1. Find the eigenvalues and eigenvectors of the following matrices. Do this by hand, but you may check your answers using Matlab.

   a) \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \)  
   b) \( \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \)  
   c) \( \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \)  
   d) \( \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \)  
   e) \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)

section 3.7, special matrices

2. Show that the following matrices are positive definite.

   a) \( \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \)  
   b) \( \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \)

section 3.8, iterative methods

3. Consider \( Ax = b \), where \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \), \( b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

   a) In class we showed that the error in the Gauss-Seidel method is given by

   \[ e_k = \left( \frac{1}{4} \right)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \]

   for \( k \geq 1 \), using the eigenvalues and eigenvectors of the iteration matrix \( B_{GS} \). Following the same procedure, derive an analogous expression for the error in Jacobi’s method. Assume that the starting guess is the zero vector.

   b) Use Matlab to plot \( \rho(B_\omega) \), the spectral radius of the SOR iteration matrix, for \( 0 \leq \omega \leq 2 \). Use the commands \texttt{eig, abs, max} to produce the plot. Make sure to use a fine enough mesh in the variable \( \omega \) to resolve the details of the function (1000 points on the interval \( 0 \leq \omega \leq 2 \) is sufficient.) This plot is typical for the type of matrices appearing in Young’s theorem. Suppose we don’t know the exact value of the optimal SOR parameter - in using an approximate value for the iterative method, is it better to overestimate or underestimate the value of \( \omega^* \)? Explain the reason behind your answer.

4. Consider the iteration \( x_{k+1} = Bx_k + c \) and assume that \( ||B|| = \alpha < 1 \). We know that the error satisfies \( ||x - x_{k+1}|| \leq \alpha ||x - x_k|| \), which is an important theoretical error bound, but the right side cannot be computed in practice because although we know \( x_k \), we don’t know \( x \). Here we derive an alternative error bound that can be computed in practice.

   a) Show that \( I - B \) is invertible and that \( ||(I - B)^{-1}|| \leq \frac{1}{1-\alpha} \).

   b) Show that \( ||x - x_{k+1}|| \leq \frac{\alpha}{1-\alpha} ||x_{k+1} - x_k|| \).

Note: this bound can be computed because we know \( x_{k+1} \) and \( x_k \).