

chapter 5 : approximation and interpolation

basic questions

How can we $\begin{cases} \text{approximate an arbitrary function } f(x) \text{ by a polynomial } p(x)? \\ \text{interpolate a set of data values } (x_i, f_i) \text{ by a polynomial } p(x)? \end{cases}$

recall

A polynomial of degree n has the form $p(x) = a_0 + a_1x + \dots + a_nx^n$.

application

$$\int_a^b f(x)dx \rightarrow \int_a^b p(x)dx \quad , \quad \dots$$

note

Given $f(x)$, there are many ways to find an approximating polynomial $p(x)$.

Taylor approximation

$$f(x) = p(x) + r(x)$$

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

: Taylor polynomial of degree n about $x = a$

$$r(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1} \quad : \quad \text{remainder} \quad , \quad \text{error}$$

note

The Taylor polynomial $p(x)$ satisfies the following conditions.

$$p(a) = f(a) \quad , \quad p'(a) = f'(a) \quad , \quad \dots \quad , \quad p^{(n)}(a) = f^{(n)}(a)$$

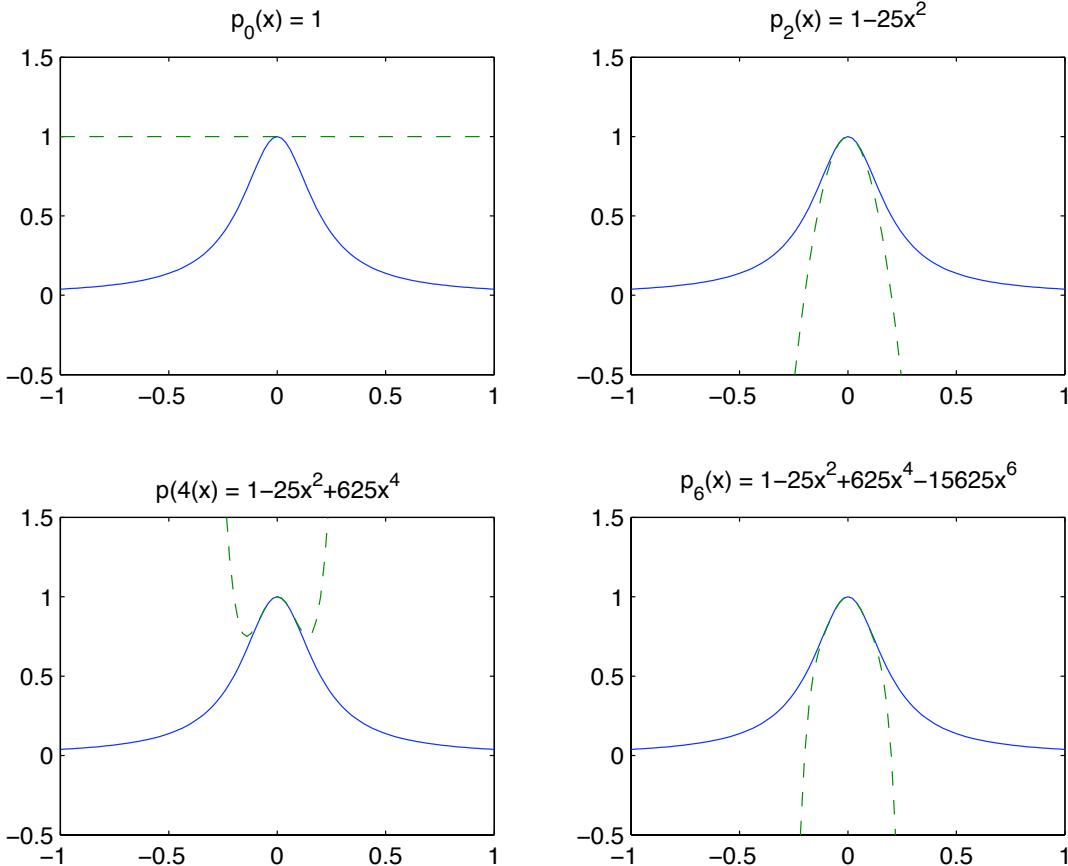
ex

$$f(x) = \frac{1}{1+25x^2} \quad , \quad a = 0$$

In this case there is a simple way to find the Taylor polynomial using the formula for the sum of a geometric series.

