4.1 method of stationary phase

$$f(\lambda) = \int_a^b g(t)e^{i\lambda h(t)} dt \ , \ \lambda \to \infty \ , \ \text{assume} \ t \ , \ g(t) \ , \ h(t) \ \text{are real}$$

$$e^{i\lambda h(t)} = \cos \lambda h(t) + i \sin \lambda h(t) \ : \ \text{oscillates} \ , \ \lambda h(t) : \ \text{phase}$$

**case 1 :** \( h'(t) \neq 0 \) for all \( t \in [a, b] \)

\( h(t) = h(t_0) + h'(t_0)(t - t_0) + \cdots \)

phase varies rapidly near \( t = t_0 \) as \( \lambda \to \infty \) , \( \pm \) contributions to integral cancel

$$f(\lambda) = \int_a^b \frac{g(t)}{i\lambda h'(t)} i\lambda h'(t)e^{i\lambda h(t)} dt = \frac{g(t)}{i\lambda h'(t)} e^{i\lambda h(t)} \bigg|_a^b - \int_a^b \left( \frac{g(t)}{i\lambda h'(t)} \right)' e^{i\lambda h(t)} dt$$

$$f(\lambda) \sim \frac{g(b)}{i\lambda h'(b)} e^{i\lambda h(b)} - \frac{g(a)}{i\lambda h'(a)} e^{i\lambda h(a)} = O(\lambda^{-1}) \ \text{as} \ \lambda \to \infty$$

alternative derivation

$$h(t) = h(a) + h'(a)(t - a) + \cdots$$

$$\int_a^{a+\epsilon} g(a)e^{i\lambda h(t)} dt \sim \int_a^{a+\epsilon} g(a)e^{i\lambda(h(a) + h'(a)(t-a))} dt = g(a)e^{i\lambda h(a)} \int_0^\epsilon e^{i\lambda h'(a)t} dt$$

$$= g(a)e^{i\lambda h(a)} \left. \frac{e^{i\lambda h'(a)t}}{i\lambda h'(a)} \right|_0^\epsilon = \frac{g(a)}{i\lambda h'(a)} e^{i\lambda h(a)} e^{i\lambda h'(a)\epsilon} - \frac{g(a)}{i\lambda h'(a)} e^{i\lambda h(a)} \ \text{ok}$$

**case 2 :** \( h'(t_0) \) for some \( t_0 \in (a, b) : \ \text{point of stationary phase} \)

\( h(t) = h(t_0) + h'(t_0)(t - t_0) + \frac{1}{2} h''(t_0)(t - t_0)^2 + \cdots = h(t_0) \pm s^2 \)

phase varies more slowly wrt \( t \) near \( t = 0 \) than near \( t \neq t_0 \) , weaker cancellation

\( \pm \) depends on \( \text{sign}(h''(t_0)) \) ,

$$t - t_0 = \left( \frac{2}{|h''(t_0)|} \right)^{1/2} s + O(s^2)$$

$$f(\lambda) \sim \int_{-\infty}^{\infty} g(t_0)e^{i\lambda(h(t_0) \pm s^2)} \left( \frac{2}{|h''(t_0)|} \right)^{1/2} ds = g(t_0)e^{i\lambda h(t_0)} \left( \frac{2}{|h''(t_0)|} \right)^{1/2} \int_{-\infty}^{\infty} e^{\pm i\lambda s^2} ds$$

$$= g(t_0) \left( \frac{2\pi}{\lambda|h''(t_0)|} \right)^{1/2} e^{i(\lambda h(t_0) \pm \pi/4)} = O(\lambda^{-1/2}) \ \text{as} \ \lambda \to \infty$$

1. These results can also be derived using the method of steepest descent; a point of stationary phase corresponds to a saddle point when \( t \) is complex.

2. In summary, the leading order terms in the asymptotic expansion of \( f(\lambda) \) as \( \lambda \to \infty \) come from points of stationary phase and the endpoints of the interval.
These are plots of $\cos \lambda h(t)$ for $\lambda = 50$ and various $h(t)$.

