Optimal Insurance in a Continuous-Time Model

Kristen S. Moore and Virginia R. Young[1]

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Keywords: optimal insurance, expected utility, stochastic control, Hamilton-Jacobi-Bellman equations, Markov Chain Approximation Method

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[1] Kristen S. Moore is an Assistant Professor, and Virginia R. Young is a Professor in the Department of Mathematics at the University of Michigan, Ann Arbor, Michigan, 48109. Moore can be reached at (734) 615-6864 or ksmoore@umich.edu, and Young can be reached at (734) 764-7227 or vryoung@umich.edu.

Moore is the corresponding author.

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Abstract: We seek the optimal dynamic consumption, investment, and insurance strategies for an individual who seeks to maximize her expected discounted utility of consumption and bequest over a fixed or random horizon, such as her random future lifetime. Thus, we incorporate an insurable loss and random horizon into the classical consumption and investment framework of Merton. We determine that if the premium is proportional to the expected payout, then the optimal per-claim insurance is deductible insurance; thus, we extend this result for static models to our dynamic setting. We compute the value function and optimal controls for many examples and contrast their qualitative properties, including the impact of the investor's horizon (or mortality) on the optimal controls and the interaction between the demand for insurance and the risky asset. We employ the Markov Chain Approximation Method of Kushner for those examples for which closed form solutions are not available.

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1. Introduction

We consider a risk-averse investor who can invest in and trade dynamically between a riskless and risky asset and who faces a random, insurable loss that is independent of the risky asset. We seek the optimal asset allocation, consumption, and insurance strategies to maximize her expected utility of terminal wealth (or bequest) and consumption over the fixed horizon T, or the random horizon τ, where τ is the time of death. The problem of optimal consumption and saving under uncertainty was studied by Merton (1969, 1971), Samuelson (1969), and others; we incorporate a random horizon and a random, insurable loss, which we model as a compound Poisson process, into the optimal investment and consumption problem.

The investor faces competing objectives. Insurance is costly and therefore reduces the utility that could be derived from saving or consuming; however, absorbing a large loss without insurance also diminishes utility. Current consumption reduces future utility from bequest.

Investing in the risky asset may yield high returns, but it may also yield losses. Moreover, in the case of a random horizon, there is an additional source of uncertainty.

Under fairly general conditions, the optimal insurance is deductible insurance (Arrow, 1963). Mossin (1968) showed that it is never optimal to purchase full insurance when the premium includes a positive loading above the actuarially fair price. However, most of the existing models for deductible insurance are static ones. In this paper, we examine a dynamic model for insurance and determine that if the premium rate is proportional to the expected payout, then optimal per-claim insurance is deductible insurance. We restrict the form of indemnity to be per-claim because we model the insurance risk as a compound Poisson process. The optimality of deductible insurance for a compound Poisson random variable is known in the static case (van Heerwaarden, 1991, Theorem 9.4.1; and Gollier and Schlesinger, 1995), so it is no surprise that it holds in the dynamic case.

By considering the dynamic setting, random horizon, and the effect of insurance on investment and consumption, this work complements and extends the existing literature on optimal insurance. Briys (1986) assumes that the loss is proportional to wealth and looks for the optimal coinsurance. It turns out that if the loss is proportional to wealth and if the utility is isoelastic, then the optimal insurance is a coinsurance, if we allow the deductible to be a function of wealth (and time); see Example 3.5. Gollier (1994) assumes that the loss is not a function of wealth—only of time—and looks for the optimal coinsurance. We show that optimal insurance in this case is deductible insurance; the two cases coincide in special cases. Plus, we extend Briys (1986) and Gollier (1994) by allowing the horizon to be random, as in the random time of death, and we consider the effect of insurance on investment and consumption decisions.

Other researchers have used stochastic control techniques to solve problems of interest in actuarial science and insurance economics. Asmussen and Taksar (1997) examine the problem of maximizing the discounted value of dividend payout for insurance companies. Højgaard and Taksar (1997) maximize the discounted value of wealth by optimizing over proportional reinsurance policies. Both papers use diffusion models, which are appropriate for modeling the cash flows of insurance companies but not for individuals—the perspective we take in this paper. Young and Zariphopoulou (2002a, b) use stochastic control techniques, similar to the ones in this paper, to find the reservation prices for a buyer and seller of insurance products.

In Section 2, we state the specifics of our problem. In Section 3, we determine optimal per-claim insurance for the case in which the horizon of the insurance buyer is fixed and finite. In Section 4, we examine the case in which the horizon of the insurance buyer is determined by the random time of death. In Section 5, we present numerical results for which the techniques of Sections 3 and 4 do not yield closed form solutions for the value function and optimal controls. Specifically, we consider the isoelastic case in which the loss amount is a fixed number. Gollier (1994) examines this case but does not fully consider the occasion when self-insurance is optimal. In each example, we examine the impact of the horizon on the optimal controls and examine the interaction between investment, consumption, and the insurable loss. We summarize the main results of our work below.

1. We show that in our dynamic setting, for “per-claim” indemnity coverage, optimal insurance is deductible insurance. This is consistent with the static results of Mossin (1968), van Heerwaarden (1991, Theorem 9.4.1), and Gollier and Schlesinger (1995).
2. We prove some general qualitative properties of the optimal deductible; namely, that, under certain conditions, it increases with wealth and the price of insurance.
3. By contrasting Examples 3.4 and 4.1, we find that introducing a random horizon eliminates the dependence of the optimal strategies on the investor’s horizon in the case of exponential utility.
4. We find that when we impose a lump-sum (versus continuous) insurance premium, the investor’s mortality (or expected horizon) affects the optimal deductible. Moreover, the parameters of the stock price process affect the optimal deductible; as the stock becomes more risky, the demand for insurance increases. (See Example 4.2.)
5. We demonstrate that under power utility, the parameters of the loss process affect the optimal consumption strategy and that this dependence may be counterintuitive, depending on the relative risk aversion. However, under logarithmic utility, the optimal consumption strategy is independent of the parameters of the loss process. See Examples 3.5, 4.3, 5.1, and 5.2
6. Finally, we observe that when we add an exogenous wage, the parameters of the loss process and the investor’s mortality (or expected horizon) affect the optimal strategies; see Examples 5.1b and 5.2.