$$1 + s + s^2 + s^3 + \dots = \begin{cases} \frac{1}{1-s} & \text{if } |s| < 1 \\ \text{diverges} & \text{if } |s| \geq 1 \end{cases}$$

$$s = -25x^2 \Rightarrow \frac{1}{1+25x^2} = 1 + (-25x^2) + (-25x^2)^2 + (-25x^2)^3 + \dots$$



note : $|x| < \frac{1}{5} \Rightarrow \lim_{n \rightarrow \infty} p_{2n}(x) = f(x)$, $|x| \geq \frac{1}{5} \Rightarrow \lim_{n \rightarrow \infty} p_{2n}(x)$ diverges

explanation : $f(x)$ has complex singularities at $x = \pm \frac{1}{5}i$, Math 555

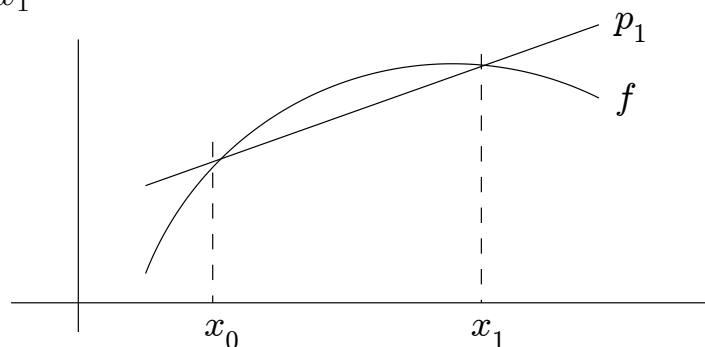
Hence, the Taylor polynomial is a good approximation to $f(x)$ for $x \approx a$, but in general we need to consider alternative methods of approximation.

polynomial interpolation

thm: Let $f(x)$ be given and let x_0, x_1, \dots, x_n be $n + 1$ distinct points. Then there exists a unique polynomial $p_n(x)$ of degree n which interpolates $f(x)$ at the given points, i.e. such that $p_n(x_i) = f(x_i), i = 0 : n$.

pf: existence : soon , uniqueness : Fundamental Thm of Algebra

ex : $n = 1$, x_0, x_1



questions

1. How can we construct $p_n(x)$ for a general value of n ?
2. Is there an efficient way to evaluate $p_n(x)$ for $x \neq x_i$?
3. How large is the error $|f(x) - p_n(x)|$ for $x \neq x_i$?

section 5.1 : Lagrange form of the interpolating polynomial

def

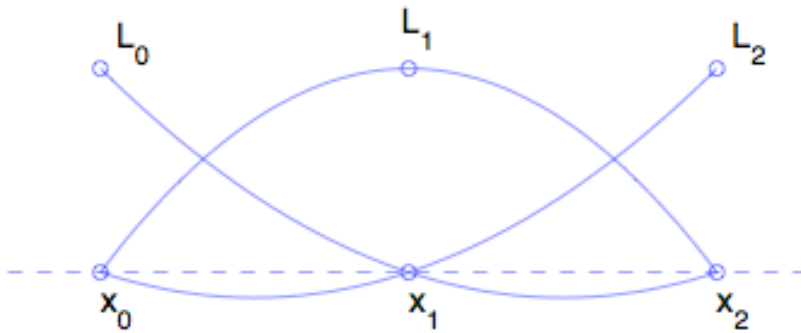
$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^n \left(\frac{x - x_i}{x_k - x_i} \right), \quad k = 0 : n : \text{Lagrange polynomial}$$

ex : $n = 2$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0)(x - 1)}{(-1 - 0)(-1 - 1)} = \frac{1}{2}x^2 - \frac{1}{2}x$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - (-1))(x - 1)}{(0 - (-1))(0 - 1)} = -x^2 + 1$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - (-1))(x - 0)}{(1 - (-1))(1 - 0)} = \frac{1}{2}x^2 + \frac{1}{2}x$$



properties : general value of n

1. $\deg L_k = n$
2. $L_k(x_i) = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$

Now let $f(x)$ be given. The Lagrange form of the interpolating polynomial is

$$p_n(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x) = \sum_{k=0}^n f(x_k)L_k(x).$$

check

1. $\deg p_n \leq n$
2. $p_n(x_i) = \sum_{k=0}^n f(x_k)L_k(x_i) = f(x_i)$, $i = 0 : n$ ok

ex

$$f(x) = \frac{1}{1 + 25x^2}, \quad x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$$

$$\begin{aligned} p_2(x) &= f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) \\ &= \frac{1}{26} \cdot \left(\frac{1}{2}x^2 - \frac{1}{2}x\right) + 1 \cdot (-x^2 + 1) + \frac{1}{26} \cdot \left(\frac{1}{2}x^2 + \frac{1}{2}x\right) = 1 - \frac{25}{26}x^2 \end{aligned}$$

check ... ok

note

The Lagrange form is useful theoretically (to show that $p_n(x)$ exists), but it has practical disadvantages.

