Math 557 Winter 2011 Homework 1 due: Thursday, January 20

0. (optional) Give a brief description of your academic background and research interests. If you work in a lab or research group, please give your supervisor’s name and describe your project. One paragraph is fine.

1. Let \( \text{Ei}(x) = s_n(x) + r_n(x) \) as in class. Show that \( r_n(x) \sim \frac{n!e^{-x}}{x^{n+1}} \) as \( x \to \infty \).

2. Find the asymptotic expansion of \( f(x) \) as \( x \to \infty \) with respect to the given sequence \( \{ \phi_n(x) \} \). Give the first three terms and the general term.
   a) \( f(x) = \log(1 + e^{-x}), \phi_n(x) = e^{-nx}, n \geq 1 \)
   b) \( f(x) = (x^2 + 1)^{-1/3}, \phi_n(x) = x^{-2(n+\frac{1}{3})}, n \geq 0 \)

3. page 11/9 (asymptotic expansion of Laplace transform of \( (1 + t^2)^{-1} \))

4. Define the error function by \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \).
   a) Find the Taylor series of \( \text{erf}(x) \) about \( x = 0 \). (hint: There is an easy method that doesn’t require computing the derivatives of \( \text{erf}(x) \).)
   b) Find the asymptotic expansion of \( \text{erf}(x) \) as \( x \to \infty \) with respect to the following sequence.

\[
\phi_n(x) = 1, \frac{e^{-x^2}}{x}, \frac{e^{-x^2}}{x^3}, \frac{e^{-x^2}}{x^5}, \ldots
\]

In (a) and (b), give the first four terms and the general term.

c) Find the interval of convergence for the series in (a) and (b).

Use Matlab to do part (d) below. Use \texttt{format long} and print all digits.

d) Make a table with the following format.
   - column 1: \( n \)
   - column 2: \( n \)-term Taylor approximation for \( \text{erf}(2) \)
   - column 3: error in column 2
   - column 4: \( n \)-term asymptotic approximation for \( \text{erf}(2) \)
   - column 5: error in column 4

Take \( n = 1 : 18 \). To compute the error you need the exact value; use \texttt{erf(2)} for that.

e) What is the smallest error in \( \text{erf}(2) \) that can be achieved using the asymptotic series? How many terms in the series are used in that case? How many terms are needed in the Taylor series to obtain the same accuracy? (By the way, the command \texttt{type erfcore} shows how Matlab computes \( \text{erf}(x) \).)

5. Find the first two terms in the asymptotic expansion as \( x \to \infty \).
   a) \( \int_0^1 e^{-xt^3} \, dt \)  
   b) \( \int_0^\infty e^{-xt} \cos t \, dt \)  
   c) \( \int_0^\infty e^{-xt} \sin t \, dt \)