Assignment #6 due: Wednesday, April 12

1. Let $A = (a_{ij}) \in \mathbb{C}^{m\times m}$. Prove the following results.

a) Gershgorin's theorem
If λ is an eigenvalue of A, then there exists an index i such that $|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{m} |a_{ij}|$.

b) Bauer-Fike theorem
Assume that A is diagonalizable with $A = XDX^{-1}$, where $D = \mathrm{diag}(\lambda_i)$, and let w be an eigenvalue of $A + \delta A$. Then there exists an index i such that $|w - \lambda_i| \leq ||\delta A||_2 \cdot \kappa_2(X)$.

c) Let $A = \begin{pmatrix} 1 & \epsilon^{-1} \\ \epsilon & 1 \end{pmatrix}$, $A + \delta A = \begin{pmatrix} 1 & \epsilon^{-1} \\ 0 & 1 \end{pmatrix}$, where $\epsilon = 10^{-2}$.
Find $\lambda_1, \lambda_2, w_1, w_2, ||\delta A||_2, \kappa_2(X)$ using Matlab. Show that the BF theorem holds.

2. Which of the following matrices are normal? For those that are normal, find a unitary diagonalization. For those that are not normal, find a Schur factorization. Do the calculations by hand. (You may check your results using the Matlab functions `eig, schur`.)

a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

3. In class we derived an algorithm for reducing an $m \times m$ matrix A to upper Hessenberg form by unitary similarity transformations with operation count $\frac{10}{3}m^3$. If A is real symmetric, then the reduced matrix is tridiagonal and the algorithm can be modified so that the operation count becomes $\frac{4}{3}m^3$. Program the two versions in Matlab (you have to derive the 2nd version and show analytically that the operation count is $\frac{4}{3}m^3$; see pages 199-200 for a hint). Apply both codes to the matrix $A = [\,8\,,\,4\,,\,2\,,1\,;\,4\,,\,8\,,\,4\,,\,2\,;\,2\,,\,4\,,\,8\,,\,4\,;\,1\,,\,2\,,\,4\,,\,8\,]$ and output the final result. Run the codes for real symmetric $m \times m$ matrices (your choice) for $m = 100:100:1000$, present cputime$/m^3$, and comment on whether the results are consistent with the theoretical operation counts.

4. Let A be the 4×4 integer matrix from problem 3. Here we will compute the e-values of A using various methods. Present the results in a table with long format floating point numbers. Explain the results using the relevant theorems from class.

a) Apply the power method, inverse iteration (unshifted, i.e. $\mu = 0$), and Rayleigh quotient iteration to A. Omit phase 1, i.e. do not reduce A to tridiagonal form. For each method, start with $v^{(0)} = (1, 1, 1, 1)^T/2$, take 10 steps, and present the iterates $\lambda^{(k)}$, $k = 0:10$.

b) Take 10 steps of the QR algorithm starting with $A^{(0)} = A$. Print the maximum off-diagonal element magnitude at each step and also print the final matrix.

announcement
The final exam is on Wednesday, April 26, 10:30am-12:30pm in 4088 East Hall. It will cover the entire course. You may use one sheet of notes, both sides. Calculators are not allowed.