1. it is expensive to evaluate $p_n(x)$ for $x \neq x_i$
2. adding an extra point x_{n+1} means that all the $L_k(x)$ must be recomputed

section 5.3 : Newton form of the interpolating polynomial

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1})$$

ex : $n = 1$, x_0 , x_1

$$\begin{aligned} p_1(x) &= f(x_0) \left(\frac{x - x_0}{x_0 - x_1} \right) + f(x_1) \left(\frac{x - x_1}{x_1 - x_0} \right) : \text{Lagrange form} \\ &= f(x_0) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0) : \text{Newton form} \end{aligned}$$

note

For $n \geq 2$, we need a method to compute a_0, \dots, a_n . Suppose that $p_{n-1}(x)$ is already known. Then define $p_n(x) = p_{n-1}(x) + a_n(x - x_0) \cdots (x - x_{n-1})$. We will derive a formula for a_n .

$$i = 0 : n - 1 \Rightarrow p_n(x_i) = p_{n-1}(x_i) + a_n(x_i - x_0) \cdots (x_i - x_{n-1}) = f(x_i)$$

$$i = n \Rightarrow p_n(x_n) = p_{n-1}(x_n) + a_n(x_n - x_0) \cdots (x_n - x_{n-1}) = f(x_n)$$

$$\Rightarrow a_n = \frac{f(x_n) - p_{n-1}(x_n)}{(x_n - x_0) \cdots (x_n - x_{n-1})}$$

This formula is correct, but there is a more efficient way to compute a_n . First define some notation.

$$a_n = f[x_0, \dots, x_n] \Rightarrow a_0 = f[x_0], \quad a_1 = f[x_0, x_1], \quad a_2 = f[x_0, x_1, x_2], \quad \dots$$

claim : $f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$: divided difference

pf

$p_{n-1}(x)$ interpolates $f(x)$ at x_0, \dots, x_{n-1} , $\deg p_{n-1} \leq n - 1$

$q_{n-1}(x)$ interpolates $f(x)$ at x_1, \dots, x_n , $\deg q_{n-1} \leq n - 1$

define $g(x) = \left(\frac{x - x_0}{x_n - x_0}\right)q_{n-1}(x) + \left(\frac{x_n - x}{x_n - x_0}\right)p_{n-1}(x)$

then $\deg g \leq n$

$g(x_0) = p_{n-1}(x_0) = f(x_0)$, $g(x_n) = q_{n-1}(x_n) = f(x_n)$

$i = 1 : n - 1 \Rightarrow g(x_i) = \left(\frac{x_i - x_0}{x_n - x_0}\right)q_{n-1}(x_i) + \left(\frac{x_n - x_i}{x_n - x_0}\right)p_{n-1}(x_i) = f(x_i)$

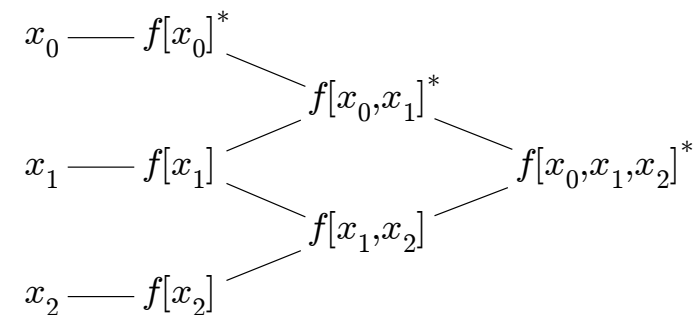
$\Rightarrow g(x) = p_n(x)$, now equate the coefficients of x^n

$\Rightarrow \frac{f[x_1, \dots, x_n]}{x_n - x_0} - \frac{f[x_0, \dots, x_{n-1}]}{x_n - x_0} = f[x_0, \dots, x_n]$ ok

Hence we can rewrite Newton's form of the interpolating polynomial as follows.

$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots + f[x_0, \dots, x_n](x - x_0) \cdots (x - x_{n-1})$

divided difference table



The starred values are the coefficients in Newton's form of $p_n(x)$.

ex: $f(x) = \frac{1}{1+25x^2}$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$$\begin{array}{cccc}
 -1 & \frac{1}{26}^* & & \\
 & & \frac{25}{26}^* & \\
 0 & 1 & -\frac{25}{26}^* & \Rightarrow p_2(x) = \frac{1}{26} + \frac{25}{26}(x - (-1)) - \frac{25}{26}(x - (-1))(x - 0) \\
 & & -\frac{25}{26} & \\
 1 & \frac{1}{26} & & = 1 - \frac{25}{26}x^2 \quad \text{ok}
 \end{array}$$

note

Using Newton's form for $p_n(x)$, if an extra interpolation point x_{n+1} is added, then the previous terms remain unchanged.

