1. Recall the expression $W(z) = U(z + \frac{R^2}{z}) + \frac{R}{2\pi R} \log z$ for potential flow past a cylinder with circulation. Consider the case $\frac{R}{2\pi R} > 2U$, where $U > 0$, $R > 0$, $\Gamma > 0$. In class it was stated that there is a single stagnation point in the interior of the flow in this case. Find the location of the stagnation point in terms of $U, R, \Gamma$. Plot some streamlines for the case $U = R = 1, \Gamma = 6\pi$.

2. Recall the expression $W(z) = U\sqrt{z^2 + a^2}$ for potential flow past a flat plate. The flow is unrealistic due to the absence of a wake, so consider a simple model in which the complex potential is modified by adding a counter-rotating pair of point vortices, one at $z = z_0$ with circulation $\Gamma$ and the other at $z = \bar{z}_0$ with circulation $-\Gamma$. Assume that $U > 0$, real $z_0 > 0$.

a) Find the complex potential of the modified flow in the form $W_1(z) = W_2(\zeta(z))$, where $\zeta = z + \sqrt{z^2 + a^2}$ is the conformal mapping from the exterior of a plate in the $z$-plane to the exterior of a cylinder in the $\zeta$-plane and $W_2(\zeta)$ is the modified complex potential in the $\zeta$-plane.

b) Find the circulation value $\Gamma^*$ satisfying the Kutta condition at the edges of the plate.

c) Plot some streamlines in the $z$-plane and $\zeta$-plane for the case $U = a = 1$, $z_0 = 1 + i$, $\Gamma = \Gamma^*$.

3. Let $(x_j(t), y_j(t)), j = 1, \ldots, N$ denote a set of point vortices. Show that the following quantities are invariant in time.

$$X = \sum_{j=1}^{N} \Gamma_j x_j \quad Y = \sum_{j=1}^{N} \Gamma_j y_j \quad R^2 = \sum_{j=1}^{N} \Gamma_j (x_j^2 + y_j^2)$$

4. Consider two point vortices $z_1, z_2$ whose strengths satisfy $\Gamma_1 > \Gamma_2 > 0$ and define the center of vorticity by $Z = (\Gamma_1 z_1 + \Gamma_2 z_2)/(\Gamma_1 + \Gamma_2)$. The previous exercise shows that $Z$ is invariant in time. Show that the point vortices $z_1, z_2$ travel on circles centered at $Z$, with the same angular velocity, but different radii $R_1, R_2$. What happens to $Z, R_1, R_2$ in the limit $\Gamma_2 \to \Gamma_1$?

5. The stream function of a vortex-blob is $\psi_\delta(x, y) = -\frac{1}{2\pi} \log \sqrt{x^2 + y^2 + \delta^2}$, where $\delta$ is a smoothing parameter. Note that $\psi_0$ is the stream function of a point vortex.

a) Find the vorticity of a vortex-blob, $\omega_\delta = -\Delta \psi_\delta$, and show that $\int_{\mathbb{R}^2} \omega_\delta(x, y) \, dx \, dy = 1$ for all $\delta$. Plot $w_\delta(x, 0)$ for $-1 \leq x \leq 1$ with $\delta = 0.2, 0.1, 0.05$ (all on the same plot).

b) The evolution equations for a set of vortex-blobs are the same as for a set of point vortices except that $\psi_\delta$ is used instead of $\psi_0$. Compute the motion of a set of vortex-blobs having initial locations $x_j(0) = \cos \theta_j$, $y_j(0) = 0$, $\theta_j = (1 + \frac{j}{N+1})\pi$ and strengths $\Gamma_j = \frac{\pi}{N+1} \cos \theta_j$, for $j = 1, \ldots, N$. Take $N = 200, \delta = 2 \cdot 10^{-1}$ and use any convenient time integration scheme (code it yourself or use Matlab). Plot the blob locations at $t = 0, 1, 2, 4, 8, 16$ (use axis square in Matlab for proper scaling). The results represent a 2D model for the wake behind an airplane; see pages 50-51 in Van Dyke’s photo album. Accounting for 3D effects is an important ongoing research problem.