Math 371 **Review Sheet for Final Exam** **Fall 2008**

The final exam is on Monday December 15, 4-6pm in 2150 Dow. You may use two pages of notes (e.g. two sides of one sheet). You may use a non-programmable calculator to do arithmetic, but to receive full credit you must show all intermediate steps. Exam booklets will be provided.

1. True or False? Give a reason to justify your answer.

a) If two floating point numbers with n significant digits are added, then the result also has n significant digits.

b) $D_+D_-y_i = D_-D_+y_i$ c) $D_+y_{i-1} = D_-y_i$ d) $D_0y_i = \frac{1}{2}(D_+y_i + D_-y_i)$

e) $D_0f(x) = f'(x) + O(h^2)$

f) Suppose $f'(x)$ is approximated by the forward difference approximation $D_+f(x)$ with mesh size h. If h is reduced by a factor of $1/2$, then the error is reduced by a factor of $1/4$.

g) The function $f(x) = x^2 - 3x + 2$ has no roots in the interval $0 \le x \le 3$.

h) Let $f(x) = \frac{1}{1-x}$. Since $f(0) \cdot f(2) < 0$, $f(x)$ has a root in the interval $0 \le x \le 2$.

i) $||x||_2 \le ||x||_\infty$ j) $\kappa(A) = \kappa(A^{-1})$ k) $\rho(A^{-1}) = 1/\rho(A)$

l) If A is invertible, then the pivots in Gaussian elimination are nonzero.

m) If the pivots in Gaussian elimination are nonzero, then A is invertible.

n) In solving an $n \times n$ system of linear equations by Gaussian elimination, if n increases by a factor of 2, then the operation count increases by a factor of 8.

o) $\begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix}$

p) In computing the solution of a linear system $Ax = b$, if the residual norm $||r||$ is small, then the error norm $||e||$ is also small.

q) In solving an $n \times n$ linear system $Ax = b$ by an iterative method $x_{k+1} = Bx_k + c$, if $||B||_\infty < 1$, then $\lim_{k\to\infty} x_k = x$ for any initial guess x_0.

r) In solving a two-dimensional boundary value problem on the unit square by the standard finite-difference scheme with mesh size $h = \frac{1}{n+1}$, the matrix A_h has $O(n^4)$ elements (including zeros), but the linear system $A_hw_h = f_h$ can be solved by Jacobi's method using only $O(n^2)$ elements of storage.

s) Shifted inverse iteration is a method for finding the inverse of a matrix.

t) If $p_n(x)$ is the Taylor polynomial for $f(x)$ at $x = 0$, then $\lim_{n\to\infty} p_n(x) = f(x)$ for all x.

u) If $p_n(x)$ is the interpolating polynomial of degree n for $f(x)$ at points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$, then $\lim_{n\to\infty} p_n(x) = f(x)$ for all x in the interval $a \le x \le b$.

v) Chebyshev points are advantageous for polynomial interpolation because they are clustered near the center of the interval.

w) Suppose that $f(x)$ is approximated by a cubic spline interpolant $s(x)$ on the interval $a \le x \le b$ with interpolation points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$. Then if n is doubled, the error $\max_{a\le x\le b} |f(x) - s(x)|$ is reduced by a factor of $1/16$.

2. Find the order of accuracy of the approximation $f'(x) \approx \dfrac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$.

3. Suppose that $f(x)=x^2-5=0$ is solved by the bisection method with $a=0$ and $b=4$.
a) Take 3 steps of the bisection method starting from $x_0=2$, i.e. compute x_1, x_2, x_3.
b) How many steps are needed to ensure that the error is less than 10^{-4}?

4. Suppose that fixed-point iteration is applied to the function $g(x)=x^2-\frac{3}{2}x+\frac{1}{2}$. Given that the fixed point occurs at $x=1$, does the method converge or diverge?

5. State one advantage of ...
a) ... Newton's method over the secant method in solving $f(x)=0$.
b) ... Gaussian elimination with pivoting over Gaussian elimination without pivoting.
c) ... optimal SOR over Gauss-Seidel in solving $Ax=b$.
d) ... polynomial interpolation over Taylor approximation.
e) ... Newton's form for the interpolating polynomial over Lagrange's form.
f) ... cubic spline interpolation over piecewise linear interpolation.

6. Let $A=\begin{pmatrix}4&1&0\\1&4&1\\0&1&4\end{pmatrix}$.
a) Find a vector x such that $||Ax||_\infty = ||A||_\infty$.
b) Approximate the largest e-value of A by taking one step of the power method with initial guess $v^{(0)}=\frac{1}{\sqrt{3}}(1,1,1)^T$.
c) If $Ax=b$ is solved by Jacobi's method, does the iteration converge for any initial guess?
d) Answer the same question for Gauss-Seidel.
e) Show that A is positive definite.
f) Find an approximation to the optimal SOR parameter ω_*.

7. A theorem was stated in class on the error in polynomial interpolation. The theorem says that given a function $f(x)$ and $n+1$ distinct points $a=x_0<\cdots<b=x_n$, then $f(x)=p_n(x)+\frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-x_0)\cdots(x-x_n)$, where $p_n(x)$ is the polynomial of degree n that interpolates $f(x)$ at the points x_i. Apply the theorem for the case $n=1$ to show that $|f(x)-p_1(x)| \le \frac{1}{8}M(b-a)^2$, where $M=\max\limits_{a\le x\le b}|f''(x)|$.

8. The two-point boundary value problem $y''-y=x, y(0)=1, y(1)=0$ for $0\le x\le 1$ is solved by the finite-difference scheme $D_+D_-w_i - w_i = 0$ for $i=1:n$ with step size $h=1/(n+1)$ and $w_0=1, w_{n+1}=0$. Using $n=3$, write down the linear system $A_h w_h = f_h$.

9. Let $p(x)=1+x+x^2$ and take $x_0=-1, x_1=0, x_2=1$. Write $p(x)$ in (a) Lagrange form, (b) Newton form, (c) nested form.

10. Let $f(x)=\frac{1}{x}$ and take $x_0=1, x_1=2, x_2=3$.
a) Find Newton's form of the interpolating polynomial $p_2(x)$.
b) Write the polynomial in standard form $p_2(x)=c_0+c_1x+c_2x^2$ and check that it interpolates $f(x)$ at the given points.

11. The outdoor temperature $T(t)$ is supposed to be recorded at two-hour intervals starting at 8am, but the 12pm measurement was accidentally omitted. The recorded temperatures are $T(\text{8am})=30^o$ F, $T(\text{10am})=40^o$ F, $T(\text{2pm})=50^o$ F. Use an interpolating polynomial to find the missing temperature $T(\text{12pm})$.

12. Determine whether the given function $s(x)$ is a natural cubic spline.

a) $s(x)=\begin{cases}0 & , 0\le x\le 1\\ x^3-3x^2+3x-1 & , 1\le x\le 2\end{cases}$ b) $s(x)=\begin{cases}-\frac{1}{2}x^3-\frac{3}{2}x^2+1 & , -1\le x\le 0\\ \frac{1}{2}x^3-\frac{3}{2}x^2+1 & , 0\le x\le 1\end{cases}$