evaluation of $p_n(x)$

$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$: Newton form , 3 mults

$= a_0 + (x - x_0)(a_1 + a_2(x - x_1))$: nested form , 2 mults

general case

$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1})$

$= a_0 + (x - x_0)(a_1 + (x - x_1)(a_2 + \cdots + a_n(x - x_{n-1}) \cdots))$

operation count : Newton form = $\frac{n(n+1)}{2}$ mults , nested form = n mults

thm (the error in polynomial interpolation)

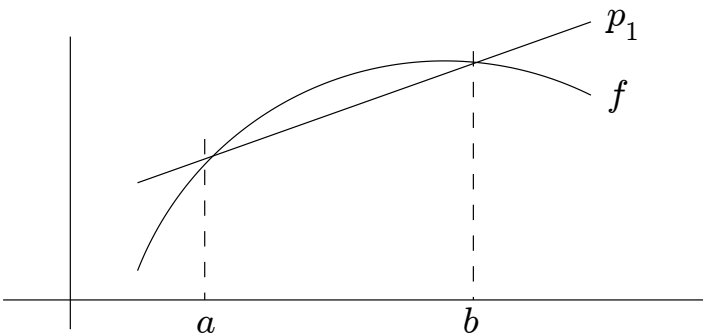
$f(x)$, x_0, \dots, x_n : given , $p_n(x)$: interpolating polynomial of degree n

$$\Rightarrow f(x) = p_n(x) + \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x - x_0) \cdots (x - x_n)$$

pf

omit , but note that this resembles Taylor approximation

application : $n = 1$, $x_0 = a$, $x_1 = b$



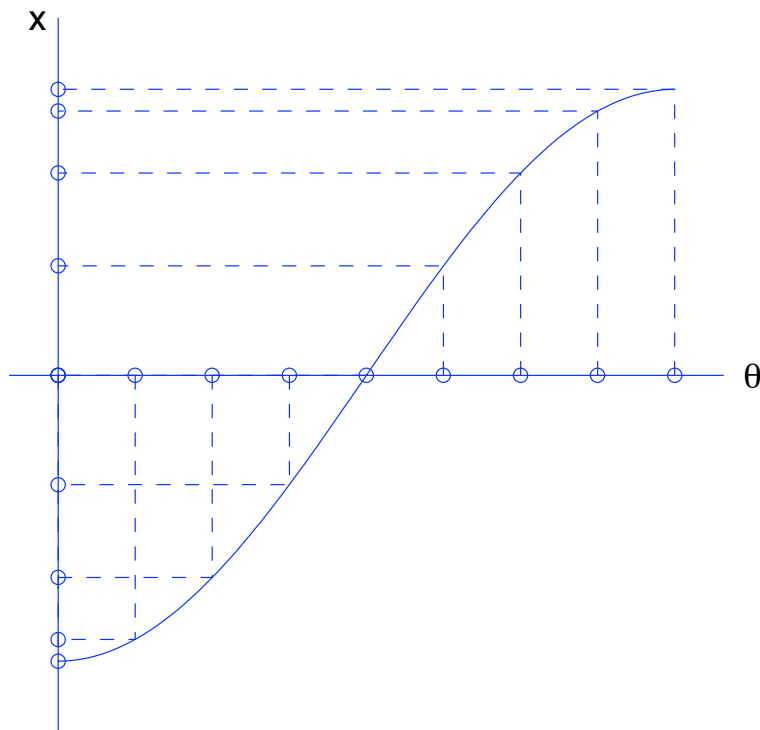
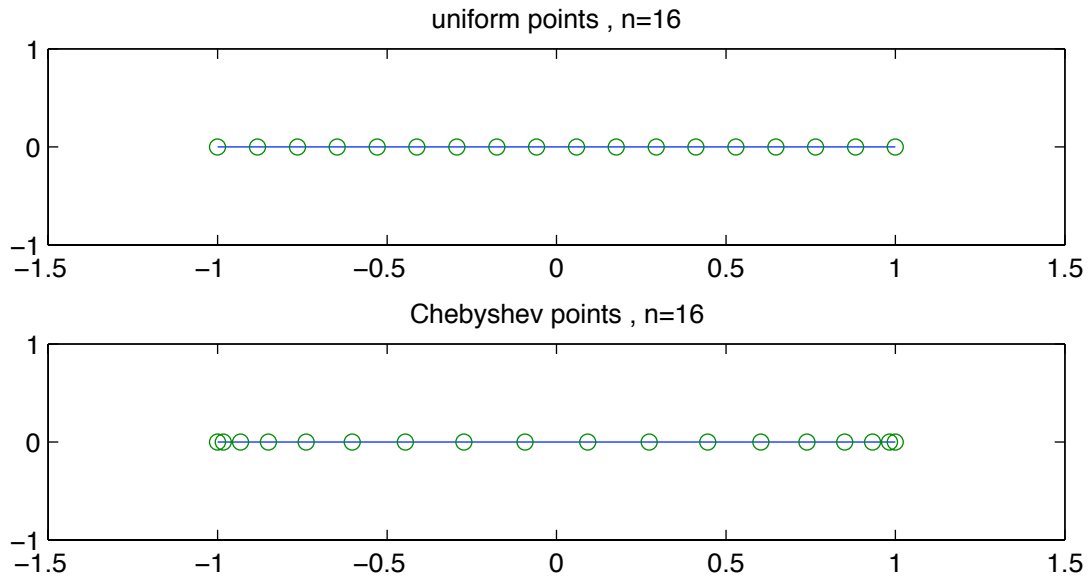
$$a \leq x \leq b \Rightarrow |f(x) - p_1(x)| \leq \frac{1}{8} M (b - a)^2 , \text{ where } M = \max_{a \leq x \leq b} |f''(x)|$$

