MATH 671 Particle Methods in Scientific Computing Winter 2007

hw#2 due: Thursday, March 8

1. Let $\rho(x)$ be a charge density on $0 \leq x \leq 1$ which satisfies charge neutrality, $\int_0^1 \rho(x)dx = 0$. In class we derived an integral expression for the associated potential function in the case of periodic boundary conditions, $\phi(x) = \int_0^1 (g(x,y) - xy)\rho(y)dy$, where $g(x,y) = -\frac{1}{2}|x-y|$ is the free-space Green's function in 1d. Verify that the integral expression for ϕ satisfies the Poisson equation, $-\phi_{xx} = \rho$, and the boundary conditions, $\phi(0) = \phi(1), \phi_x(0) = \phi_x(1)$.

2. The convolution of two functions $f(x)$ and $g(x)$ is defined by $(f * g)(x) = \int_{-\infty}^{\infty} f(x-s)g(s)ds$.

a) Let $W_0(x)$ be the nearest-mesh-point weight function. Show that $W_1 = W_0 * W_0$ is the cloud-in-cell weight function. (Here and below take $\Delta x = 1$ for simplicity.)

b) Find and plot the function $W_2 = W_1 * W_0$. Show that W_2 is continuously differentiable.

c) Show that $\int_{-\infty}^{\infty} W_i(x)dx = 1$ for $i = 0, 1, 2$.

3. Show that $\Phi(x,y,z) = 1/4\pi\sqrt{x^2+y^2+z^2}$ is a Green's function for the Laplacian in 3d.

4*. Set up a 1d Vlasov-Poisson PIC simulation with periodic boundary conditions. Use the code to simulate the two-stream instability, i.e. the interaction of two streams of identically charged particles flowing in opposite directions through each other. The initial particle distribution is uniform in space, with half the particles at velocity $v = 1 + \epsilon \sin 2\pi x$ and the other half of the particles at velocity $v = -(1 + \epsilon \sin 2\pi x)$, where ϵ is a perturbation amplitude. You may use any of the weighting functions discussed in class for charge assignment and force interpolation. Take $\epsilon = 10^{-1}$, $N_p = 500$, $N_m = 64$, $\Delta t = 10^{-3}$, and use the leap-frog method discussed in class for time-stepping. Display the particle locations in phase space at $t = 0, 1, 2$.

5*. The following code computes the electrostatic energy and forces induced by a set of charged particles in 3d with uniform charge. The coordinates of the particles are $(x_i, y_i, z_i), i = 1 : N$. The code uses direct summation.

```
phi = 0;
fx(1:N) = zeros(1:N); fy(1:N) = zeros(1:N); fz(1:N) = zeros(1:N);
for i=1:N
  for j=1:N
    if j==i;
    else;
    rij = sqrt((x(i)-x(j))^2+(y(i)-y(j))^2+z(i)-z(j))^2);
    phi = phi + 1/rij;
    fx(i) = fx(i) - (x(i)-x(j))/rij^3;
    fy(i) = fy(i) - (y(i)-y(j))/rij^3;
    fz(i) = fz(i) - (z(i)-z(j))/rij^3;
    end
  end
end
```

Run the code with $N = 2^p$ for $p = 1 : 10$. Use a random set of points in a cube of side $N^{1/3}$ (thus the particle density is uniform as N varies, e.g. as in a liquid). Plot the cpu time vs. N on log-log scales and verify the rate $O(N^2)$. Optimize the code as much as possible and plot the cpu time for the optimized code (make sure that the optimized code gives the same results as the original code, up to roundoff error). Submit a copy of the optimized code along with the cpu plots. Discuss the results.

6*. The stream function of a vortex-blob is $\psi_\delta(x,y) = -\frac{1}{2\pi}\log(x^2+y^2+\delta^2)^{1/2}$.

a) Find the associated vorticity $\omega_\delta(x,y)$. Show that $\int_{\mathbf{R}^2} \omega_\delta(x,y)\,dx\,dy = 1$ for all δ. Plot $w_\delta(x,0)$ for $-4 \le x \le 4$ with $\delta = 0.2,\ 0.1,\ 0.05$.

b) Solve the vortex-blob equations with initial data $x_j(0) = \cos\theta_j$, $y_j(0) = 0$, $\theta_j = (-1 + \frac{j}{N+1})\pi$ and strength $\Gamma_j = \frac{\pi}{N+1}\cos\theta_j$ for $j = 1:N$. Take $N = 200$, $\delta = 2\cdot 10^{-1}$. Plot the location of the vortex-blobs at $t = 0, 1, 2, 4, 8, 16$ using square axes. You may use any time integration scheme you find convenient as long as the results are correct to within plotting accuracy. Evaluate the velocity using direct summation. (Note: this is a 2d model for the trailing vortex sheet representing the wake of an aircraft; an important research problem is to add 3d effects).