functions : \( f(x) = \frac{1}{x+1} - \frac{1}{x-1} \), \( x \to \infty \)

integrals : \( \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx} dx \), \( k \to \infty \)

ODE : \( \frac{dy}{dt} = f(y) \), \( t \to \infty \)

PDE : \( u_t = f(u) + \epsilon u_{xx} \), \( \epsilon \to 0 \)

fluid dynamics : \( u_t + u \cdot \nabla u = -\nabla p + \frac{1}{Re} \Delta u \), \( \nabla \cdot u = 0 \), \( Re \to 0, \infty \)

1.1, 1.2 asymptotic expansions

\textbf{def}

1. \( f(z) = O(g(z)) \) as \( z \to z_0 \) in \( D \) \( \iff \) \( \left| \frac{f(z)}{g(z)} \right| \) is bounded as \( z \to z_0 \)

2. \( f(z) = o(g(z)) \) as \( z \to z_0 \) in \( D \) \( \iff \) \( \left| \frac{f(z)}{g(z)} \right| \to 0 \) as \( z \to z_0 \)

\textbf{ex}

\( \sin z = O(1) \) as \( z \to 0 \)

\( \sin z = o(1) \)

\( \sin z = O(z) \)

\( \sin z \neq o(z) \)

\( \sin z = z + O(z^3) \)

\textbf{note} : We can also consider \( z \to \infty \).

\textbf{ex}

\( e^{-z} \), \( D = \{ z : |z| > 0, |\arg z| < \frac{\pi}{4} \} \)

Then \( e^{-z} = o(z^{-n}) \) as \( z \to \infty \) in \( D \) for any \( n \geq 0 \).

\textbf{pf}

\( z = x + iy \), \( z \in D \) \( \Rightarrow \) \( x > 0 \)

\[ \left| \frac{e^{-z}}{z^{-n}} \right| = e^{-x}(x^2 + y^2)^{n/2} \leq e^{-x}(2x^2)^{n/2} = 2^{n/2}e^{-x}x^n \to 0 \) as \( z \to \infty \) in \( D \) \( \quad \text{ok} \)
def

\( f(z) \) is asymptotic to \( g(z) \) as \( z \to z_0 \) in \( D \iff \lim_{z \to z_0} \frac{f(z)}{g(z)} = 1 \)

In this case we write \( f(z) \sim g(z) \) as \( z \to z_0 \).

ex

\( \sin z \sim z \) as \( z \to 0 \)

\( \sin z - z \sim -\frac{1}{3!}z^3 \) as \( z \to 0 \)

\( \sinh z \sim \frac{1}{2}e^z \) as \( z \to \infty \), \( |\arg z| < \frac{\pi}{4} \)

\( \sinh z \sim \frac{1}{2}e^{-z} \) as \( z \to \infty \), \( |\arg z - \pi| < \frac{\pi}{4} \)

def

\( f(z) \) is analytic at \( z = z_0 \iff f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) for \( |z - z_0| < R \)

\( \iff f(z) = \lim_{N \to \infty} s_N(z) \) for \( |z - z_0| < R \), where \( s_N(z) = \sum_{n=0}^{N} a_n(z - z_0)^n \)

\( \iff f(z) = \text{convergent power series} \)

note: We can also consider \( z_0 = \infty \).

ex

\( \frac{1}{z^2 + 1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - \cdots \) in \( |z| < 1 \), \( z_0 = 0 \)

\( = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \cdots \) in \( |z| > 1 \), \( z_0 = \infty \)

\( \frac{1}{z^2 + 1} \sim 1 \) as \( z \to 0 \)

\( \frac{1}{z^2 + 1} - 1 \sim -z^2 \) as \( z \to 0 \)

\( \frac{1}{z^2 + 1} \sim \frac{1}{z^2} \) as \( z \to \infty \)

\( \frac{1}{z^2 + 1} - \frac{1}{z^2} \sim -\frac{1}{z^4} \) as \( z \to \infty \)