pf : review sheet

section 5.4 : optimal points for interpolation

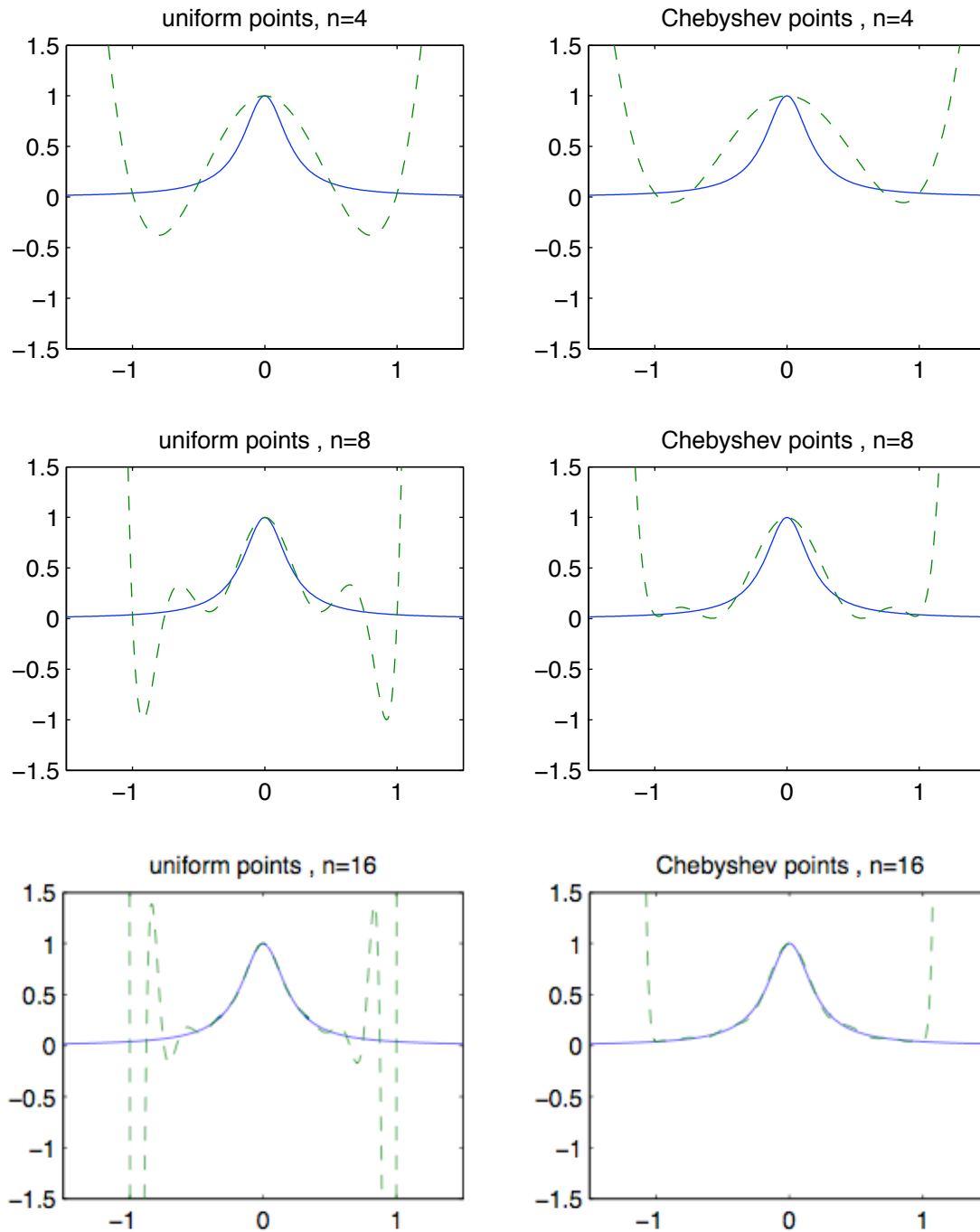
Given $f(x)$ for $-1 \leq x \leq 1$, consider two choices for the interpolation points x_0, \dots, x_n .

