Math 371 **Review Sheet for 2nd Midterm Exam** **Fall 2008**

The 2nd midterm exam is on Thursday, November 20, in class. You may use one sheet of notes (i.e. one side of one page, 8.5 in × 11 in). You may use a non-programmable calculator to do arithmetic, but to receive full credit you must show all intermediate steps. Exam booklets will be provided. The exam will focus on material since the 1st exam: 2-point BVP (section 8.1), special matrices (section 3.7), iterative methods (section 3.8), two-dimensional BVP (section 9.1), computing eigenvalues (sections 4.1, 4.2), and the review sheet.

1. True or False? Give a reason to justify your answer.

a) The spectral radius of a matrix satisfies the properties of a matrix norm.

b) $\rho(B) \leq ||B||_\infty$ for any matrix B

c) In solving a two-point boundary value problem using a 2nd order accurate finite-difference scheme, if the mesh size h is reduced by one half, then the norm of the error is also reduced by one half.

d) Suppose that a two-point boundary value problem is discretized using a finite-difference scheme and the resulting linear system is solved by Jacobi's method with stopping criterion $||r_k||_\infty \leq 10^{-2}$. If the mesh size h is made smaller, then the number of iterations needed to satisfy the stopping criterion is reduced.

e) In solving a two-dimensional boundary value problem using a finite-difference scheme and an iterative solver, the ratio of successive residual norms tends to zero as the mesh size h is reduced.

f) Consider an iterative method $x_{k+1} = Bx_k + c$ for solving $Ax = b$. If $||B||_\infty < 1$, then the method converges for all initial vectors x_0.

g) Wilkinson's example shows that the e-values of a matrix can depend very sensitively on perturbations of the matrix elements.

h) If A is invertible, symmetric, and positive definite, and x is any nonzero vector, then the Rayleigh quotient $R_A(x)$ is positive.

2. Consider a finite-difference approximation for the first derivative of a function,

$$f'(x) \approx \frac{af(x+h) + bf(x) + cf(x-h)}{h},$$

where a, b, c are unknown coefficients. The forward difference approximation D_+f, given by $(a, b, c) = (1, -1, 0)$, is 1st order accurate, and the central difference approximation $D_0 f$, given by $(a, b, c) = (\frac{1}{2}, 0, -\frac{1}{2})$, is 2nd order accurate. Are there values of (a, b, c) that yield 3rd order accuracy?

3. Consider the linear system, $2x_1 - x_2 = 1, -x_1 + 2x_2 - x_3 = 0, -x_2 + 2x_3 = 1$, with solution $x_1 = x_2 = x_3 = 1$. a) Write out Jacobi's method in component form and take one step starting from the zero vector. Compute the error norms $||e_0||_\infty, ||e_1||_\infty$. b) Repeat part (a) for Gauss-Seidel.

4. Consider the matrices $A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. a) For which of these does Jacobi's method converge? b) For which of these does Gauss-Seidel converge?

5. The formulas $\rho(B_J) = \cos \pi h$, $\rho(B_{GS}) = \cos^2 \pi h$, $\rho(B_{\omega_*}) = \frac{1-\sin \pi h}{1+\sin \pi h}$ were given in class for the spectral radius of the iteration matrix in the case of a finite-difference scheme applied to a boundary value problem. Graph these functions on the same plot for $0 \leq h \leq \frac{1}{2}$. Label each function. What do the results say about the convergence of the methods?

6. Which of these matrices is positive definite? a) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, b) $\begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}$

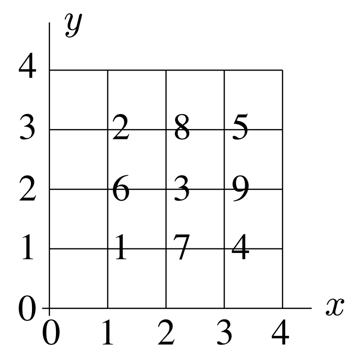
7. Prove the following results.

a) If A is positive definite, then A is invertible.

b) If A is positive definite, then the diagonal elements of A are positive.

c) If A is invertible, then A^TA is symmetric and positive definite.

8. Consider the Poisson equation $-\Delta\phi = f$ with boundary condition $\phi = g$, on the unit square $0 \leq x, y \leq 1$. Let the domain be discretized with mesh size $h = \frac{1}{4}$. Then there are nine unknown values in the interior of the domain, w_{ij}, for $i, j = 1, 2, 3$. Suppose the equation is discretized using the finite-difference scheme discussed in class and the linear system is written as $A_h w_h = f_h$, where the elements of w_h have the ordering shown in the figure (this is called the red-black ordering and it is different than the ordering considered in class). Write down the matrix A_h in this case.



9. Find the eigenvalues and eigenvectors of the matrix.

a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ d) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

10. Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

a) Take two steps of the power method starting from the initial vector $v^{(0)} = \frac{1}{\sqrt{3}}(1, 1, 1)^T$, i.e. compute $v^{(k)}, R_A(v^{(k)})$ for $k = 1, 2$.

b) Show that $\lambda = 2 + \sqrt{2}$ is an e-value of A. Is it the largest e-value of A?

11. Let A be a real symmetric matrix.

a) Show that the eigenvalues of A are real.

b) Let λ_1, λ_2 be distinct eigenvalues of A, i.e. $\lambda_1 \neq \lambda_2$. Show that the corresponding eigenvectors q_1, q_2 are orthogonal.

12. Let $Ax = b$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. In class we computed the e-values λ_1, λ_2 of A and the corresponding orthonormal e-vectors q_1, q_2.

a) Show that $x = \lambda_1^{-1}(q_1^T b)q_1 + \lambda_2^{-1}(q_2^T b)q_2$ by computing the right hand side of this formula. (This illustrates the spectral method, a method that expresses the solution of $Ax = b$ in terms of the e-values and e-vectors of A.)

b) Apply the spectral method to solve $Ax = b$, where A is the same as above and $b = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$.