1. Here there is no restriction on \( z \to \infty \).

2. Convergent power series are not the only examples of asymptotic relations.
\[
\text{Ei}(x) = \int_x^\infty e^{-t} t^{-1} dt = \int_1^\infty e^{-xt} t^{-1} dt : \text{improper integral, converges for } x > 0
\]

pf

\[
\int_x^\infty e^{-t} t^{-1} dt \leq \int_x^\infty e^{-t} x^{-1} dt = -e^{-t} x^{-1}\bigg|_x^\infty = \frac{e^{-x}}{x} < \infty \quad \text{ok}
\]

This shows that \( \lim_{x \to \infty} \text{Ei}(x) = 0 \), but precisely how fast does \( \text{Ei}(x) \to 0 \) as \( x \to \infty \)?

\[
\text{Ei}(x) = \int_x^\infty e^{-t} t^{-1} dt = -e^{-t} t^{-1}\bigg|_x^\infty - \int_x^\infty e^{-t} t^{-2} dt
\]

\[
= \frac{e^{-x}}{x} - \int_x^\infty e^{-t} t^{-2} dt
\]

\[
= \frac{e^{-x}}{x} - \left(-e^{-t} t^{-2}\bigg|_x^\infty - 2\int_x^\infty e^{-t} t^{-3} dt\right)
\]

\[
= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\int_x^\infty e^{-t} t^{-3} dt \quad \ldots
\]

\[
\text{Ei}(x) = s_n(x) + r_n(x)
\]

\[
s_n(x) = e^{-x} \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^{n+1}(n-1)!}{x^n}\right)
\]

\[
r_n(x) = (-1)^n n! \int_x^\infty e^{-t} t^{-(n+1)} dt
\]

note

1. The series \( \sum_{n=1}^\infty \frac{(-1)^{n+1}(n-1)!}{x^n} \) diverges for all \( x \neq 0 \). \quad pf \ldots

2. Fix \( n \geq 1 \).

Then \( |r_n(x)| \leq n! \frac{e^{-x}}{x^{n+1}} \to 0 \) as \( x \to \infty \), so \( s_n(x) \) is an approximation to \( \text{Ei}(x) \) for large \( x \); in fact, it can be shown that \( \frac{\text{Ei}(x)}{s_n(x)} = 1 + \frac{r_n(x)}{s_n(x)} \to 1 \) as \( x \to \infty \), so we have \( \text{Ei}(x) \sim s_n(x) \) as \( x \to \infty \) for all \( n \geq 1 \).

3. In other words, even though the series diverges for all \( x \neq 0 \), the partial sums \( s_n(x) \) approximate \( \text{Ei}(x) \) better and better as \( x \to \infty \).
question
Given $x$, we want to approximate $\text{Ei}(x)$ by $s_n(x)$; what is the best choice of $n$?

answer
The best choice of $n$ is the one that minimizes $|r_n(x)|$.

note: we know that $|r_n(x)| \leq n! \frac{e^{-x}}{x^{n+1}}$, but in fact $|r_n(x)| \sim n! \frac{e^{-x}}{x^{n+1}}$ as $x \to \infty$

pf: hw

$\Rightarrow$ the error in $s_n(x) \sim$ the 1st neglected term in the series as $x \to \infty$

$\Rightarrow \frac{r_n(x)}{r_{n-1}(x)} \sim \frac{n! \frac{e^{-x}}{x^{n+1}}}{(n-1)! \frac{e^{-x}}{x^n}} = \frac{n}{x} \leq 1 \iff n \leq x$

$\Rightarrow$ the best choice is $n = n(x) = \lfloor x \rfloor$ : largest integer $\leq x$

ex: $\text{Ei}(5) = 0.001148$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$s_n(5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.001348</td>
</tr>
<tr>
<td>2</td>
<td>0.001078</td>
</tr>
<tr>
<td>3</td>
<td>0.001186</td>
</tr>
<tr>
<td>4</td>
<td>0.001121</td>
</tr>
</tbody>
</table>
| 5   | 0.001173 | ←
| 6   | 0.001121 |
| 7   | 0.001183 |
| 14  | -0.003846 |
| 19  | 1.775902 |

note
Given $x$, the error cannot be made arbitrarily small by retaining more terms in the series; however as $x$ increases,