These results are detailed in Sections 3 through 5 and summarized in the conclusion in Section 6.

2. The Problem

We assume that an individual can invest in a riskless asset (bond) whose price at time t, X_t, follows the process $dX_t = rX_t\,dt$, for some fixed $r \geq 0$. Also, this individual can invest in a risky asset (stock) whose price at time t, S_t, follows the process

$$\begin{cases} dS_s = S_s(\mu ds + \sigma dB_s), \quad s \geq t, \\ S_t = S, \end{cases}$$

in which $\mu > r$ and $\sigma > 0$ are constants, and B is a standard Brownian motion with respect to a filtration $\{F_t\}$ of the probability space $(\Omega, F, \Pr)$. Note that S follows a geometric Brownian motion.

Let W_t be the wealth at time t of the decision maker, and let π_t be the amount that the decision maker invests in the risky asset at time t. The decision maker earns an exogeneously given wage rate of $\omega(t)$ and consumes at a rate of c_t at time t. Finally, the decision maker is subject to an insurable risk modeled as a compound Poisson process, in which N is a Poisson process with deterministic parameter $\phi(t)$, and the random loss amount is $L(W_t, t)$. Assume that N is independent of B, the Brownian motion of the process for the risky asset. Also, the (random) loss amount L is independent of N. Note that we allow the loss to depend on the wealth at time t. Then, wealth without insurance follows the process

$$\begin{cases} dW_s = \left[rW_s + (\mu - r)\pi_s + \omega(s) - c_s\right]ds + \sigma\pi_s dB_s - L(W_s, s)dN_s, \quad s \geq t, \\ W_t = w. \end{cases} \tag{2.1}$$

Next, consider per-claim insurance of I_t. That is, if the individual suffers a loss of $L = l$ at time t, the indemnity pays $I_t(l)$. Such indemnity arrangements are common in private passenger automobile insurance and homeowners insurance, among others. Assume that the premium is payable continuously, so we subtract the premium rate from the first term on the right-hand side of expression (2.1). Assume that the premium rate $P(t)$ is proportional to the expected payout; specifically, $P(t) = [1 + \theta(t)]\phi(t)E[I_t(L)]$, in which $\theta(t) \geq 0$. Then, wealth with insurance follows

$$\begin{cases} dW_s = \left[rW_s + (\mu - r)\pi_s + \omega(t) - c_s - (1+\theta(s))\phi(s)E[I_s(L)]\right]ds + \sigma\pi_s dB_s \\ \qquad\qquad - (L(W_s,\ s) - I_s(L(W_s,\ s))dN_s, \quad s \geq t, \\ W_t = w. \end{cases} \tag{2.2}$$[2]

3. Fixed Time Horizon T

We assume that the decision maker seeks to maximize (over allowable $\{c_t, \pi_t, I_t\}$) his or her expected utility of discounted consumption during a fixed time interval $[t, T]$ and final wealth at time T. Allowable $\{c_t, \pi_t, I_t\}$ are those that are measurable with respect to the information available at time t, namely F_t, and that result in (2.2) having a unique solution.

Thus, the associated value function of this problem is given by

$$V(w,\ t) = \sup_{\{c_t,\, \pi_t,\, I_t\}} E\left[\int_t^T e^{-\rho(s-t)}u_1(c_s)\ \ ds + e^{-\rho(T-t)}u_2(W_T)\ \middle|\ W_t = w\right], \tag{3.1}$$

in which u_1 and u_2 are twice-differentiable, increasing, concave utility functions, and $\rho > 0$ is the rate at which consumption and terminal wealth are discounted. The parameter ρ is a personal discount rate; it is not a market parameter, but rather part of the utility functional. The function u_1 measures the utility of consumption, while u_2 measures the utility of terminal wealth. Note also that we assume that the utility for consumption and the utility for terminal wealth are additively separable. This is not necessarily true for all decision makers under expected utility, but we assume it here.

V solves the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{cases} V_t + \max\limits_{\pi}[(\mu - r)\pi V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww}] + \max\limits_{c}[u_1(c) - cV_w] + (rw + \omega(t))V_w \\ \quad + \max\limits_{I}[\phi(t)\{EV(w - (L - I(L)),\ t) - V(w,\ t)\} - (1+\theta(t))\phi(t)E[I(L)]V_w] \\ \quad = \rho V \\ V(w,\ T) = u_2(w). \end{cases} \tag{3.2}$$

V_t, V_w, and V_{ww} denote partial derivatives of V with respect to the given variable. For example, V_{ww} is the second partial derivative of V with respect to w. See Merton (1992, Section 5.8) and Björk (1998) for the derivation of related HJB equations. There also exist verification theorems that tell us if the value function V is smooth and if $\tilde{V}$ is a smooth solution of the associated HJB

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[2] We use "subscript t or s" to represent (possibly) random processes, while we use "parentheses t or s" to denote deterministic functions of time.

equation, then under certain regularity conditions, $\tilde{V} = V$. In those cases for which there is no smooth solution, we rely on the theory of viscosity solutions; see Young and Zariphopoulou (2002a) for further details and references.

