hw2 , due: Wednesday, February 8

page 30 (singular value decomposition) 4.1 , 4.4

1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find $||A||_p$ for $p = 1,\ 2,\ \infty$.

2. Prove.

a) $||x||_\infty \le ||x||_2 \le ||x||_1$

b) If $A$ is hermitian with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$, then $\lambda_n = \min_{x \ne 0} R_A(x)$.

c) If $Q$ is orthogonal, then $||Qx||_2 = ||x||_2$, $||Q||_2 = 1$, $||QA||_2 = ||A||_2 = ||AQ||_2$.

d) If $||A|| < 1$ for some induced matrix norm, then $I + A$ is invertible.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

a) Use Matlab's command `svd` to find the SVD of $A$. State $U, \Sigma, V$ (4-digit format is fine).

b) In one plot draw the unit circle $C$ and indicate the vectors $v_1$, $v_2$, and in another plot draw the ellipse $AC$ (i.e. the image of the circle under the map $x \to Ax$) and indicate the vectors $Av_1 = \sigma_1 u_1$, $Av_2 = \sigma_2 u_2$. Use the `axis('square')` command in Matlab to ensure that the horizontal and vertical axes have the same scale.

c) Find $A_1$, the best rank-1 approximation to $A$ in the 2-norm. Find $||A - A_1||_2$.

4. Solve the two-point boundary value problem from hw1 using the compact 4th order finite-difference scheme derived in class,

$$-D_+D_-\Big(1 - \frac{h^2}{12}c_i\Big)u_i + c_i u_i = \Big(1 + \frac{h^2}{12}D_+D_-\Big)f_i \ .$$

Use meshsize $h = 1/2^p$, where $p$ is a positive integer. Solve the linear system using the tridiagonal $LU$ method derived in class (you get partial credit if you use a full matrix; to get full credit use only vectors). Turn in the program listing. For $p = 1:4$, plot the exact solution ($y(x)$ vs $x$) and the numerical solution ($u_i$ vs $x_i$, including the boundary points). For $p = 1:20$, present a table with the following data. column 1: $h$; column 2: $||u_h - y_h||_\infty$; column 3: $||u_h - y_h||_\infty / h^4$; column 4: cpu time; column 5: (cpu time)$/n$ (where $h = 1/(n+1)$ as in class). Explain the trends in each column. Among the given values of $h$, which value ensures that the error $||u_h - y_h||_\infty$ is less than $10^{-3}$ for the 4th order scheme? for the 2nd order scheme from hw1?