

hw#2 due : Tuesday , February 8

1. Find the local truncation error for the backward Euler method $u_{n+1} = u_n + hf(u_{n+1})$.
2. Consider the midpoint method $u_{n+1} = u_n + hf(u_n + \frac{h}{2}f(u_n))$ applied to the test equation $y' = \lambda y$.
 - a) Assume that λ is a negative real number. Show that $h\lambda$ is contained in the region of absolute stability if and only if $-2 \leq h\lambda \leq 0$.
 - b) Show that no imaginary value of $h\lambda$ is contained in the region of absolute stability.
3. Consider the trapezoid method $u_{n+1} = u_n + \frac{h}{2}(f(u_n) + f(u_{n+1}))$. Find the local truncation error and show that the scheme is A-stable.
4. Consider the initial value problem $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -11 & 9 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 = \begin{pmatrix} 1.0 \\ 1.2 \end{pmatrix}$.
 - a) Find an analytic expression for the exact solution of the problem and also for the numerical solutions given by the forward Euler and backward Euler methods. In each case show analytically that $\lim_{h \rightarrow 0} u_{1,n} = y_{1,n}$ at a fixed time $t = t_n = nh$.
 - b) Compute the numerical solution for $0 \leq t \leq 2$ using forward Euler and backward Euler with step size $h = 0.11, 0.10, 0.09$. In each case plot the first component of the exact solution $y_1(t)$ and the numerical solution $u_{1,n}$ vs. time. Display the results as in the class handout (use the `subplot` command in Matlab). Discuss the results. Which cases are absolutely stable and which are not absolutely stable?
5. a) Find the coefficients γ_i, β_i in the 4-step Adams-Bashforth method,

$$u_{n+1} = u_n + h(\gamma_0 f_n + \gamma_1 \nabla f_n + \gamma_2 \nabla^2 f_n + \gamma_3 \nabla^3 f_n),$$

$$u_{n+1} = u_n + h(\beta_0 f_n + \beta_1 f_{n-1} + \beta_2 f_{n-2} + \beta_3 f_{n-3}).$$
 b) Find the coefficients γ_i^*, β_i^* in the 3-step Adams-Moulton method,

$$u_{n+1} = u_n + h(\gamma_{-1}^* f_{n+1} + \gamma_0^* \nabla f_{n+1} + \gamma_1^* \nabla^2 f_{n+1} + \gamma_2^* \nabla^3 f_{n+1}),$$

$$u_{n+1} = u_n + h(\beta_{-1}^* f_{n+1} + \beta_0^* f_n + \beta_1^* f_{n-1} + \beta_2^* f_{n-2}).$$
 c) Find the local truncation error for each method. Which method is more accurate?
6. Consider the problem $y'' + y = 0, y(0) = 1, y'(0) = 0$. Express the problem as a first order linear system, $y' = Ay$. Solve the system using (a) forward Euler, (b) backward Euler, and (c) RK4, with step size $h = 0.1$ up to time $t = 4\pi$. Plot the solution in phase space, i.e. plot $y_1(t)$ on the x -axis and $y_2(t)$ on the y -axis, for the exact solution and the numerical solutions. Present all these results in a single Matlab plot which is correctly scaled (use the commands `axis('equal')` and `hold on`). Which of the three methods gives the best results? Why? Does absolute stability play a role?