The value function V is increasing and concave with respect to wealth w because the utility functions u_1 and u_2 are increasing and concave and because the differential equation for wealth is linear with respect to the controls. Thus, the optimal investment strategy $\{\pi^*_t\}$ is given by

$$\pi^*_t = -\frac{(\mu - r)}{\sigma^2} \frac{V_w(W^*_t, t)}{V_{ww}(W^*_t, t)}, \tag{3.3}$$

in which W^*_t is the optimally controlled wealth. Also, the optimal consumption $\{c^*_t\}$ solves the equation

$$u_1'(c^*_t) = V_w(W^*_t, t). \tag{3.4}$$

Concerning the optimal indemnity process $\{I^*_t\}$, we have the following proposition via standard arguments from insurance economics, as in Arrow (1963).

3.1 Proposition: *The optimal indemnity process* $\{I^*_t\}$ *is either no insurance or per-claim deductible insurance, in which the deductible may vary with respect to time. Specifically, at a given time, the optimal deductible* d^*_t *(if it exists) solves*

$$(1+\theta(t))V_w(W^*_t,\ t) = V_w(W^*_t - d_t,\ t). \tag{3.5}$$

No insurance is optimal at time t if and only if

$$(1+\theta(t))V_w(W^*_t,\ t) \geq V_w(W^*_t - \operatorname{ess\,sup} L(W^*_t,\ t),\ t). \qquad \square$$

We remark that we do not necessarily assume that L is a continuous random variable in this proposition. Also, if $\theta(t) = 0$, then $d^*_t = 0$; that is, full insurance is optimal if the premium rate is actuarially fair.

The result in Proposition 3.1 is well known in static models; namely, if insurance is per-claim insurance and if the premium is proportional to the expected payout, then optimal insurance is deductible insurance. Indeed, van Heerwaarden (1991, Theorem 9.4.1) proves it in her dissertation, and Schlesinger and Gollier (1995) prove it in the economics literature. Our contribution is to show that the same result holds in a dynamic setting, a not-so-surprising result. Note that the condition in (3.5) can be interpreted economically. Indeed, the left-hand-side is the marginal cost of decreasing d_t (or increasing insurance coverage), while the right-hand-side is the marginal benefit of increasing insurance coverage. Thus, optimality occurs when the marginal cost equals the marginal benefit, an intuitively pleasing result that parallels the static case.

A well-known property of the demand for insurance in the static case is that insurance can be a Giffen good if the relative risk aversion is greater than one. If the wealth effect is greater than the substitution effect, insurance is a Giffen good (Briys, Dionne and Eeckhoudt, 1989). However, insurance is not a Giffen good in the continuous-time model, as stated in the following corollary to Proposition 3.1. Note that this identical to Proposition 1 of Gollier (1994), in which he assumes insurance is proportional coverage.

3.2 Corollary: *An increase in the instantaneous price of insurance reduces the instantaneous demand for insurance, if* d^*_t *exists*.

Proof: Suppose $\theta(t)$ increases at time t for an infinitesimal length of time, then this change has no effect on the characteristics of the value function V. Because $V_{ww} < 0$, d^*_t also increases at time t for an infinitesimal length of time. □

Note that there is only a substitution effect in the instantaneous case. If there were a permanent increase in the loading, then demand for insurance might increase or decrease due to the wealth effect. We examine this problem in specific examples.

In the next corollary of Proposition 3.1, we show that if the value function V exhibits decreasing absolute risk aversion with respect to wealth, then the demand for insurance decreases with increasing wealth. Here, we follow Pratt (1964) and define the absolute risk aversion of V with respect to w by $-\frac{V_{ww}(w,t)}{V_w(w,t)}$. Corollary 3.3 directly parallels the corresponding result in the static case.

3.3 Corollary: *If V exhibits decreasing absolute risk aversion with respect to wealth, then the demand for insurance decreases with increasing wealth*, *if* d^*_t *exists*.

Proof: Differentiate equation (3.5) with respect to w to get that

$$\begin{aligned}\frac{\partial d^*_t}{\partial w} &\propto (1+\theta(t))V_{ww}(w,t) - V_{ww}(w-d^*_t, t)\\ &= (1+\theta(t))V_{ww}(w,t) - V_{ww}(G[(1+\theta(t))V_w(w,t),t],t),\end{aligned}$$

in which G is the inverse of V_w with respect to wealth. Let $w = G(y, t)$; then,

$$\begin{aligned}\frac{\partial d^*_t}{\partial w} &\propto (1+\theta(t))V_{ww}(G(y,t),t)-V_{ww}(G[(1+\theta(t))V_w(G(y,t),t),t],t)\\ &\propto (1+\theta(t))\frac{V_{ww}(G(y,t),t)}{V_w(G(y,t),t)}-\frac{V_{ww}(G[(1+\theta(t))y,t],t)}{V_w(G(y,t),t)}\\ &\propto \frac{V_{ww}(G(y,t),t)}{V_w(G(y,t),t)}-\frac{V_{ww}(G[(1+\theta(t))y,t],t)}{(1+\theta(t))y}\\ &\propto -\frac{V_{ww}(G[(1+\theta(t))y,t]}{V_{ww}(G[(1+\theta(t))y,t],t)}-\frac{V_{ww}(G(y,t),t)}{V_w(G(y,t),t)}\geq 0\end{aligned}$$

because G decreases with increasing wealth and V exhibits decreasing absolute risk aversion with respect to wealth. □

Next, we consider several examples.

