1. Show that the z-component of vorticity in polar coordinates is \( \omega_z = \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \).

Problems 2–5 are from chapter 3 of Drazin & Reid

2. **Oscillations of a swirl flow in a cylinder.** The velocity field \((u_r, u_\theta, u_z) = (0, r\Omega_0, 0)\) defines an inviscid equilibrium swirl flow with constant angular velocity \(\Omega_0\) in a cylinder of radius \(R_0\). In class we showed that the pressure perturbation \(p(r)\) satisfies the equation \( (D_\ast D - (n^2/r^2))p = k^2(1 + (4\Omega_0^2/\gamma^2))p \), where \(n\) and \(k\) are the azimuthal and axial wavenumbers, \(\gamma = s + in\Omega_0\), and \(s\) is the growth rate. The boundary conditions are that \(p(r)\) has a finite value at \(r = 0\) and \(\gamma Dp + (2in\Omega_0/r)p = 0\) at \(r = R_0\). Show that \(s = \pm 2i\Omega_0/(1 + \alpha^2/a^2)^{1/2} - in\), where \(\alpha = k/R_0\) and \(\alpha\) is any root of the equation \(\alpha J_\gamma'(\alpha) \pm n(1 + \alpha^2/a^2)^{1/2}J_\gamma(\alpha) = 0\). Hence the flow is marginally stable and \(\text{imag}(s)\) gives the oscillation frequency of the normal mode. Show that when \(n = 0\), this becomes \(s = \pm 2i\Omega_0/(1 + j_{1,m}^2/a^2)\), where \(j_{1,m}\) is the \(m\)th positive zero of \(J_1(\alpha)\).

3. **Rayleigh’s theorem for swirl flow.** Consider a general inviscid swirl flow with velocity \((u_r, u_\theta, u_z) = (0, V(r), 0)\) between two cylinders, \(R_1 \leq r \leq R_2\). In class we showed that a 2D perturbation satisfies the equation \( (s + in\Omega)(D_\ast D - n^2/r^2)\phi - inr^{-1}(DD_\ast V)\phi = 0 \), where \(\phi = ru\) and \(u = u(r)\) is the radial perturbation velocity. Multiply the equation by \(r\phi^\ast/(s + in\Omega)\) (where \(\phi^\ast\) is the complex conjugate of \(\phi\)), integrate from \(r = R_1\) to \(R_2\), apply the boundary conditions \(\phi(R_1) = \phi(R_2) = 0\), take the imaginary part of the result, and hence derive the relation \(\text{real}(s) \cdot n \int_{R_1}^{R_2} (|DD_\ast V|^2/s + in\Omega)^2 r\, dr = 0\). This yields Rayleigh’s theorem for 2D perturbations of a swirl flow, i.e. a necessary condition for inviscid instability is that the basic vorticity profile should have a local maximum or local minimum somewhere in the flow domain.

4. **Oscillations of a columnar vortex (Kelvin modes).** Consider the basic inviscid swirl flow given by \(\Omega(r) = \Omega_0\) for \(r < R_0\), \(\Omega(r) = \Omega_0(R_0/r)^2\) for \(r > R_0\). Show that the flow has constant vorticity for \(r < R_0\) and is irrotational for \(r > R_0\). Show that the flow is marginally stable with respect to both axisymmetric and 2D perturbations, and that in the latter case the oscillation frequency is \(\text{imag}(s) = -\Omega_0(n-1)\). Assume that the equation of the boundary of the perturbed vortex is \(r = R_0(1 + \epsilon \exp(st + in\theta))\) and show that the perturbation represents a sequence of waves traveling around the vortex with angular velocity \(\omega(n) = \Omega_0(1 - n^{-1})\). Note that \(\omega(1) = 0\) and \(|\omega(n)| < |\Omega_0|\) for \(n \geq 2\); explain what this means geometrically.

5. **Instability of a cylindrical vortex sheet (Rotunno).** Consider the basic inviscid swirl flow given by \(\Omega(r) = 0\) for \(r < R_0\), \(\Omega(r) = \Omega_0(R_0/r)^2\) for \(r > R_0\), which represents a cylindrical vortex sheet of radius \(R_0\). Show that the flow is marginally stable with respect to axisymmetric perturbations, but unstable with respect to 2D perturbations with growth rate \(s = \pm (\Omega_0/2)((n^2 - 2n)^{1/2} - in)\).