1. uniform points : $x_i = -1 + ih$, $h = \frac{2}{n}$, $i = 0 : n$
2. Chebyshev points : $x_i = -\cos(ih)$, $h = \frac{\pi}{n}$, $i = 0 : n$



note : The Chebyshev points are clustered near the endpoints of the interval.

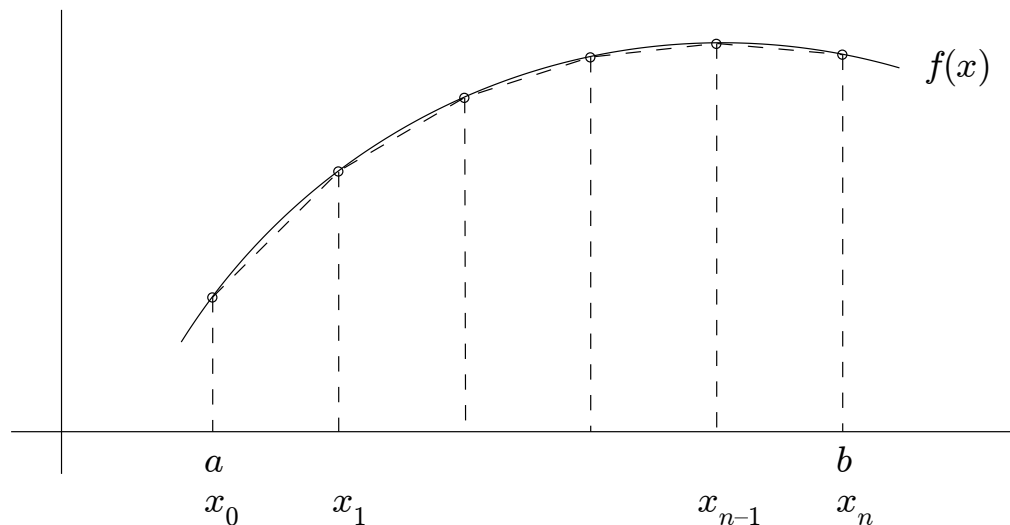
ex : $f(x) = \frac{1}{1 + 25x^2}$, polynomial interpolation on the interval $-1 \leq x \leq 1$



1. Interpolation at the uniform points gives a good approximation near the center of the interval, but it gives a bad approximation near the endpoints.
2. Interpolation at the Chebyshev points gives a good approximation on the entire interval.

section 5.5 : piecewise linear interpolation

Given $f(x)$, $a \leq x \leq b$, $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.



The interpolating polynomial $p_n(x)$ may not be a good approximation to $f(x)$ on the entire interval, so instead we consider $g(x)$, the piecewise linear interpolant.

$x_i \leq x \leq x_{i+1} \Rightarrow g(x) = f[x_i] + f[x_i, x_{i+1}](x - x_i) \Rightarrow g(x_i) = f(x_i)$, $i = 0 : n$

error : $|f(x) - g(x)| \leq \frac{M}{8} \max_i |x_{i+1} - x_i|^2$, where $M = \max_{a \leq x \leq b} |f''(x)|$

note : $g(x)$ is continuous, but it is not differentiable at $x = x_i$.

section 5.6 : cubic spline interpolation

A cubic spline is a function $s(x)$ satisfying the following conditions.