3.4 Example (exponential utility): Suppose $u_1(c) \equiv 0$ and $u_2(w) = -\frac{1}{\alpha}e^{-\alpha w}$, for some $\alpha > 0$. Also, assume that the random loss L is independent of wealth, although we allow L's probability distribution to vary deterministically with respect to time. From $u_1(c) \equiv 0$, it follows that the optimal consumption is identically 0. A straightforward, but tedious, calculation shows that

$$V(w,t) = -\frac{1}{\alpha}\exp\left(-\alpha w e^{r(T-t)} - \frac{(\mu-r)^2}{2\sigma^2}(T-t)\right)\psi(t),$$

in which ψ solves

$$\begin{cases}\psi'(t)+\psi(t)\left[\phi(t)\left\{M_{L(t)\wedge d_t}(\alpha e^{r(T-t)})-1\right\}+\phi(t)(1+\theta(t))E(L(t)-d_t)_+\alpha e^{r(T-t)}-\omega(t)\alpha e^{r(T-t)}-\rho\right]=0,\\ \psi(T)=1\ ,\end{cases}$$

where $M_{L(t)\wedge d_t}$ is the moment generating function of $L(t) \wedge d_t = \min(L(t), d_t)$. It follows that the optimal deductible is given by

$$d^*(t) = \min\left[\frac{1}{\alpha}e^{-r(T-t)}\ln(1+\theta(t)), ess\sup L(t)\right],$$

and the optimal allocation to the risky asset is

$$\pi^* = \frac{(\mu-r)}{\sigma^2\alpha}e^{-r(T-t)}.$$

These results are consistent with those of Merton (1969; 1992, Section 4.9). Observe that $d^*(t)$ and $\pi^*(t)$ are not stochastic; in particular, they are independent of wealth. Also, the optimal investment strategy is independent of wealth and, thus, independent of whether the individual buys insurance. Such independence from wealth is generally observed in calculations with exponential utility because the absolute risk aversion (Pratt, 1964) is constant (equal to α). Moreover, note that $d^*(t)$ is independent of the parameters of the stock process, another artifact of exponential utility.

Note that as the risk aversion of the decision maker increases, as measured by α, the allocation to the risky asset decreases. Moreover, the deductible decreases, i.e, the demand for insurance increases. Also, as the load $\theta(t)$ increases, the demand for insurance decreases because it becomes more expensive. Thus, Corollary 3.2 holds for permanent increases in $\theta(t)$ because the wealth effect is zero.

Note also that as the horizon T increases, the deductible and the allocation to the risky asset decrease; thus, an investor with a longer horizon behaves more conservatively. This contradicts the conventional wisdom that investors with longer horizons can be more aggressive in assuming risk (see Siegel, 1994, for example). Samuelson (1963, 1989a,b) challenged this folk wisdom by pointing out that, although the standard deviation of the annual average return decreases with time, the standard deviation of the total return increases. $\square$

3.5 Example (power and logarithmic utility): Suppose $u_1(c) = \dfrac{c^\gamma}{\gamma}$ and $u_2(w) = b\dfrac{w^\gamma}{\gamma}$, for some $\gamma < 1$, $\gamma \neq 0$, and for some $b \geq 0$. The parameter b represents the weight that the individual gives to terminal wealth versus consumption and can be viewed as a measure of her propensity toward saving. Also, suppose the wage rate is zero, and assume that the loss is proportional to wealth, namely, $L(W_t, t) = \beta(t)\, W_t$, for some deterministic function $\beta(t)$ between 0 and 1. Then,

$$V(w, t) = \frac{w^\gamma}{\gamma}\xi(t),$$

in which ξ is given by

$$\xi(t) = \left[b^{\frac{1}{1-\gamma}} \exp\left(-\int_t^T \frac{H(s)}{1-\gamma}\, ds \right) + \int_t^T \exp\left(-\int_t^s \frac{H(u)}{1-\gamma}\, du \right) ds \right]^{1-\gamma},$$

with H given by

$$H(t) = \rho + \phi(t) - \delta\gamma + (1+\theta(t))\phi(t)\gamma \max\left(0,\, (1+\theta(t))^{\frac{1}{\gamma-1}} - (1-\beta(t)) \right)$$
$$- \phi(t) \max\left((1+\theta(t))^{\frac{1}{\gamma-1}},\, 1-\beta(t) \right)^\gamma,$$

and $\delta = r + \frac{(\mu - r)^2}{2\sigma^2(1-\gamma)}$. Thus, the optimal deductible is

$$d^*_t = \min\left(1-(1+\theta(t))^{-\frac{1}{1-\gamma}},\ \beta(t)\right)W^*_t,$$

the optimal consumption is

$$c^*_t = V_w^{\frac{1}{\gamma-1}} = \xi(t)^{-\frac{1}{1-\gamma}} W^*_t,$$

and the optimal investment in the risky asset is

$$\pi^*_t = \frac{\mu - r}{\sigma^2(1-\gamma)} W^*_t,$$

all multiples of wealth. This linearity in wealth occurs because the power utility function exhibits constant relative risk aversion (Pratt, 1964), that is, $-\frac{w\,u''(w)}{u'(w)} = 1-\gamma.$ Note that as the relative risk aversion increases, the proportion of wealth invested in the risky asset decreases and the demand for insurance increases (d^*_t decreases).

We see that ξ, and hence V and c^* are affected explicitly by the horizon T and that π^* and d^* are affected by T only through c^*'s impact on wealth. Note also that ξ, and hence c^* and V, depend on the frequency and severity parameters ϕ and β of the loss process. Moreover, we see that the optimal deductible d^* is independent of the stock price parameters and the optimal allocation π^* to the risky asset is independent of the insurable loss.