$h(t) = t$: no point of stationary phase

$h(t) = t^2$, $t_0 = 0$

$h(t) = t + t^2$, $t_0 = -\frac{1}{2}$

$h(t) = t - \frac{1}{3}t^3$, $t_0 = \pm 1$: two points of stationary phase

$h(t) = t^3$, $t_0 = 0$: a higher order point of stationary phase
\textbf{claim} : \( \int_{-\infty}^{\infty} e^{\pm i \lambda s^2} ds = \left( \frac{\pi}{\lambda} \right)^{1/2} e^{\pm i \pi/4} \) for \( \lambda > 0 \)

\textbf{pf} : we will show that \( \int_{0}^{\infty} e^{is^2} ds = \frac{\sqrt{\pi}}{2} e^{\pi i/4} \), claim follows, other case on hw4

\[
(\int_{0}^{R} + \int_{CR} + \int_{C}) e^{iz^2} dz = 0
\]

1. \( z \in C \Rightarrow z = re^{\pi i/4} \Rightarrow iz^2 = -r^2 \)

\[
\lim_{R \to \infty} \int_{C} e^{iz^2} dz = -\int_{0}^{\infty} e^{-r^2} e^{\pi i/4} dr = -\frac{\sqrt{\pi}}{2} e^{\pi i/4}
\]

2. \( z \in CR \Rightarrow z = Re^{i\theta} \Rightarrow iz^2 = iR^2(\cos 2\theta + i \sin 2\theta) \)

\[
\left|\int_{CR} e^{iz^2} dz\right| \leq \int_{0}^{\pi/4} e^{-R^2 \sin 2\theta} R d\theta \leq R \int_{0}^{\pi/4} e^{-R^2 \cdot 4\theta/\pi} d\theta
\]

\[
\text{note : } 0 \leq \theta \leq \pi/4 \Rightarrow \sin 2\theta \geq 4\theta/\pi
\]

\[
= Re^{-R^2 \cdot 4\theta/\pi} \cdot \frac{\pi}{-4R^2} \bigg|_{0}^{\pi/4} = \frac{\pi}{4R} (1 - e^{-R^2}) \to 0 \text{ as } R \to \infty \quad \text{ok}
\]

\textbf{ex} : \( J_n(\lambda) = \frac{1}{\pi} \int_{0}^{\pi} \cos(n t - \lambda \sin t) dt \) : Bessel function

\[
= \frac{1}{\pi} \text{Re} \int_{0}^{\pi} e^{i(nt - \lambda \sin t)} dt = \frac{1}{\pi} \text{Re} \int_{0}^{\pi} e^{-i\lambda \sin t} e^{int} dt
\]

\( h(t) = -\sin t \) , \( h(t_0) = -1 \)

\( h'(t) = -\cos t = 0 \Rightarrow t_0 = \frac{\pi}{2} \) : point of stationary phase

\( h''(t) = \sin t \) , \( h''(t_0) = 1 \)

\( h(t) = h(t_0) + h'(t_0)(t - t_0) + \frac{1}{2} h''(t_0)(t - t_0)^2 + \cdots = -1 + \frac{1}{2}(t - \frac{\pi}{2})^2 + \cdots \)

\( J_n(\lambda) \sim \frac{1}{\pi} \text{Re} \int_{-\infty}^{\infty} e^{i\lambda(-1 + \frac{1}{2}t^2)} e^{in\pi/2} dt = \frac{1}{\pi} \text{Re} \left( e^{-i(\lambda - n\pi/2)} \int_{-\infty}^{\infty} e^{\frac{1}{2}i\lambda t^2} dt \right) \)

\[
= \frac{1}{\pi} \text{Re} \left( e^{-i(\lambda - n\pi/2)} \left( \frac{2\pi}{\lambda} \right)^{1/2} e^{\pi i/4} \right) = \left( \frac{2}{\pi \lambda} \right)^{1/2} \cos(\lambda - n\pi/2 - \pi/4) \text{ as } \lambda \to \infty
\]
derivation by method of steepest descent
assume \( h'(t_0) = 0, h''(t_0) > 0, \ a < t_0 < b \)

\[ h(t) = h(t_0) + z^2 \Rightarrow z = \pm (h(t) - h(t_0))^{1/2}, \ \text{choose } z = \left( \frac{1}{2} h''(t_0) \right)^{1/2} (t - t_0) + \cdots \]

\[ z(t_0) = 0, \ z(a) = \tilde{a} < 0, \ z(b) = \tilde{b} > 0 \]

\[ f(\lambda) = \int_{a}^{b} g(t) e^{i\lambda h(t)} dt = \int_{a}^{b} g(t(z)) e^{i\lambda h(t(z))} t'(z) dz = e^{i\lambda h(t_0)} \int_{a}^{b} g(t(z)) e^{i\lambda z^2 t'(z)} dz \]

apply method of steepest descent, saddle point: \( z_0 = 0 \)

\[ iz^2 = i(x^2 - y^2 + 2ixy) = -2xy + i(x^2 - y^2) \]
\[ \psi(x, y) = x^2 - y^2 = \text{cnst: hyperbolas, asymptotes: } y = \pm x \]