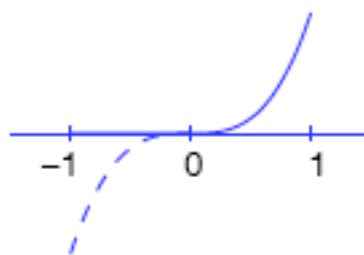
1. For each interval $x_i \leq x \leq x_{i+1}$, $s(x)$ is a cubic polynomial.
2. $s(x)$, $s'(x)$, $s''(x)$ are continuous at the interior points x_1, \dots, x_{n-1}

ex : $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$$s(x) = \begin{cases} 0 & , -1 \leq x \leq 0 \\ x^3 & , 0 \leq x \leq 1 \end{cases}$$

$$s'(x) = \begin{cases} 0 & , -1 \leq x \leq 0 \\ 3x^2 & , 0 \leq x \leq 1 \end{cases}$$

$$s''(x) = \begin{cases} 0 & , -1 \leq x \leq 0 \\ 6x & , 0 \leq x \leq 1 \end{cases}$$



We see that $s(x)$ is a cubic spline.

problem

Given $f(x)$, $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. Find a cubic spline $s(x)$ such that $s(x_i) = f(x_i)$, $i = 0 : n$.

$n + 1$ points $\Rightarrow n$ intervals $\Rightarrow 4n$ unknown coefficients

interpolation conditions $\Rightarrow 2n$ equations

continuity of $s'(x)$, $s''(x)$ at interior points $\Rightarrow 2(n - 1)$ equations

We can choose 2 more equations. A popular choice is to set $s''(x_0) = s''(x_n) = 0$, which gives the natural cubic spline interpolant.

how to find $s(x)$

ex : $-1 \leq x \leq 1$, $x_i = -1 + ih$, $h = \frac{2}{n}$, $i = 0 : n$

step 1 : 2nd derivative conditions

$x_i \leq x \leq x_{i+1} \Rightarrow s(x) = s_i(x)$, $i = 0 : n - 1$

$s_i(x)$ is cubic polynomial

$s_i''(x)$ is a linear polynomial $\Rightarrow s_i''(x) = a_i \left(\frac{x_{i+1} - x}{h} \right) + a_{i+1} \left(\frac{x - x_i}{h} \right)$

$\Rightarrow s_i''(x_i) = a_i$, $s_i''(x_{i+1}) = a_{i+1} \Rightarrow s_{i-1}''(x_i) = a_i = s_i''(x_i)$

Hence $s''(x)$ is continuous at the interior points.

step 2 : interpolation

integrate twice

$$s_i(x) = \frac{a_i(x_{i+1} - x)^3}{6h} + \frac{a_{i+1}(x - x_i)^3}{6h} + b_i \left(\frac{x_{i+1} - x}{h} \right) + c_i \left(\frac{x - x_i}{h} \right)$$

$$s_i(x_i) = \frac{a_i h^2}{6} + b_i = f_i \Rightarrow b_i = f_i - \frac{a_i h^2}{6}$$

$$s_i(x_{i+1}) = \frac{a_{i+1} h^2}{6} + c_i = f_{i+1} \Rightarrow c_i = f_{i+1} - \frac{a_{i+1} h^2}{6}$$

$$s_i(x) = \frac{a_i(x_{i+1} - x)^3}{6h} + \frac{a_{i+1}(x - x_i)^3}{6h} + \left(f_i - \frac{a_i h^2}{6} \right) \left(\frac{x_{i+1} - x}{h} \right) + \left(f_{i+1} - \frac{a_{i+1} h^2}{6} \right) \left(\frac{x - x_i}{h} \right)$$

step 3 : 1st derivative conditions

$$s'_i(x) = -\frac{a_i(x_{i+1} - x)^2}{2h} + \frac{a_{i+1}(x - x_i)^2}{2h} \\ + \left(f_i - \frac{a_i h^2}{6}\right) \cdot \frac{-1}{h} + \left(f_{i+1} - \frac{a_{i+1} h^2}{6}\right) \cdot \frac{1}{h}$$

$$s'_i(x_i) = -\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}$$

$$s'_i(x_{i+1}) = \frac{a_{i+1} h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}$$

we require $s'_{i-1}(x_i) = s'_i(x_i)$

$$\frac{a_i h}{2} - \frac{f_{i-1}}{h} + \frac{a_{i-1} h}{6} + \frac{f_i}{h} - \frac{a_i h}{6} = -\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}$$

$$\frac{a_{i-1} h}{6} + a_i \left(\frac{h}{2} - \frac{h}{6} + \frac{h}{2} - \frac{h}{6}\right) + \frac{a_{i+1} h}{6} = \frac{f_{i-1} - 2f_i + f_{i+1}}{h}$$

$$a_{i-1} + 4a_i + a_{i+1} = \frac{6}{h^2} (f_{i-1} - 2f_i + f_{i+1}), \quad i = 1 : n - 1$$

step 4 : BC

$$s''_0(x_0) = 0 \Rightarrow a_0 = 0, \quad s''_{n-1}(x_n) = 0 \Rightarrow a_n = 0$$

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} f_0 - 2f_1 + f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix}$$

