Math 371 **Review Sheet for 1st Midterm Exam** **Fall 2008**

The 1st midterm exam is on Thursday Oct 16 in class. You may use one sheet of notes (i.e. one side of one page, 8.5 in $\times$ 11 in). You may use a non-programmable calculator to do arithmetic, but to receive full credit you must show all intermediate steps. The exam will cover up to section 3.5 (LU factorization). Exam booklets will be provided.

1. True or False? Give a reason to justify your answer.

(a) If two floating point numbers with n significant digits are subtracted, then the result also has n significant digits.

(b) Suppose the derivative $f'(x)$ is approximated by the forward difference approximation $D_+f(x)$ with step size h. Then for large h, the roundoff error dominates the truncation error, but for small h, the truncation error dominates the roundoff error.

(c) $D_+D_-f = D_-D_+f$

(d) If A is invertible, then the pivots in Gaussian elimination are nonzero.

(e) In solving an $n \times n$ system of linear equations by Gaussian elimination, if n is increased by a factor of 10, then the operation count increases by a factor of 10^3.

(f) In solving a linear system of equations by Gaussian elimination, partial pivoting is often used even if the pivots are nonzero, in order to reduce the operation count.

(g) If A is invertible, then $\kappa_\infty(A) \geq 1$.

chapter 1 : finite precision arithmetic

2. Let $f(x) = \sqrt{1+x^2} - 1$. (a) Evaluate $f(x)$ for $x = 0.1$ using 4-digit arithmetic; (b) Show that $f(x) = x^2/(\sqrt{1+x^2}+1)$ and repeat part (a); (c) Find the quadratic Taylor approximation for $f(x)$ about $x = 0$ and evaluate it for $x = 0.1$ using 4-digit arithmetic; (d) Matlab gives $f(0.1) = 0.004987562112089$. Which of the methods agrees best with Matlab's result? Explain why the other methods are less accurate.

3. Show that the central difference approximation defined by $D_0f(x) = \frac{f(x+h)-f(x-h)}{2h}$ is 2nd order accurate, i.e. show that $D_0f(x) = f'(x) + O(h^2)$.

chapter 2 : rootfinding

4. Suppose the equation $f(x) = x^2 - 5 = 0$ is solved by the bisection method with $a = 0$ and $b = 3$. How many steps are needed to ensure that the error is less than 10^{-3}?

5. The screened Coulomb potential is given by $\phi(r) = \frac{e^{-\kappa r}}{r}$, where r is the distance from a charged particle to a point in space and κ controls the screening effect. Let $\kappa = \frac{1}{2}$. Use Newton's method to find the value of r for which $\phi(r) = 0.005$. Let $r_0 = 1$ be the starting value and take two steps. How many digits in the final result can you trust?

6. Below is the algorithm for the bisection method. Find and correct any errors.

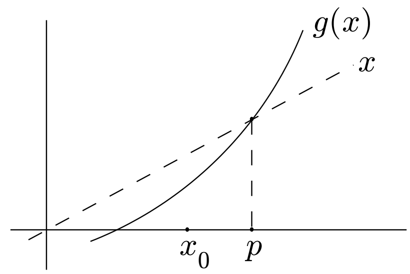
<u>bisection method</u> (assume $f(a) \cdot f(b) > 0$)

1. $n = 0$, $a_0 = a$, $b_0 = b$
2. $x_n = \frac{a_n - b_n}{2}$: current estimate of the root
3. if $f(x_n) \cdot f(a_n) < 0$, then $a_{n+1} = x_n$, $b_{n+1} = b_n$
4. else $a_{n+1} = a_n$, $b_{n+1} = x_n$
5. set $n = n + 1$ and go to line 1

7. Consider the problem of solving $f(x) = 0$. (a) State one advantage of Newton's method over the bisection method; (b) State advantage of the bisection method over Newton's method.

8. Suppose that fixed-point iteration is applied to the function $g(x) = x^2 - \frac{3}{2}x + \frac{1}{2}$. Given that the fixed point occurs at $x = 1$, does the method converge or diverge?

9. The plot on the right shows a function $g(x)$ and its fixed point p. Consider fixed-point iteration given by $x_{n+1} = g(x_n)$. (a) Given the initial guess x_0 as indicated, find x_1. (b) Does the sequence x_n converge to p?



chapter 3 : linear algebra

10. Solve $Ax = b$ by Gaussian elimination with partial pivoting.

(a) $A = \begin{pmatrix} 0 & 4 & -15 \\ 10 & 0 & 15 \\ 1 & -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} -12 \\ 100 \\ 0 \end{pmatrix}$; (b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

11. Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find a vector x such that $||Ax||_\infty = ||A||_\infty$.

12. Show that if $Ax = b$ and $\tilde{A}\tilde{x} = b$, then $\dfrac{||x - \tilde{x}||}{||\tilde{x}||} \leq \kappa(A)\dfrac{||A - \tilde{A}||}{||A||}$. (This result says that the condition number controls the size of perturbations in the solution of $Ax = b$ due to perturbations in the matrix.)

13. Let $A = \begin{pmatrix} 1{\cdot}2969 & 0{\cdot}8648 \\ 0{\cdot}2161 & 0{\cdot}1441 \end{pmatrix}$, $b = \begin{pmatrix} 0{\cdot}8642 \\ 0{\cdot}1440 \end{pmatrix}$, $x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\tilde{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\tilde{x}_1 = \begin{pmatrix} 0{\cdot}9911 \\ -0{\cdot}4870 \end{pmatrix}$.

(a) Show that A is invertible; (b) Show that x is the exact solution of $Ax = b$; (c) Find $||e||_\infty$, $||r||_\infty$ for the vectors $\tilde{x}_1, \tilde{x}_2$; (d) Find $\kappa_\infty(A)$; (This example shows that for an ill-conditioned matrix, if the residual is small, there's no guarantee that the error is also small.)

14. Let $A \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Solve $Ax = b$ for x by LU factorization.

15. Let A be a 3×3 matrix. Suppose we apply LU factorization with partial pivoting and obtain the relation $E_2P_2E_1P_1A = U$, where U is upper triangular and

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Compute $\tilde{E}_1 = P_2E_1P_2$; (b) Show that $P_2^2 = I$; (c) Show that $P_2E_1 = \tilde{E}_1P_2$; (d) Compute $P = P_2P_1$ and $L = \tilde{E}_1^{-1}E_2^{-1}$; (e) Show that $PA = LU$; (f) Find P, L, U such that $PA = LU$ for the following matrix A, and use the factorization to solve $Ax = b$.

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$