1. The following Matlab code computes the QR factorization of a matrix $A$ by Householder’s method. Copy the code into an m-file and fill in the missing variables (denoted by \cdot\cdot\cdot). Submit the completed code. Print out the resulting $Q,R$ using format long. Also print out the $Q,R$ from Matlab’s command $\text{qr}(A)$ and comment on any differences between the results.

```matlab
clear
A = [1 1; -1 0; 0 1];
[m,n] = size(A);
for k = 1:\cdot\cdot\cdot
    x = A(k:m,k);
    e = zeros(\cdot\cdot\cdot,1); e(1) = 1;
    v = norm(x)*e - x;
    v = v/norm(v);
    for j = k:\cdot\cdot\cdot
        A(k:\cdot\cdot\cdot,j) = A(k:\cdot\cdot\cdot,j) - 2*v*(v'*A(k:\cdot\cdot\cdot,j));
    end
    H = eye(\cdot\cdot\cdot) - 2*v*v';
    Q(:,:,k) = zeros(m,m); Q(1:\cdot\cdot\cdot,1:\cdot\cdot\cdot,k) = eye(\cdot\cdot\cdot);
    Q(k:\cdot\cdot\cdot,k:\cdot\cdot\cdot,k) = H;
end
temp = eye(\cdot\cdot\cdot); for k=1:\cdot\cdot\cdot; temp = temp*Q(:,:,k); end
Q = temp; R = A;
```

2. Consider the overdetermined linear system: $x - y = 1, x + y = 0, x = 1$. Sketch the lines in the $xy$-plane. Find and plot the least squares solution.

3. The molecular weights of six nitric oxides ($N_aO_b$) were measured experimentally, yielding the results below. Using this data, perform a least squares fit to estimate the atomic weight of nitrogen and oxygen. You may use any method to solve the least squares problem.

NO (30.006), $N_2O$ (44.013), NO$_2$ (46.006), NO$_2$O$_3$ (76.012), NO$_2$O$_4$ (92.011), NO$_2$O$_5$ (108.010)

4. Prove.
   a) $\kappa(A) \geq 1$ for any induced matrix norm
   b) If $U$ is unitary, then $\kappa_2(U) = 1, \kappa_2(UA) = \kappa_2(AU) = \kappa_2(A)$.
   c) $\kappa_2(A) = \sigma_{\max}/\sigma_{\min}$
   d) If $A$ is hermitian, then $\kappa_2(A) = |\lambda_{\max}|/|\lambda_{\min}|$.
   e) If $Ax = b$ and $(A + \delta A)(x + \delta x) = b$, then $\|\delta x\|/\|x + \delta x\| \leq \kappa(A)$.
   f) Consider the following example of $Ax = b, (A + \delta A)(x + \delta x) = b$.

\[
\begin{pmatrix}
10 & 7 & 8 & 7 \\
7 & 5 & 6 & 5 \\
8 & 6 & 10 & 9 \\
7 & 5 & 9 & 10
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
= \begin{pmatrix}
32 \\
23 \\
33 \\
31
\end{pmatrix},
\begin{pmatrix}
10 & 7 & 8.1 & 7.2 \\
7.08 & 5.04 & 6 & 5 \\
8 & 5.98 & 9.89 & 9 \\
\end{pmatrix}
\begin{pmatrix}
-81 \\
137 \\
-34 \\
22
\end{pmatrix}
= \begin{pmatrix}
32 \\
23 \\
33 \\
31
\end{pmatrix}
\]

Verify the arithmetic. Compute $\|\delta x\|_\infty/\|x + \delta x\|_\infty$, $\kappa_\infty(A)$. 

5. Consider the problem \( f(x_1, x_2, x_3) = (x_1 + x_2) \times x_3 \), evaluated by the algorithm \( \tilde{f}(x_1, x_2, x_3) = (\text{fl}(x_1) \oplus \text{fl}(x_2)) \otimes \text{fl}(x_3) \). Show that the algorithm is backward stable. In deriving this result, you may assume that each individual floating point operation is backward stable, as shown in class. (More generally, it can be shown that the composition of two backward stable algorithms is backward stable.)

6. (a) Repeat the Matlab steps on page 41 of the lecture notes illustrating the backward stability of Householder’s method for QR factorization using Matlab’s \texttt{qr} command. Give the six numerical answers as in the notes. Do your results agree with those in the notes? Discuss and explain.

   b) Repeat part (a) replacing Matlab’s \texttt{qr} command by the modified Gram-Schmidt method using the code given below. Give the six numerical answers. Discuss and explain the similarities and differences in comparison with the results you obtained in part (a).

\begin{verbatim}
function [Q,R] = mgs(A)
    [m,n] = size(A);
    for i = 1:n
        v(:,i) = A(:,i);
    end
    for i = 1:n
        R(i,i) = norm(v(:,i),2);
        Q(:,i) = v(:,i)/R(i,i);
        for j = i+1:n
            R(i,j) = dot(Q(:,i),v(:,j));
            v(:,j) = v(:,j) - R(i,j) * Q(:,i);
        end
    end
end
\end{verbatim}

7. Solve the two-point boundary value problem from hw1 using the compact 4th order finite-difference scheme derived in class,

\[ -D_+D_- \left( 1 - \frac{h^2}{12} d_i \right) u_i + d_i u_i = \left( 1 + \frac{h^2}{12} D_+D_- \right) f_i, \]

with mesh size \( h = 1/2^p \), where \( p \) is a positive integer. Solve the linear system using the tridiagonal \( LU \) method derived in class (use vectors, not full matrices). Turn in the program listing. For \( p = 1:4 \), plot the exact solution \((y(x) \text{ vs. } x)\) and the numerical solution \((u_i \text{ vs. } x_i\), including the boundary points\). For \( p = 1:20 \), present a table with the following data. column 1: \( h \); column 2: \( ||u_h - y_h||_\infty \); column 3: \( ||u_h - y_h||_\infty / h^4 \). Discuss and explain the trends in each column. Among the given values of \( h \), which value ensures that the error \( ||u_h - y_h||_\infty \) is less than \( 10^{-3} \) for the 4th order scheme? ... for the 2nd order scheme from hw1?