1. the minimum error decreases,

2. the optimal $n$ increases,

3. the error curve becomes flatter near the minimum, so a small error is obtained even for $n << \lfloor x \rfloor$, e.g. even $n = 1$ may be adequate in some applications.
exponential integral and some asymptotic approximations

error and asymptotic approximation of error

\( E_i(x), s_n(x), n=1:4 \)

\( r_n(x) \)

\( n!e^{-x}x^{-(n+1)} \)

\( x=2 \)

\( x=5 \)

\( x=10 \)
\textbf{def:} \{\phi_n(z) : n = 1, 2, \ldots \} is an asymptotic sequence as \( z \to z_0 \)
\[ \iff \phi_{n+1}(z) = o(\phi_n(z)) \text{ as } z \to z_0 \]

\textbf{ex:} \( \phi_n(z) = (z - z_0)^n \) as \( z \to z_0 \)
\[ \phi_n(z) = e^{-nz} \text{ as } z \to \infty \]
\[ \phi_n(z) = z^n \log z \text{ as } z \to 0 \]

\textbf{def:} \( f(z) \sim \sum_{n=1}^{\infty} a_n \phi_n(z) \) as \( z \to z_0 \) : asymptotic expansion \( \text{wrt} \ \{\phi_n(z)\} \)
\[ \iff f(z) = \sum_{n=1}^{N} a_n \phi_n(z) + o(\phi_N(z)) \text{ as } z \to z_0 \text{ for all } N \geq 1 \]
\[ \iff f(z) = \sum_{n=1}^{N} a_n \phi_n(z) + O(\phi_{N+1}(z)) \text{ as } z \to z_0 \text{ for all } N \geq 1 \]

In this case, the error has the same order of magnitude as the first term omitted.

\textbf{ex 1:} If \( f(z) \) is analytic at \( z = z_0 \), then \( f(z) \sim \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n \) as \( z \to z_0 \).

A convergent power series is also an asymptotic expansion.

\textbf{ex 2:} \( \text{Ei}(x) = \int_{x}^{\infty} e^{-t}t^{-1}dt \sim e^{-x} \sum_{n=1}^{\infty} (-1)^{(n+1)}(n-1)! \frac{n!}{x^n} \) as \( x \to \infty \)

\textbf{pf:} \( \phi_n(x) = e^{-x}x^{-n} \) : asymptotic sequence as \( x \to \infty \)
\[ \left| \frac{\text{Ei}(x) - s_n(x)}{\phi_n(x)} \right| = \left| \frac{r_n(x)}{\phi_n(x)} \right| \leq \frac{n!e^{-x}x^{-(n+1)}}{e^{-x}x^{-n}} = \frac{n!}{x} \to 0 \text{ as } x \to \infty \quad \text{ok} \]

Hence even if a series diverges, it may still be an asymptotic expansion.

\textbf{note:} If \( f(z) \sim \sum_{n=1}^{\infty} a_n \phi_n(z) \) as \( z \to z_0 \), then the \( a_n \) are unique.

\textbf{pf}
\[ f(z) = a_1 \phi_1(z) + o(\phi_1(z)) \Rightarrow \lim_{z \to z_0} \frac{f(z)}{\phi_1(z)} = a_1 \]
\[ f(z) = a_1 \phi_1(z) + a_2 \phi_2(x) + o(\phi_2(z)) \Rightarrow \lim_{z \to z_0} \frac{f(z) - a_1 \phi_1(z)}{\phi_2(z)} = a_2 \quad \ldots \quad \text{ok} \]

However if \( f_1(z) \) and \( f_2(z) \) have the same asymptotic expansion, it doesn’t follow that \( f_1(z) = f_2(z) \).

\textbf{ex:} \( f_1(z) = 0, f_2(z) = e^{-z} \sim \sum_{n=1}^{\infty} 0 \cdot z^{-n} \) as \( z \to \infty \), but \( f_1(z) \neq f_2(z) \)