functions : \[ f(x) = \frac{1}{x+1} - \frac{1}{x-1} , \quad x \to \infty \]

integrals : \[ \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx} \, dx \quad , \quad k \to \infty \]

ODE : \[ \frac{dy}{dt} = f(y) \quad , \quad t \to \infty \]

PDE : \[ u_t = f(u) + \epsilon u_{xx} \quad , \quad \epsilon \to 0 \]

fluid dynamics : \[ u_t + u \cdot \nabla u = -\nabla p + \frac{1}{Re} \Delta u \quad , \quad Re \to 0 \quad , \quad \infty \]

1.1, 1.2 asymptotic expansions

def
1. \( f(z) = O(g(z)) \) as \( z \to z_0 \) in \( D \) \iff \[ \left| \frac{f(z)}{g(z)} \right| \text{ is bounded as } z \to z_0 \]

2. \( f(z) = o(g(z)) \) as \( z \to z_0 \) in \( D \) \iff \[ \left| \frac{f(z)}{g(z)} \right| \to 0 \text{ as } z \to z_0 \]

ex
sin \( z = O(1) \) as \( z \to 0 \)

sin \( z = o(1) \)

sin \( z = O(z) \)

sin \( z \neq o(z) \)

sin \( z = z + O(z^3) \)

note : We can also consider \( z \to \infty \).

ex

\( e^{-z} \) , \( D = \{ z : |z| > 0 , \ | \arg z | < \frac{\pi}{4} \} \)

Then \( e^{-z} = o(z^{-n}) \) as \( z \to \infty \) in \( D \) for any \( n \geq 0 \).

pf

\( z = x + iy \) , \( z \in D \implies x > 0 \)

\[ \left| \frac{e^{-z}}{z^{-n}} \right| = e^x (x^2 + y^2)^{n/2} \leq e^{-x} (2x^2)^{n/2} = 2^{n/2}e^{-x}x^n \to 0 \text{ as } z \to \infty \text{ in } D \quad \text{ ok} \]
def
\( f(z) \) is asymptotic to \( g(z) \) as \( z \to z_0 \) in \( D \) \iff \( \lim_{z \to z_0} \frac{f(z)}{g(z)} = 1 \)

In this case we write \( f(z) \sim g(z) \) as \( z \to z_0 \).

ex
\( \sin z \sim z \) as \( z \to 0 \)
\( \sin z \sim z - \frac{1}{3!} z^3 \) as \( z \to 0 \)
\( \sinh z \sim \frac{1}{2} e^z \) as \( z \to \infty \) , \( |\arg z| < \frac{\pi}{4} \)

def
\( f(z) \) is analytic at \( z = z_0 \) \iff \( f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) for \( |z - z_0| < R \)
\( \iff f(z) = \lim_{N \to \infty} s_N(z) \) for \( |z - z_0| < R \), where \( s_N(z) = \sum_{n=0}^{N} a_n(z - z_0)^n \)
\( \iff f(z) = \text{convergent power series} \)

note : We can also consider \( z_0 = \infty \).

ex
\( \frac{1}{z^2 + 1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - \cdots \) in \( |z| < 1 \) , \( z_0 = 0 \)
\( = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \cdots \) in \( |z| > 1 \) , \( z_0 = \infty \)
\( \frac{1}{z^2 + 1} \sim 1 \) as \( z \to 0 \)
\( \frac{1}{z^2 + 1} \sim 1 - z^2 \) as \( z \to 0 \)
\( \frac{1}{z^2 + 1} \sim \frac{1}{z^2} \) as \( z \to \infty \)
\( \frac{1}{z^2 + 1} \sim \frac{1}{z^2} - \frac{1}{z^4} \) as \( z \to \infty \)

note
Convergent power series are not the only way to obtain asymptotic relations.
\[ \text{Ei}(z) = \int_z^\infty e^{-t}t^{-1}dt : \text{not analytic at } z_0 = \infty \]

\[ \text{Ei}(x) = \int_x^\infty e^{-t}t^{-1}dt \to 0 \text{ as } x \to \infty , \text{ but precisely how fast?} \]

\[ \text{Ei}(x) \leq \frac{e^{-x}}{x} , \text{ pf } \ldots \]

\[ \text{Ei}(x) = \int_x^\infty e^{-t}t^{-1}dt = -e^{-t}t^{-1}\bigg|_x^\infty - \int_x^\infty e^{-t}t^{-2}dt = \frac{e^{-x}}{x} - \int_x^\infty e^{-t}t^{-2}dt \]

\[ = \frac{e^{-x}}{x} - \left( -e^{-t}t^{-2}\bigg|_x^\infty - 2\int_x^\infty e^{-t}t^{-3}dt \right) = \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\int_x^\infty e^{-t}t^{-3}dt \ldots \]

\[ \text{Ei}(x) = s_n(x) + r_n(x) \]

\[ s_n(x) = e^{-x}\left( \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^{n+1}(n - 1)!}{x^n} \right) \]

\[ r_n(x) = (-1)^n n! \int_x^\infty e^{-t}t^{-(n+1)}dt \]

\[ \text{note} \]