\( z_0 = 0 \Rightarrow \text{Im}(iz^2) = 0 \Rightarrow \psi(x, y) = x^2 - y^2 = 0 \Rightarrow y = \pm x : \text{sa/sd} \)
on \( y = x, \phi(x, y) = -2xy = -2x^2: \text{sd} \), similarly on \( y = -x, \cdots: \text{sa} \)

\( C_{\tilde{a}}, C_{\tilde{b}}: \text{sd paths through } \tilde{a}, \tilde{b} \Rightarrow \int_{a}^{b} = \int_{\text{sd}} + \int_{C_{\tilde{a}}} + \int_{C_{\tilde{b}}} \)
on \( \text{sd: } z = (1 + i)s, \ iz^2 = i(1 + i)^2 s^2 = -2s^2 \)

\[ \int_{\text{sd}} \sim \int_{-\infty}^{\infty} g(t_0) e^{-2\lambda s^2} t'(z_0)(1 + i) ds = g(t_0) \left( \frac{2}{h''(t_0)} \right)^{1/2} \sqrt{2} e^{\pi i/4} \left( \frac{\pi}{2\lambda} \right)^{1/2} \text{as } \lambda \rightarrow \infty \]
on \( C_{\tilde{a}}: iz^2 = i\tilde{a}^2 - s, \ 0 < s < \infty \)

\[ h(t) = h'(a)(t - a) + O((t - a)^2) = h(t_0) + z^2 = h(t_0) + \tilde{a}^2 + is \]

\[ t = a + \frac{is}{h'(a)} + O(s^2) \]

\[ \int_{C_{\tilde{a}}} \sim \int_{0}^{\infty} g(a) e^{i\tilde{a}^2 - \lambda s} \frac{ids}{h'(a)} = g(a) e^{i\tilde{a}^2} \frac{i}{h'(a)} \lambda \text{ as } \lambda \rightarrow 0, \ \text{similarly on } C_{\tilde{b}} \ldots \]

\[ f(\lambda) \sim g(t_0) \left( \frac{2\pi}{\lambda h''(t_0)} \right)^{1/2} e^{i(\lambda h(t_0) + \pi/4)} + g(t) \left. e^{i\lambda h(t)} \right|_{a}^{b} \text{ as } \lambda \rightarrow \infty \ \text{ok} \]
4.2 linear dispersive waves

**ex 1 :** $\phi_t + c\phi_x = 0$ : 1st order wave equation

look for $\phi(x, t) = a \cos(kx - \omega t)$ : elementary wave

$a$ : amplitude

$k$ : wavenumber , $k = 2\pi/L$ , $L$ : wavelength

$\omega$ : frequency , $\omega = 2\pi/T$ , $T$ : period

$kx - \omega t$ : phase

If $x$ and $t$ vary so that $kx - \omega t = \text{cnst}$, this defines a line in $xt$-space on which $\phi(x, t) = \text{cnst}$; in this case the wave travels without change of shape at the phase speed given by $dx/dt = \omega/k$, which is found by substituting $\phi(x, t)$ into the PDE.

$\Rightarrow \omega - ck = 0 \Rightarrow \omega = ck \Rightarrow \omega/k = c \Rightarrow \phi(x, t) = a \cos k(x - ct)$

For general initial data $\phi(x, 0) = f(x)$, the solution is $\phi(x, t) = f(x - ct)$, which travels at the phase speed $c$ without change of shape. The solution can also be expressed as a superposition of elementary waves,

$$\phi(x, t) = \int_0^\infty a(k) \cos(kx - \omega t)dk = \int_0^\infty a(k) \cos k(x - ct)dk,$$

where the amplitude $a(k)$ is determined by the initial condition,

$$f(x) = \int_0^\infty a(k) \cos(kx)dk.$$

**ex 2 :** $\phi_t + c\phi_x = \phi_{xxx}$ : linearized KdV equation

look for $\phi(x, t) = a \cos(kx - \omega t) \Rightarrow \omega = ck + k^3$

In contrast with example 1, now the phase speed $\omega/k = c + k^2$ depends on the wavenumber; hence waves with different wavenumbers travel at different speeds; this is called dispersion and $\omega = \omega(k)$ is the dispersion relation.