Next, consider the effect of insurance on the optimal consumption. If we solve the problem in (3.2) with I_t restricted to be identically zero, then we find that the value function $\hat{V}$ is similar to the value function V above with the function H replaced by

$$\hat{H}(t) = \rho + \phi(t) - \delta\gamma - \phi(t)(1-\beta(t))^{\gamma}.$$

When $\gamma < 0$, we can use the fact that $g(x) = x^{\gamma}$ is decreasing and convex to show that $H(t) \geq \hat{H}(t)$, for all $t \leq T$, from which it follows that the relative consumption with insurance is greater than the relative consumption without insurance. Here, we measure relative consumption with respect to the optimally controlled wealth. Similarly, when $\gamma \in (0,1)$, we can use the fact that $g(x) = x^{\gamma}$ is increasing and concave to show that $H(t) \leq \hat{H}(t)$, for all $t \leq T$, from which it follows that the relative consumption with insurance is less than the relative consumption without insurance. Thus, when $\gamma \in (0,1)$, an investor with no insurance consumes more.

To explain this counterintuitive behavior, we note that when $\gamma < 0$, the utility approaches $-\infty$ as $w \to 0^+$; however, when $\gamma \in (0,1)$, the utility approaches 0 as $w \to 0^+$. In other words, when $\gamma < 0$, there is an infinite penalty for losing everything. When $\gamma \in (0,1)$, one can increase utility in the face of a larger potential loss by consuming more; the penalty for losing everything is finite.

Young and Zariphopoulou (2002b) find similar counterintuitive results when $0 < \gamma < 1$. They speculate that such results for $0 < \gamma < 1$ are counterintuitive because a variety of studies have estimated the value of the relative risk aversion, $1-\gamma$, to lie between 1 and 2 (or equivalently, $-1 < \gamma < 0$). See, for example, the well-cited article by Friend and Blume (1975) that provides an empirical justification for constant relative risk aversion (CRRA), as well as the more recent Mitchell et al. (1999) paper in which the CRRA value is taken between 1 and 2. In the context of estimating the present value of a variable annuity for Social Security, Feldstein and Ranguelova (2001) argue that the value of CRRA is less than 3 and probably even less than 2. Indeed, this is standard in macroeconomics and finance; see Cochrane (2001) and Campbell and Viceira (2002).

Suppose we let $\gamma \to 0$, so that $u_1(c) = \ln c$ and $u_2(w) = b \ln w.$ Then,

$$V(w,t) = \xi(t) \ln w + \psi(t),$$

in which

$$\xi(t) = \left(b - \frac{1}{\rho} \right) e^{-\rho(T-t)} + \frac{1}{\rho}.$$

The optimal controls are consistent with our results for $\gamma \neq 0$. However, we remark that the optimal consumption c^* is independent of the frequency and severity parameters ϕ and β of the loss process; this was not the case when $\gamma \neq 0$. As in the case of power utility ($\gamma \neq 0$), we see that ξ, and hence V and c^*, are affected explicitly by the horizon T and that π^* and d^* are affected by T only through c^*'s impact on wealth. More specifically, if $\rho b > 1$, ξ is a decreasing function of T and, therefore, an individual with a longer horizon consumes a larger proportion of wealth. Similarly, if $\rho b < 1$, an individual with a longer horizon consumes a smaller proportion of wealth. The intuition behind this result is not clear. On the one hand, b measures the investor's propensity toward saving. On the other hand, ρ measures her preference for immediate, versus deferred, consumption. □

4. Random Time of Death τ

Now, suppose that the time horizon is not fixed, but rather it is determined by the random time of death τ of the decision maker. We still assume that the decision maker seeks to maximize (over allowable $\{c_t, \pi_t, I_t\}$) her expected utility of discounted consumption and final wealth. That is, the associated value function of this problem is given by

$$V(w, t) = \sup_{\{c_t,\, \pi_t,\, I_t\}} E\left[\int_t^\tau e^{-\rho(s-t)} u_1(c_s)\, ds + e^{-\rho(\tau-t)} u_2(W_\tau) \,\middle|\, W_t = w\right]. \tag{4.1}$$

The value function V solves the HJB equation:

$$\begin{cases} V_t + \max_\pi [(\mu - r)\pi V_w + \frac{1}{2}\sigma^2 \pi^2 V_{ww}] + \max_c [u_1(c) - cV_w] + (rw + \omega(t))V_w \\ \quad + \max_I [\phi(t)\{EV(w - (L - I(L)),\ t) - V(w,\ t)\} - (1+\theta(t))\phi(t)E[I(L)])V_w] \\ \quad + \lambda_x(t)\left[u_2(w) - V\right] = \rho V, \\ \\ \lim_{s\to\infty} E\left[e^{-\int_t^s (\rho + \lambda_x(u))du} V(W^*_s,\, s) \,\middle|\, W^*_t = w \right] = 0, \end{cases} \tag{4.2}$$

in which $\lambda_x(t)$ is the hazard rate, or force of mortality, for a person (x) at time t, or at age $x+t$. See Merton (1992, Section 4.6) for a derivation of the boundary condition. As before, V is increasing and concave with respect to wealth w. Thus, the optimal controls are as in Section 3; in particular, Propositions 3.1 and Corollaries 3.2 and 3.3 hold in this case.

4.1 Example (exponential utility): In this example, we reconsider Example 3.4, but with a random horizon and with the riskless rate of return $r = 0$. Suppose $u_1(c) \equiv 0$ and $u_2(w) = -\frac{1}{\alpha} e^{-\alpha w}$, for some $\alpha > 0$, and suppose the random loss L is independent of wealth. As is Example 3.4, we can show that

$$V(w,t) = f(t) e^{-\alpha w},$$

in which $f(t)$ solves

$$f' + G(t) f = \frac{\lambda_x(t)}{\alpha},$$

and

$$G(t) = \phi(t)\left[e^{\alpha d^*} - 1 + (1+\theta)(L(t) - d^*)\alpha\right] - \frac{1}{2}\frac{\mu^2}{\sigma^2} + \omega(t)\alpha + \lambda_x(t) + \rho.$$

We assume that the parameter values are such that $f(t) < 0$. Moreover, we have that the optimal consumption, deductible, and allocation to the risky asset are given by

$$c^* = 0,$$

$$d^*(t) = \min\left\{\frac{1}{\alpha}\ln\left(1+\theta(t)\right), ess\sup L(t)\right\},$$

and

$$\pi^* = \frac{\mu}{\sigma^2\alpha},$$

respectively. Thus, as in Example 3.4, the optimal investment strategy is independent of the parameters of the insurable loss and the optimal deductible is independent of the stock price process.

