1. Solve the difference equation for $u_n$. Check and make sure your answer satisfies the initial conditions and difference equation.
   a) $u_n - 2u_{n-1} - 3u_{n-2} = 0$, $u_0 = 0$, $u_1 = 1$
   b) $u_n - 3u_{n-1} + 3u_{n-2} - u_{n-3} = 0$, $u_0 = 1$, $u_1 = 0$, $u_2 = -3$

2. Consider the following 2-step methods for $y' = f(y)$,
   \begin{align*}
   \text{scheme 1} : & \quad u_n - u_{n-2} - \frac{h}{3}(f(u_n) + 4f(u_{n-1}) + f(u_{n-2})) = 0 \quad \text{(Milne's method)} \\
   \text{scheme 2} : & \quad u_n + 4u_{n-1} - 5u_{n-2} - h(4f(u_{n-1}) + 2f(u_{n-2})) = 0
   \end{align*}
   Answer the following questions for each scheme.
   a) Find $\rho(\zeta)$, $\sigma(\zeta)$. Is the scheme consistent? Is the root condition satisfied?
   b) Find the local truncation error in the form $\tau_n = Cy^{(r+1)}(t)h^{r+1} + O(h^{r+2})$.
   c) Consider the equation $y' = \lambda y$. Find the characteristic roots and show that they satisfy
      \begin{align*}
      \text{scheme 1} : & \quad \zeta_1(h) = e^{h\lambda} + O(h^5) , \ \zeta_2(h) = -e^{-h\lambda/3} + O(h^3) \\
      \text{scheme 2} : & \quad \zeta_1(h) = e^{h\lambda} + O(h^4) , \ \zeta_2(h) = -5e^{-3h\lambda/5} + O(h^2)
      \end{align*}
   d) Take $y(0) = 1$ for the exact solution and $u_0 = 1$, $u_1 = e^{h\lambda}$ for the numerical solution. Derive the following expressions for $u_n$. Are the schemes convergent? Explain.
      \begin{align*}
      \text{scheme 1} : & \quad u_n = (1 + O(h^5)) \cdot (e^{\lambda t} + O(h^4)) + O(h^5) \cdot ((-1)^n e^{-\lambda t/3} + O(h^2)) \\
      \text{scheme 2} : & \quad u_n = (1 + O(h^4)) \cdot (e^{\lambda t} + O(h^3)) + O(h^4) \cdot ((-5)^n e^{-3\lambda t/5} + O(h))
      \end{align*}

3. Consider the \textit{van der Pol equation}, $y'' + (y^2 - 1)y' + y = 0$, a model for nonlinear oscillations in an electrical circuit. The distinguishing feature of the equation is that it has a stable limit cycle, i.e., an attracting periodic solution. Express the equation as a first-order system, take $y(0) = 0.1$, $y'(0) = 0.1$ as initial data, and compute the numerical solution on the interval $0 \leq t \leq 40$ using the following schemes.
   a) 2-step Adams-Bashforth
   b) a predictor-corrector scheme using 2-step Adams-Bashforth as the predictor and one iteration of 1-step Adams-Moulton as the corrector

Take $h = 0.3$, 0.15, 0.075. Use the exact initial data and one step of Euler’s method for the starting values of the numerical solution. Plot the numerical solution two ways: (1) in physical space ($y$ versus $t$), (2) in the phase plane ($y'$ versus $y$). Discuss the results. How do the two methods compare?

\textbf{Announcement.} The midterm exam is on Thursday, February 24, in class. It will cover all the material on ODEs. You may use one page of notes (one side) and a calculator.