MATH 654 Introduction to Fluid Dynamics Fall 2008

homework 1 , due: Thurs, Sept 25

0. Give a brief description of your research or scientific interests (one paragraph is fine).

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2. Derive the following identities.

a) $\nabla \cdot (\mathbf{F}\mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\nabla \cdot \mathbf{F})\mathbf{G}$

b) $\dfrac{D(\rho f)}{Dt} = \rho \dfrac{Df}{Dt} + \dfrac{D\rho}{Dt} f$

c) $\dfrac{D}{Dt}(\frac{1}{2}|\mathbf{u}|^2) = \mathbf{u} \cdot \dfrac{D\mathbf{u}}{Dt}$

d) $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u} \times (\nabla \times \mathbf{u})$

e) $\mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{u} \cdot \nabla(\frac{1}{2}|\mathbf{u}|^2)$

3. a) Let $\mathbf{u} = (u, v) = u\mathbf{e_x} + v\mathbf{e_y}$ be the velocity field of a 2D flow expressed in Cartesian coordinates. Note that $u = u(x, y, t)$, $v = v(x, y, t)$ and $\mathbf{e_x} = (1, 0)$, $\mathbf{e_y} = (0, 1)$. Show that the acceleration is

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = (u_t + uu_x + vu_y)\mathbf{e_x} + (v_t + uv_x + vv_y)\mathbf{e_y}.$$

b) Let $\mathbf{u} = (u, v) = u\mathbf{e_r} + v\mathbf{e}_\theta$ be the velocity field of a 2D flow expressed in polar coordinates. Note that $u = u(r, \theta, t)$, $v = v(r, \theta, t)$ and $\mathbf{e_r} = \mathbf{e_r}(r, \theta)$, $\mathbf{e}_\theta = \mathbf{e}_\theta(r, \theta)$. Show that the acceleration is

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \Big(u_t + uu_r + \frac{vu_\theta}{r} - \frac{v^2}{r}\Big)\mathbf{e_r} + \Big(v_t + uv_r + \frac{vv_\theta}{r} + \frac{uv}{r}\Big)\mathbf{e}_\theta.$$

4. Complete the proof of the lemma $J_t = (\nabla \cdot u)J$ by deriving the following results.

a) $\begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_{xt} & \eta_{yt} & \eta_{zt} \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} = v_y J$, b) $\begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_{xt} & \zeta_{yt} & \zeta_{zt} \end{vmatrix} = w_z J$

c) In class we derived the relation $\rho(\phi(x, t), t)J(x, t) = \rho(x, 0)$ using the transport theorem. Differentiate this relation wrt time t and rederive the relation $J_t = (\nabla \cdot u)J$.

5. Derive an evolution equation for $\frac{1}{2}\rho|u|^2$, the kinetic energy density per unit volume at a fixed point in space, for the case of incompressible stratified flow. Use this result to derive an expression for $\frac{d}{dt}\int_W \frac{1}{2}\rho|u|^2\, dV$, the rate of change of total kinetic energy in a fixed volume W. Interpret the expression in words.

6. For each flow given below, find the flow map $\phi(x, t)$, find the set $C_1 = \phi(C_0, 1)$ (i.e. the image of the set C_0 under the flow map at time $t = 1$, where C_0 is the unit circle centered at the origin), plot C_0 and C_1, and describe in words what the flow does to C_0.

a) $(u, v) = (x, y)$, b) $(u, v) = (x, -y)$, c) $(u, v) = (y, -x)$