Recall that in Example 3.4, the optimal deductible and allocation to the risky asset were decreasing functions of the horizon *T*. In contrast, in this example, these strategies are independent of the force of mortality λ_x, and thus of the investor's horizon. However, as in Example 3.4, the value function *V* is driven by the horizon through the dependence of *f* and *G* on λ_x; however, the impact is ambiguous. For example, in the simpler case in which $L(t) = 1$ and the model parameters θ, ϕ, λ_x, and ω are constant, we have that

$$V(w,t) = V(w) = -\frac{1}{\alpha}\frac{\lambda}{B}e^{-\alpha w},$$

where

$$B = \lambda + \rho + \phi + \frac{1}{2}\frac{\mu}{\sigma^2} + \omega\alpha + (1+\theta)\phi\left[\ln(1+\theta) - (1+\alpha)\right];$$

thus, $V(w)$ increases with λ if and only if

$$\alpha\left[\omega - (1+\theta)\phi\right] < (1+\theta)\phi\left[1 - \ln(1+\theta)\right] - \left(\rho + \phi + \frac{1}{2}\frac{\mu}{\sigma^2}\right). \qquad \square$$

4.2 Example (exponential utility with lump-sum premium): Suppose we have the conditions given in Example 4.1. Also, suppose we have a stationary model; in particular, $\omega(t) \equiv \omega$, $\phi(t) \equiv \phi$, $\theta(t) \equiv \theta$, $\lambda_x(t) \equiv \lambda$, and the loss random variable *L* is independent of wealth *and* time. Suppose also that the premium is payable as a lump sum at time t equal to the expected present value of the payout times the loading $(1 + \theta)$. Specifically, the premium payable at time t is given by

$$\frac{1}{\lambda}(1+\theta)\,\phi\,E(L-d)_+ .$$

Because the model is stationary, the premium payable at time t is independent of t. For a given level of deductible d, the value function $V(w)$ solves

$$\begin{cases} \omega V' - \dfrac{\mu^2}{2\sigma^2}\dfrac{(V')^2}{V''} + \phi\{EV(w - L \wedge d) - V(w)\} + \lambda\left[u_2(w) - V\right] = \rho V, \\ \\ \lim\limits_{s\to\infty} E\left[e^{-(\lambda+\rho)(s-t)}V(W^*_s) \middle| W^*_t = w\right] = 0. \end{cases}$$

It follows that V is given by

$$V(w) = -\frac{1}{\alpha}e^{-\alpha w}\frac{\lambda}{\lambda + \rho + \delta + \alpha\omega - \phi\left[M_{L\wedge d}(\alpha) - 1\right]},$$

in which $\delta = \dfrac{\mu^2}{2\sigma^2}$. We assume that $\lambda + \rho + \delta + \alpha\omega - \phi\left[M_{L\wedge d}(\alpha) - 1\right] > 0$. Therefore, as in Example 4.1, the optimal allocation to the risky asset is $\pi^* = \dfrac{\mu}{\alpha\sigma^2}$.

To find the optimal deductible d^*, we maximize

$$V\left(w - \frac{1}{\lambda}(1+\theta)\,\phi\,E(L-d)_+\right)$$

with respect to d. It follows that the optimal d^* solves

$$(1+\theta)\left(\lambda + \rho + \delta + \alpha\omega - \phi\left[M_{L\wedge d}(\alpha) - 1\right]\right) = \lambda e^{\alpha d}.$$

This is our first example in which the optimal deductible depends on the parameters of the stock price process and on the Poisson frequency parameter ϕ. Moreover, d^* depends on the investor's horizon through its dependence on the force of mortality λ.

It is not difficult to determine the following properties of the optimal deductible:

$$\frac{\partial d^*}{\partial\theta} > 0,\ \frac{\partial d^*}{\partial\delta} > 0,\ \frac{\partial d^*}{\partial\lambda} \propto (1+\theta) - e^{\alpha d^*},\ \frac{\partial d^*}{\partial\rho} > 0,\ \frac{\partial d^*}{\partial\phi} < 0,\ \frac{\partial d^*}{\partial\omega} > 0, \text{and}$$

$$\frac{\partial d^*}{\partial\alpha} \propto (1+\theta)\omega - (1+\theta)\phi\int_0^{d^*}(1+\alpha x)\,e^{\alpha x}S_L(x)\,dx - \lambda\,d^*e^{\alpha d^*}.$$

Thus, as the insurance gets relatively more expensive (θ increases), the demand for insurance decreases (d^* increases). As the stock process gets more risky (δ decreases), the demand for insurance increases. As the force of mortality increases, (i.e., as the horizon of the investor decreases), the demand for insurance is ambiguous. As the value of future wealth decreases (ρ increases), the demand for insurance decreases. As claims become more frequent (ϕ increases), the demand for insurance increases (d^* decreases). As the wage rate increases, the demand for insurance decreases; this is a type of wealth effect that is a bit of a surprise with exponential utility. As the buyer becomes more risk averse (α increases), the demand for insurance is ambiguous. On the one hand, the demand could increase because the buyer is more risk averse; on the other hand, if the wage rate is great enough, future wages can act as a type of insurance. Finally, we can show that if the loss L increases in the ordering of first stochastic dominance (Wang and Young, 1998), then the demand for insurance increases. ☐

