

Assignment #5 due: Wednesday, March 29

1. Consider $Ax = b$. Let x be the exact solution and let \tilde{x} be an approximate solution. The error is $e = x - \tilde{x}$ and the residual is $r = b - A\tilde{x}$.

(a) Show that $Ae = r$ and $\frac{\|e\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}$.

It follows that if A is invertible, then $e = 0$ if and only if $r = 0$, but if A is ill-conditioned, then the relative error $\frac{\|e\|}{\|x\|}$ may be large even if the relative residual $\frac{\|r\|}{\|b\|}$ is small. This occurs in the following example (due to W. Kahan).

$$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \quad b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}, \quad x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \tilde{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{x}_2 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}$$

(b) Show that $Ax = b$ (using exact arithmetic). Consider \tilde{x}_1 and \tilde{x}_2 as approximate solutions and for each one compute the corresponding $\frac{\|e\|_\infty}{\|x\|_\infty}, \frac{\|r\|_\infty}{\|b\|_\infty}$. Find $\kappa_\infty(A)$.

2. Let $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$. Using Gaussian elimination and partial pivoting, find P, L, U such that $PA = LU$.

3. Let A be the 5-point discrete Laplacian on the unit square with $h = \frac{1}{4}$ (so that A is a 9×9 matrix, the same as in class). Note that A is a band matrix and furthermore it's sparse within the band. Suppose that Gaussian elimination without pivoting is performed, and assume that the L, U factors are stored in the original matrix A . Draw a box around each zero element of A that gets filled in during the elimination. (Hint: this requires no computation.) We already observed in class that Gaussian elimination preserves the bandwidth of a band matrix, but this exercise shows that sparsity within the band may be lost. In this case we say that fill-in occurs within the band.

4. Let $A \in \mathbb{C}^{m \times m}$ be hermitian. Show that the following statements are equivalent by proving $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow a$.

a) The eigenvalues of A are positive.

b) A is positive definite.

c) Δ_k is positive definite for $k = 1, \dots, m$, where $\Delta_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$.

d) $\det \Delta_k > 0$ for $k = 1, \dots, m$

e) A has a Cholesky factorization.

5. Show that $A = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix}$ is positive definite and find its Cholesky factorization.

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Use $m = 1000$ instead of $m = 200$. Record the elapsed times. For each part (a) to (g), answer questions (i) and (ii) in the text.

7. Let $u(x, y)$ be the temperature in a square plate that is heated on one side and cooled on the other three sides. The temperature $u(x, y)$ satisfies the Laplace equation $\Delta u = 0$ for (x, y) in $(0, 1) \times (0, 1)$, with Dirichlet boundary conditions $u(x, 1) = 1, u(0, y) = u(1, y) = u(x, 0) = 0$.

a) Solve for $u(x, y)$ using the 5-point discrete Laplacian with meshsize $h = 2^{-p}$, $p = 2, 3, 4, 5$. Use Cholesky factorization to solve the linear system for $u_{ij} \approx u(x_i, y_j)$. To simplify the programming, you may use a full matrix (as opposed to a sparse representation). Display the numerical solution u_{ij} for each value of h using a contour plot and a surface plot (include the boundary values). The relevant commands in Matlab are **contour** and **mesh** (or **surf**).

b) Present a table with the following data. column 1: h ; column 2: $u_h(\frac{1}{4}, \frac{1}{4})$. Does the numerical solution converge as $h \rightarrow 0$? At what rate?

c) Something interesting happens at the northeast and northwest corners of the plate, where the temperature jumps from $u = 0$ to $u = 1$. Based on the numerical results, what is the limiting value of the exact solution $u(x, y)$ as (x, y) approaches one of those corners along various directions?