^{-ct}\Big|_0^1 = -e^{-c} + 1 = 1 - e^{-\frac{1}{5}} = 0.1813$

Hence 18% of shoppers wait one minute or less.

2. Find the probability that a shopper waits 5 minutes or more.

$\text{prob}(T \geq 5) = \int_5^{\infty} f(t)\,dt = \int_5^{\infty} c\,e^{-ct}\,dt = -e^{-ct}\Big|_5^{\infty} = 0 - (-e^{-5c}) = e^{-1} = 0.3679$

Hence even though the <u>average</u> waiting time is 5 minutes, only 37% of shoppers actually wait 5 minutes or more. In fact, the <u>median</u> waiting time is only 3.5 minutes (hw7), i.e. half the shoppers wait less than 3.5 minutes and half wait more.




The average waiting time ($\mu = 5$) is greater than the median waiting time ($m = 3.5$) because some of the shoppers who wait more than 3.5 minutes actually wait a lot longer (e.g. 10 minutes).