4.3 Example (power and logarithmic utility): We revisit Example 3.5, but now incorporate a random horizon. Suppose $u_1(c) = \frac{c^\gamma}{\gamma}$ and $u_2(w) = b\frac{w^\gamma}{\gamma}$, for some $\gamma < 1$, $\gamma \neq 0$ and for some $b \geq 0$, and suppose the random loss L is proportional to wealth, namely, $L(W_t, t) = \beta(t)\, W_t$, for some deterministic function $\beta(t)$ between 0 and 1. Also, assume that the wage rate is identically 0. Then, V is given by

$$V(w,t) = \frac{w^\gamma}{\gamma}\xi(t),$$

in which ξ solves the non-linear, non-homogeneous differential equation

$$0 = \xi' - \tilde{H}(t)\xi + (1-\gamma)\xi^{\frac{\gamma}{\gamma-1}} + b\lambda_x(t),$$

with $\tilde{H}(t) = H(t) + \lambda_x(t)$, in which H is given in Example 3.5. We assume that the parameters are such that $\xi(t) > 0$. Thus, the optimal deductible is

$$d^*_t = \min\left(1 - (1+\theta(t))^{\frac{1}{\gamma-1}},\ \beta(t)\right) W^*_t,$$

the optimal consumption is

$$c^*_t = V_w^{\frac{1}{\gamma-1}} = \xi(t)^{\frac{1}{\gamma-1}} W^*_t,$$

and the optimal investment in the risky asset is

$$\pi^*_t = \frac{\mu - r}{\sigma^2(1-\gamma)} W^*_t,$$

all multiples of wealth, as in Example 3.5. Recall that in Example 3.5, ξ, and hence V and c^*, were affected explicitly by the fixed horizon T. A similar phenomenon occurs here; these quantities are impacted by the force of mortality λ_x. Moreover, as in Example 3.5, π^* and d^* are affected by mortality only through c^*'s impact on wealth. Finally, we note that the optimal deductible d^* is independent of the stock price parameters and the optimal allocation π^* to the risky asset is independent of the insurable loss.

Suppose now that $\gamma \to 0$, so that $u_1(c) = \ln c$ and $u_2(w) = b \ln w$. Then, the optimal deductible and allocation to the risky asset are consistent with our results for $\gamma \neq 0$. The optimal consumption is

$$c^*_t = V_w^{-1} = \xi(t)^{-1} W^*_t .$$

Here ξ is given by

$$\xi(t) = \bar{a}^{\rho}_{x+t} + b\bar{A}^{\rho}_{x+t},$$

in which $\bar{a}^{\rho}_{x+t}$ equals the actuarial present value of a life annuity issued to a person aged $x + t$ that pays 1 per year continuously with the rate of discount equal to ρ (Bowers et al., 1997). Similarly, $\bar{A}^{\rho}_{x+t}$ equals the net single premium of a whole life insurance policy issued to a person aged $x + t$ that pays 1 at the moment of death with the rate of discount equal to ρ. The results in Example 3.5 concerning how the optimal controls are affected by the parameters of the loss process hold in this case as well. In particular, in the special case of constant hazard rate λ, we have that $\xi^{-1}(t) = \dfrac{\lambda + \rho}{1 + b\lambda}$. Thus, if $\rho b < 1$, ξ^{-1} increases with λ; therefore, investors with a longer horizon (smaller λ) consume a smaller proportion of wealth. This is consistent with the result of Example 3.5. □

5. Numerical Results

In this section, we present numerical results for examples for which the techniques of Sections 3 and 4 do not yield closed form solutions for the value function and optimal controls. We restrict our attention to the stationary problem in which the model parameters λ, ϕ, θ, and ω

are constant. In this case, the HJB equation is an ordinary differential equation. We employ the Markov Chain Approximation Method (MCAM) of Kushner (2000) to numerically approximate the value function and controls. We refer the reader to Sections 2.1 and 5.6 of Kushner and Dupuis (2001) and Fitzpatrick and Fleming (1991) for details on the implementation of the MCAM and to Chapter 13 of Kushner and Dupuis (2001) for a discussion of convergence of the method.

We describe the intuition behind and some of the details of the MCAM in the context of a specific example, for which a closed form solution is available, in Appendix 1. We demonstrate the convergence of the method for this same problem in Example 5.1a. We then apply the method to more complicated examples for which closed form solutions are not available.

5.1a Example: *Convergence of the MCAM and Impact of Varying the Model Parameters*

In this example, we consider an individual who stands to lose a constant proportion $\beta \in (0,1]$ of her wealth and who seeks to maximize her expected utility of consumption and terminal wealth over her lifetime. We assume that her utility of consumption and wealth are given by $u_1(c) = \frac{c^\gamma}{\gamma}$ and $u_2(w) = b\frac{w^\gamma}{\gamma}$, respectively. Thus, we revisit the stationary case of Example 4.3. In the first part of the example, we set the wage rate $\omega = 0$. In this case, one can verify that the function

$$V(w) = a\frac{w^\gamma}{\gamma} \tag{5.1}$$

solves the HJB equation where a solves

$$a[r\gamma - \rho + \frac{(\mu - r)^2}{2\sigma^2}\frac{\gamma}{1-\gamma} + \phi Q - \lambda] + (1-\gamma)a^{\frac{\gamma}{\gamma-1}} + \lambda b = 0 \tag{5.2}$$

and

$$Q = \max\{1-\beta, (1+\theta)^{\frac{1}{\gamma-1}}\}^\gamma - 1 - (1+\theta)\gamma(\beta - [1-(1+\theta)^{\frac{1}{\gamma-1}}])_+. \tag{5.3}$$

Moreover, the optimal controls are given by

$$\pi^*_t = \frac{\mu - r}{\sigma^2(1-\gamma)}W^*_t, \tag{5.4}$$

$$c^*_t = a^{\frac{1}{\gamma-1}} W^*_t, \tag{5.5}$$

and

$$d^*_t = \min\{\beta, 1-(1+\theta)^{\frac{1}{\gamma-1}}\} W^*_t\,. \tag{5.6}$$

In the experiments that follow, we demonstrate the convergence of the MCAM to the solution given above and we examine the impact of varying the model parameters. In Example 5.1b, we include a nonzero exogenous wage ω. In this case a closed form solution is not available; we use the MCAM to compute the value function and optimal controls and we contrast the qualitative properties of the resulting solutions with the $\omega = 0$ case.

