

Assignment #6 due : Thursday, April 7

1. Consider the heat equation in two dimensions, $v_t = v_{xx} + v_{yy}$, on the domain $0 \leq x, y \leq 1$ with Dirichlet boundary conditions $v(0, y, t) = v(1, y, t) = v(x, 0, t) = 0$, $v(x, 1, t) = 1$ and initial condition $v(x, y, 0) = 0$. The solution $v(x, y, t)$ represents the temperature of a square plate that is heated on one side and cooled on the other three sides. Solve the problem numerically up to time $t = 2$ using the explicit scheme $u_{j,l}^{n+1} = u_{j,l}^n + k(D_+^x D_-^x + D_+^y D_-^y)u_{j,l}^n$. Take $h = 0.1$ and $k = 0.0025$. Make a contour plot and a surface plot of the numerical solution at time $t = 2$ (including the boundary values). The relevant commands in Matlab are `contour` and `mesh` (or `surf`).

2. Fritz John wrote in his textbook on partial differential equations, "Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation." He gave the following example to illustrate this principle.

a) Consider the wave equation $v_t + cv_x = 0$, $v(x, 0) = f(x)$, $c > 0$. Show that the numerical solution determined by the upwind scheme can be expressed as

$$u_j^n = ((1 - c\lambda)I + c\lambda S_-)^n f_j = \sum_{l=0}^n \binom{n}{l} (1 - c\lambda)^{n-l} (c\lambda)^l f_{j-l},$$

and the numerical solution determined by the downwind scheme can be expressed as

$$w_j^n = ((1 + c\lambda)I - c\lambda S_+)^n f_j = \sum_{l=0}^n \binom{n}{l} (1 + c\lambda)^{n-l} (-c\lambda)^l f_{j+l}.$$

b) Let $f(x) = e^{\alpha x}$ and take $t = t_n = nk$, $x = x_j = jh$ with $\lambda = \frac{k}{h}$ fixed. Using the formulas derived above, show that the numerical solutions v_j^n, w_j^n converge to the correct value $v(x, t) = e^{\alpha(x-ct)}$ as $n \rightarrow \infty$ for any value of λ .

In other words, the numerical solution converges even when the scheme is unstable. Fritz John noted that despite appearances, this result is in fact consistent with the CFL condition since an analytic function (such as $f(x) = e^{\alpha x}$) is determined by its values in any interval. It should further be noted that in computing the solution of a problem that is physically unstable (e.g. Kelvin-Helmholtz instability of a shear layer), it is necessary to use an unstable scheme (to ensure consistency).

3. Consider the wave equation $v_t + v_x = 0$ with two cases of initial data $v(x, 0)$ given by

$$f_1(x) = \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ -1, & x > 0 \end{cases}, \quad f_2(x) = \begin{cases} -1, & x < 0 \\ 1 - 2|x - 1|, & 0 \leq x \leq 2 \\ -1, & x > 2 \end{cases}.$$

Compute the solution for $-1 \leq x \leq 5$, $0 \leq t \leq 2$ using the upwind scheme and the downwind scheme, with $h = 0.05$ and $k = 0.04, 0.06$. For each scheme, plot the numerical solution and the exact solution (on the same plot) at $t = 0$ and $t = 2$. Discuss the results.