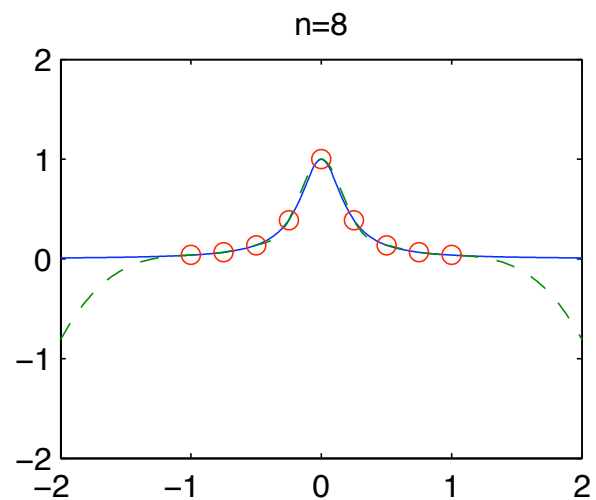
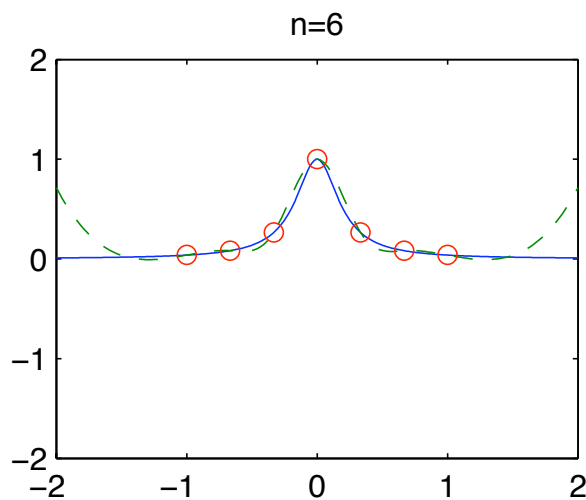
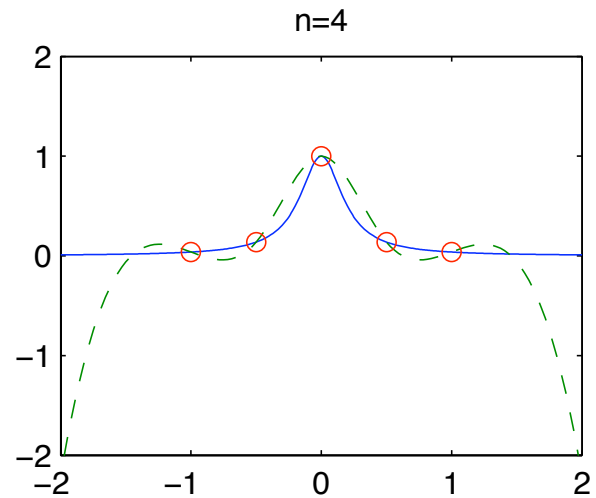
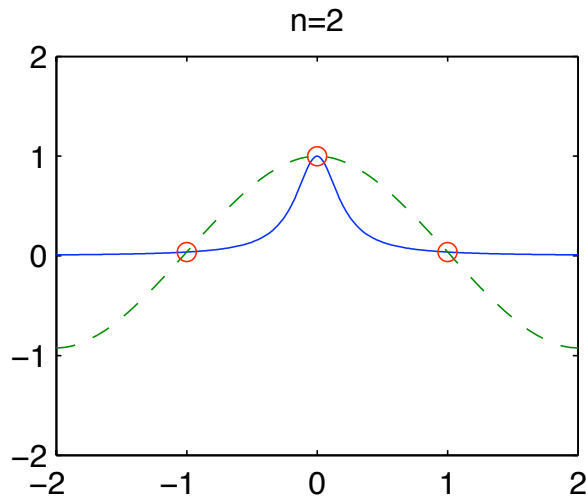
tridiagonal

symmetric

positive definite

ex : natural cubic spline interpolation

$$f(x) = \frac{1}{1 + 25x^2}, \quad -1 \leq x \leq 1, \quad x_i = -1 + ih, \quad h = \frac{2}{n}, \quad i = 0 : n$$



note

1. The natural cubic spline has inflection points at the endpoints of the interval, in keeping with the boundary conditions $s''(x_0) = s''(x_n) = 0$.
2. There are additional inflection points in the interior of the interval.
3. The spline approximations $s(x)$ converge uniformly to $f(x)$ on the interval $-1 \leq x \leq 1$ as the number of interpolation points increases.

4. $|f(x) - s(x)| \leq \frac{5}{384} \max_{a \leq x \leq b} |f^{(4)}(x)| h^4$: 4th order accurate