

Assignment #5 due : Thursday March 24

1. Given $u_j, j = 0, \pm 1, \dots$, define $\widehat{u}(\xi h) = \sum_{j=-\infty}^{\infty} u_j e^{ij\xi h}$. Verify the following formulas.

$$\text{a) } u_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{u}(\xi h) e^{-ij\xi h} d(\xi h) \quad , \quad \text{b) } \sum_{j=-\infty}^{\infty} |u_j|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{u}(\xi h)|^2 d(\xi h)$$

2. Consider the difference scheme $u_j^{n+1} = u_j^n + k D_+ D_- u_j^{n+1}$ for the heat equation $v_t = v_{xx}$, corresponding to backward Euler in time and 2nd order central differencing in space. Assume free-space boundary conditions.

a) Find the amplification factor $\rho(\xi h)$ and show that $0 \leq \rho(\xi h) \leq 1$ for all ξh . Plot $\rho(\xi h)$ for $0 \leq \xi h \leq \pi$ and $\lambda = \frac{1}{2}, 1, 10$.

b) Use Fourier analysis to show that the scheme is unconditionally stable in the 2-norm.

c) Use the energy method to show that the scheme is unconditionally stable in the 2-norm.

3. Consider the heat equation $v_t = v_{xx}$ on the interval $0 \leq x \leq 1$ with zero Dirichlet boundary conditions $v(0, t) = v(1, t) = 0$ and initial condition $v(x, 0) = 1 - 2|x - \frac{1}{2}|$ (problem 5a on hw4). The problem was previously solved using forward Euler in time and 2nd order central differencing in space. Now solve the problem by the Crank-Nicolson method with the same values of h, k as before. Solve the linear system at each time step by the tridiagonal Gaussian elimination algorithm given in class. Compare the Crank-Nicolson and forward Euler results, and explain any differences.

4. Consider the heat equation $v_t = v_{xx}$ on the interval $0 \leq x \leq 1$ with zero-flux boundary conditions $v_x(0, t) = v_x(1, t) = 0$. In this case the boundary values, $v(0, t)$ and $v(1, t)$, will vary in time. Consider the method of lines, $u'_j = D_+ D_- u_j$ for $j = 0, \dots, N, Nh = 1$, with $D_0 u_0 = D_0 u_N = 0$ to account for the boundary conditions, where $D_0 = \frac{1}{2}(D_+ + D_-)$.

a) Write the system in the form $u' = Au$, where $u = (u_0, \dots, u_N)^T$. Find the eigenvalues and eigenvectors of A . (Check your answer after deriving it.)

b) Suppose the forward Euler method is used to solve $u' = Au$. Show that the scheme is absolutely stable if the timestep satisfies $k \leq h^2/2$.

c) Consider the initial condition $v(x, 0) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$ This problem models the diffusion of gas molecules in a closed container, where $v(x, t)$ is the density of the gas at point x and time t . The initial condition defines a state in which the left half of the container is a vacuum and the right half of the container contains a gas of uniform density. Solve the problem numerically using the scheme described above, with $h = 0.05$ and $\lambda = \frac{1}{2}, \frac{1}{4}$. Plot the results for $t = 0, 1, 2$. Based on the numerical results, what is the long-time asymptotic state of the system?

d) (optional) Find an analytical expression for the numerical solution u_j^n . Study the convergence of this expression: (1) for fixed $t = nk$ as $k, h \rightarrow 0$, (2) for fixed k, h as $t = nk \rightarrow \infty$. In both cases assume that $\lambda = k/h^2$ is fixed.