1. The series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n - 1)!}{x^n} \) diverges for all \( x \neq 0 \). \( \text{pf } \ldots \)

2. Fix \( n \geq 0 \). Then \( |r_n(x)| \leq n! \frac{e^{-x}}{x^{n+1}} \). It follows that \( \frac{\text{Ei}(x)}{s_n(x)} = 1 + \frac{r_n(x)}{s_n(x)} \to 1 \) as \( x \to \infty \), so \( \text{Ei}(x) \sim s_n(x) \) as \( x \to \infty \).

\[ \text{note : Suppose we want to approximate } \text{Ei}(x) \text{ by } s_n(x) \text{ for a given value of } x. \]

The best choice of \( n \) is the one that minimizes \( r_n(x) \).

\[ \text{hw : } |r_n(x)| \sim n! \frac{e^{-x}}{x^{n+1}} \text{ as } x \to \infty \]

\[ \Rightarrow \text{the error in } s_n(x) \sim \text{the 1st neglected term in the series as } x \to \infty \]

\[ \Rightarrow \left| \frac{r_n(x)}{r_{n-1}(x)} \right| \sim \frac{n! \frac{e^{-x}}{x^{n+1}}}{(n - 1)! \frac{e^{-x}}{x^n}} = \frac{n}{x} \leq 1 \iff n \leq x \]

\[ \Rightarrow \text{the best choice is } n = [x] : \text{the largest integer } \leq x \]
\textbf{ex} : \( \text{Ei}(5) = 0.001148 \)

\begin{center}
\begin{tabular}{|c|c|}
\hline
\( n \) & \( s_n(5) \) \\
\hline
1 & 0.001348 \\
2 & 0.001078 \\
3 & 0.001186 \\
4 & 0.001121 \\
5 & 0.001173 \leftarrow \\
6 & 0.001121 \\
7 & 0.001183 \\
14 & -0.003846 \\
19 & 1.775902 \\
\hline
\end{tabular}
\end{center}

\textbf{note} : Given \( x \), the error cannot be made arbitrarily small. As \( x \) increases,
1. the optimal \( n \) increases and the minimum error decreases,
2. the error curve becomes flatter near the minimum, so a small error is obtained even for \( n \ll [x] \), e.g. even \( n = 1 \) may be adequate in some applications.

\textbf{def} : \( \{ \phi_n(z) : n = 0, 1, 2, \ldots \} \) is an \textbf{asymptotic sequence} as \( z \to z_0 \)
\( \Leftrightarrow \phi_{n+1}(z) = o(\phi_n(z)) \) as \( z \to z_0 \)

\textbf{ex}
\( \phi_n(z) = (z - z_0)^n \) as \( z \to z_0 \)
\( \phi_n(z) = e^{-nz} \) as \( z \to \infty \)
\( \phi_n(z) = 1, z \log z, z, z^2 \log z, z^2, z^3 \log z, z^3, \ldots \) as \( z \to 0 \)

\textbf{def} : \( f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \) as \( z \to z_0 \) : \textbf{asymptotic expansion}
\( \Leftrightarrow f(z) = \sum_{n=0}^{N} a_n \phi_n(z) + o(\phi_N(z)) \) as \( z \to z_0 \) for all \( N \geq 0 \)

\textbf{ex 1} : If \( f(z) \) is analytic at \( z = z_0 \), then \( f(z) \sim \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n \) as \( z \to z_0 \).

Hence a convergent power series is an example of an asymptotic expansion.

\textbf{ex 2} : \( \text{Ei}(x) = \int_{x}^{\infty} e^{-t}t^{-1}dt \sim e^{-x} \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{(n-1)!}{x^n} \) as \( x \to \infty \)

This is a \textbf{divergent} series which nonetheless is an asymptotic expansion.

\textbf{pf} : \( \phi_n(x) = e^{-x}x^{-n} \), \ldots