Consider steady homogeneous ideal 2D flow with zero body force.

a) Consider the velocity field \((u, v) = \gamma(x, -y)\) with strain rate \(\gamma\). Show that the pressure has the form \(p(r) = p_0 - \frac{1}{2}\rho_0\gamma^2 r^2\), where \(r^2 = x^2 + y^2\).

b) Consider the velocity field \((u, v) = \frac{\omega}{2}(-y, x)\) with vorticity \(\omega\). Show that the pressure has the form \(p(r) = p_0 + \frac{1}{2}\rho_0\omega^2 r^2\).

Note that the pressure is maximum at \(r = 0\) in (a) and minimum at \(r = 0\) in (b).

2. Let \(S = \frac{1}{2}(\nabla u - \nabla u^T)\) and \(\omega = \nabla \times u\). Show that \(S\omega = 0\). Find the eigenvalues of \(S\).

3. Prove the following identities.

a) \(\nabla \times (F \times G) = F(\nabla \cdot G) - G(\nabla \cdot F) + (G \cdot \nabla)F - (F \cdot \nabla)G\)

b) \(\nabla \cdot \nabla u = \Delta u\), \(\nabla \cdot \nabla u^T = \nabla (\nabla \cdot u)\)

c) \(u \cdot \Delta u = \nabla \cdot ((\nabla u)u) - |\nabla u|^2\), where \(|\nabla u|^2 = |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\)

d) \(\nabla \times u = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) e_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) e_\theta + \left(\frac{1}{r} \frac{\partial (ru_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta}\right) e_z\)

4. Show that the vorticity relation \(\omega(\phi(x, t), t) = \phi_x(x, t)\omega(x, 0)\) in 3D ideal flow reduces to \(\omega(\phi(x, t), t) = \omega(x, 0)\) in 2D ideal flow.

5. Consider 2D incompressible flow with stream function \(\psi(x, y, t)\) in a flow domain \(D\). Let \((x_1, y_1), (x_2, y_2)\) be two points in \(D\) and assume \(\rho = 1\). Show that \(\psi(x_2, y_2, t) - \psi(x_1, y_1, t)\) is the total mass flux across any curve in \(D\) connecting the two points.

6. page 45 / 2 (Navier-Stokes equations in cylindrical coordinates; use results from hw#1)

7. Consider steady incompressible viscous flow between two cylinders, \(R_1 \leq r \leq R_2\), in the following two cases. Find the velocity \(u(r)\) and vorticity \(\omega(r)\) in each case.

a) The cylinders are stationary and the flow is driven by a constant pressure gradient \(p_z\) in the axial direction (axisymmetric annular Poiseuille flow). Assume that \(u(r, \theta, z) = u(r)e_z\).

b) The cylinders are rotating (axisymmetric Couette flow). Assume that \(p = p(r), u(r, \theta, z) = u(r)e_\theta, u(R_1) = U_1, u(R_2) = U_2\).

8. Plot the streamlines associated with the following streamfunctions. Use a contour plot command in Matlab. Use subplot to conserve paper and axis square to get correct scaling.

a) \(\psi(x, y) = \gamma xy - \frac{\omega}{2}(x^2 + y^2)\)

This is a strained vortex. Take \(\gamma = 1, \omega = 0, 0.4, 1, 2, 3, 40\).

b) \(\psi(x, y) = \frac{\Gamma_1}{2\pi} \log((x - x_1)^2 + (y - y_1)^2)^{1/2} + \frac{\Gamma_2}{2\pi} \log((x - x_2)^2 + (y - y_2)^2)^{1/2} + Vx\)

This is a pair of point vortices. Take \(x_1 = -x_2 = 1, y_1 = y_2 = 0\).

case 1: \(\Gamma_1 = \Gamma_2 = 2\pi, V = 0\) (co-rotating)

case 2: \(\Gamma_1 = -\Gamma_2 = 2\pi, V = 0\) (counter-rotating)

case 3: \(\Gamma_1 = -\Gamma_2 = 2\pi, V = 0.5\) (counter-rotating in a uniform stream)