$k \to 0 \Rightarrow \omega \sim ck \Rightarrow$ long waves travel at speed $c$

$k \to \infty \Rightarrow \omega \sim k^3 \Rightarrow$ short waves travel arbitrarily fast

The solution can still be expressed as a superposition of elementary waves,

$$\phi(x, t) = \int_0^\infty a(k) \cos(kx - \omega(k)t)dk.$$
Consider a superposition of two elementary waves with wavenumbers \( k_1, k_2 \) and frequencies \( \omega_1 = \omega(k_1), \omega_2 = \omega(k_2) \).

\[
\cos a + \cos b = \cos\left[\frac{1}{2}(a - b) + \frac{1}{2}(a + b)\right] + \cos\left[\frac{1}{2}(a - b) - \frac{1}{2}(a + b)\right] = 2 \cos\left[\frac{1}{2}(a - b)\right] \cos\left[\frac{1}{2}(a + b)\right]
\]

\[
\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) = 2 \cos\left[\frac{1}{2}((k_1 - k_2)x - (\omega_1 - \omega_2)t\right] \cos\left[\frac{1}{2}((k_1 + k_2)x - (\omega_1 + \omega_2)t\right]
\]

The 1st term on the right is a slowly varying amplitude for the rapidly varying 2nd term; the product can be interpreted as a series of wave packets traveling with speed \( \frac{\omega_2 - \omega_1}{k_2 - k_1} \approx \omega'(k_{12}) \), where \( k_{12} = \frac{1}{2}(k_1 + k_2) \).

The speed \( \omega'(k) \) is called the **group velocity**: for non-dispersive equations (ex 1), the phase speed and group velocity are the same, but for dispersive equations (ex 2), they are different.

Return to the general solution for a dispersive equation; since fast waves overtake slow waves, wave packets can appear and disappear; “an arbitrary initial disturbance disperses into a slowly varying wave train”; can we analyze that?

\[
\phi(x, t) = \int_0^\infty a(k) \cos(kx - \omega(k)t) dk \quad \cos(kx - \omega(k)t) = \frac{1}{2}(e^{ith(k)} + e^{-ith(k)})
\]

\[
h(k) = k \frac{x}{t} - \omega(k)
\]

\[
h'(k) = \frac{x}{t} - \omega'(k) = 0 \Rightarrow \omega'(k_0) = \frac{x}{t} : \text{point of stationary phase}
\]

Set \( x = vt \) for \( v \in \text{range } \omega'(k) \); this defines a line in \( xt \)-space; consider \( \phi(x, t) \) on such lines as \( t \to \infty \).

\[
\phi(x, t) \sim a(k_0) \left(\frac{2\pi}{|\omega''(k_0)|}ight)^{1/2} \cos(k_0 x - \omega(k_0) t + \pi/4) \text{ as } t \to \infty , \text{ where } \omega'(k_0) = \frac{x}{t}
\]

1. \( \phi(x, t) = O(t^{-1/2}) : \text{dispersive decay} , \text{ unlike ex 1} \)

2. The asymptotic result resembles an elementary wave, but it is not uniform; the coefficients depend on \( x/t \).

\[
\omega'(k_0(x, t)) = \frac{x}{t} \Rightarrow \frac{dw'}{dk_0} \cdot \frac{\partial k_0}{\partial x} = 1 \Rightarrow \frac{\partial k_0}{\partial x} = O(t^{-1}) , \frac{\partial k_0}{\partial t} = O(t^{-1})
\]

An observer traveling at the group velocity \( \omega'(k_0) \) will see waves with wavenumbers near \( k_0 \); an observer standing at a fixed location, will see wave packets with different wavenumbers appear and disappear.
$y_1 = \cos(k_1 x), \ k_1 = 4.75$

