MATH 572 Numerical Methods for Scientific Computing II Winter 2005

Assignment #4 due : Thursday , March 10

1. Consider the 2-step BDF scheme, $\nabla u_n + \frac{1}{2}\nabla^2 u_n = hf(u_n)$.

a) Find the characteristic roots $\zeta_1(h), \zeta_2(h)$ for the test equation and plot them using Matlab over the interval $-10 \le h\lambda \le 0$.

b) Show analytically that the negative real axis is contained in the region of absolute stability. (Note: the scheme is actually A-stable, but it is not required to show that here.)

2. The Lorenz system is defined by $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} \sigma(y_2 - y_1) \\ ry_1 - y_2 - y_1y_3 \\ y_1y_2 - by_3 \end{pmatrix}$.

These equations were originally derived as a model for thermal convection; the variables represent the temperature, density, and velocity in a certain fluid flow. It was discovered by numerical computations that the parameters $\sigma = 10, b = 8/3, r = 28$ yield a system with chaotic dynamics. Solve this system with initial conditions $y_1(0) = 0, y_2(0) = 1, y_3(0) = 0$ up to time $t = 100$ using Matlab's `ode45` solver. (Part of the exercise is to read the online documentation and learn how this command works.) Plot the solution two ways: (a) projection of the orbit in the y_1y_3-plane, (b) full orbit in $y_1y_2y_3$-space using `plot3`. The object displayed in the plots is a strange attractor.

3. Consider the finite-difference scheme for the heat equation $u_j^{n+1} = u_j^n + kD_+D_-u_j^n$ on the whole real line with free-space boundary conditions. Show that the numerical solution u_j^n has an asymptotic expansion of the form $u_j^n = v_j^n + kE_j^n + O(k^2)$ for $k \to 0$, where $v_j^n = v(x_j, t_n)$ is the exact solution of the heat equation $v_t = v_{xx}$ and $E_j^n = E(x_j, t_n)$ is the principal error function. There are two steps: (a) heuristically derive the equation satisfied by $E(x,t)$, (b) prove the validity of the expansion.

4. Consider the following difference scheme for the heat equation $v_t = v_{xx}$ on the whole real line.

$$\frac{u_j^{n+1} - u_j^n}{k} = D_+D_-u_j^n - \frac{h^2}{12}(D_+D_-)^2u_j^n$$

a) Show that the local truncation error is $O(k) + O(h^4)$.

b) Find the amplification factor $\rho(\xi h)$.

c) For what values of $\lambda = \frac{k}{h^2}$ is the scheme stable in the 2-norm?

5. Compute the solution of the heat equation $v_t = v_{xx}$ on $0 \le x \le 1$ with boundary conditions $v(0,t) = v(1,t) = 0$ and initial condition (a) $v(x,0) = 1 - 2|x - \frac{1}{2}|$, (b) $v(x,0) = \sin \pi x$. Use 2nd order central differencing in space and forward Euler in time, with $h = 0.05$, $k = 0.0013, 0.0012$. Plot the solution at time $t = 0, k, 25k, 50k$ (in case (a) this reproduces the results on the class handout). Explain the results.