$y_2 = \cos(k_2 x), \ k_2 = 5.25$

$y = \cos(k_1 x) + \cos(k_2 x), \ k_1 = 4.75, \ k_2 = 5.25$

$y = \cos(k_1 x) + \cos(k_2 x), \ k_1 = 4.5, \ k_2 = 5.5$

$y = \cos(k_1 x) + \cos(k_2 x), \ k_1 = 4.9, \ k_2 = 5.1$
\textbf{ex}: \( \phi_t + c \phi_x = \phi_{xxx} \)

\[ \phi(x, t) = \int_{0}^{\infty} a(k) \cos(kx - \omega(k)t)dk, \quad \omega(k) = ck + k^3 \]

\( \phi(x, 0) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \)

\[ \hat{\phi}(k, 0) = \int_{-\infty}^{\infty} \phi(x, 0)e^{ikx}dx : \text{Fourier transform} \]

\[ = \int_{-1}^{1} e^{ikx}dx = \frac{e^{ik} - e^{-ik}}{ik} = 2\sin k \]

\( \phi(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(k, 0)e^{-ikx}dk : \text{inverse Fourier transform} \)

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin k}{k}e^{-ikx}dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin k}{k} (\cos kx - i \sin kx)dk = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin k}{k} \cos(kx)dk \]

\[ \phi(x, t) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin k}{k} \cos(kx - (ck + k^3)t)dk \]

\[ h(k) = k\frac{x}{t} - \omega(k) = k\frac{x}{t} - (ck + k^3) \]

\[ h'(k) = 0 \Rightarrow \omega'(k) = c + 3k^2 = \frac{x}{t} \Rightarrow k_0 = \pm \left( \frac{1}{3} \left( \frac{x}{t} - c \right) \right)^{1/2}, \quad \omega''(k_0) = 6k_0 \]

\textbf{case 1}: \( x/t > c \)

\[ \phi(x, t) \sim \frac{\sin k_0}{k_0} \left( \frac{4}{3\pi|k_0|t} \right)^{1/2} \cos(k_0x - (ck_0 + k_0^3)t - \pi/4) \] as \( t \to \infty \)

\textbf{case 2}: \( x/t = c \)

\( k_0 = 0 \), cannot take \( k_0 \to 0 \) in case 1

\( h(k) = -k^3 \Rightarrow h(k_0) = \max h(k) = 0 \), \( h'(k_0) = h''(k_0) = 0 \), \( h'''(k_0) \neq 0 \)

\textbf{hw2 (#5)} \( \Rightarrow \phi(x, t) = O(t^{-1/3}) \) as \( t \to \infty \): decays more slowly than case 1

\textbf{case 3}: \( x/t < c \)

\[ h'(k) = \frac{x}{t} - (c + 3k^2) < 0 \] for all real \( k \); hence there are no points of stationary phase; can integrate by parts and show \( \phi(x, t) = O(t^{-n}) \) as \( t \to \infty \) for all \( n \geq 1 \); however \( h'(k) = 0 \) has imaginary solutions \( k_0 = \pm i \left( \frac{1}{3} \left( c - \frac{x}{t} \right) \right)^{1/2} \), which are saddle points of \( h(k) \) in the complex \( k \)-plane, so the method of steepest descent can be applied to show that \( \phi(x, t) \) is exponentially small as \( t \to \infty \).

\textbf{recall}: 1. \( \text{Ai}(\lambda) = \frac{1}{2\pi i} \oint_{C_1} e^{\lambda z - \frac{1}{2}z^2}dz \): connects case 1 and case 3

2. \( \phi(x, t) = t^{-1/3} \text{Ai}(x(3t)^{-1/3}) \) is a solution of \( \phi_t + \phi_{xxx} = 0 \)
\[
\begin{align*}
\text{ex: } \quad \phi_t + c\phi_x &= \phi_{xxx}, \quad c = 1 \\
\phi(x, 0) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases} \Rightarrow \phi(x, t) &= \frac{2}{\pi} \int_0^\infty \frac{\sin k}{k} \cos(kx - (ck + k^3)t) dk
\end{align*}
\]
\[ \begin{align*}
\text{ex: } \phi_t + c \phi_x &= \phi_{xxx}, \quad c = 1, \quad \text{case 1: } x/t > c, \quad k_0 = \pm \left(\frac{1}{3} \left(\frac{x}{t} - c\right)\right)^{1/2} \\
\phi(x, t) &\sim \frac{\sin k_0}{k_0} \left(\frac{4}{3\pi |k_0| t}\right)^{1/2} \cos(k_0 x - (c k_0 + k_0^3) t - \pi/4) \text{ as } t \to \infty
\end{align*} \]