We fix as our base scenario the following choice of parameters:

Premium load θ	0.25	Utility parameter γ	0.50
Loss frequency ϕ	0.10	Force of mortality λ	0.05
Risk free rate r	0.03	Subjective discount ρ	0.05
Mean stock return μ	0.07	Propensity to save b	1.00
Stock volatility σ	0.30	Loss proportion β	0.50

***Table 1**: Parameters for the base scenario in Examples 5.1a and 5.1b*

We choose the parameter values, and γ in particular, to ensure convergence of the numerical method; see Fitzpatrick and Fleming (1991). We comment on more realistic values of γ in Example 3.5. Observe that by (5.6),

$$d^*_t = \min\{\beta, 1-(1+\theta)^{\frac{1}{\gamma-1}}\} W^*_t = \min\{0.50, 0.36\} W^*_t = 0.36 W^*_t\,;$$

thus, the optimal strategy to insure against a loss of 50% of one's wealth is 36% coinsurance.

Convergence of the MCAM

Figure 1 shows the approximate and actual value function and optimal controls for $w \in [0, 300]$. We note that the error at the right hand boundary is caused by the fact that the transition probability there prohibits wealth from exceeding 300. In effect, we are truncating the domain on which the HJB equation is defined and imposing an artificial boundary condition at $w = 300$; see Appendix 1 for details. However, the convergence is excellent away from the boundary; for that reason, in the experiments that follow, we approximate the value function and optimal controls on [0, 300], but restrict our attention to the interval [0, 100]. We used a mesh size of $dw = 0.10$.

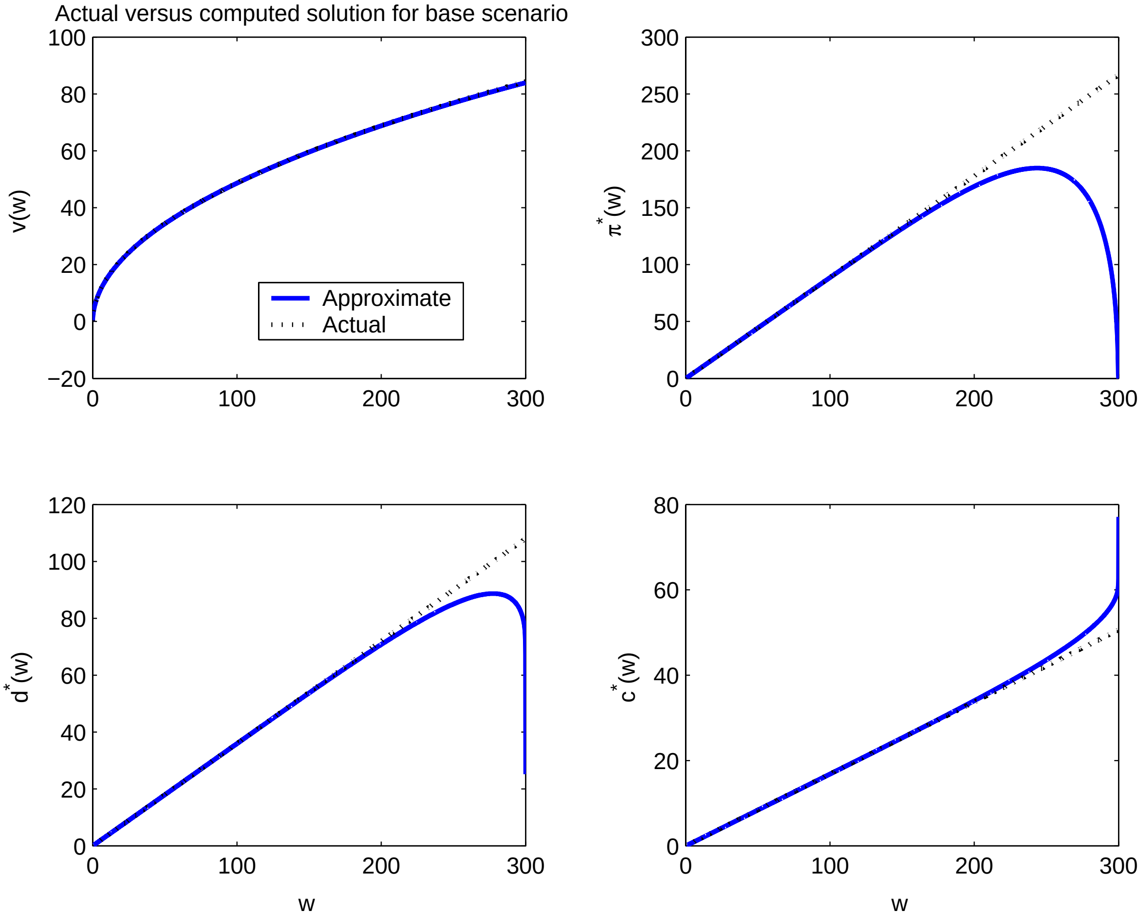

Figure 1*: Convergence of the MCAM to the actual solution for Example 5.1a*

Impact of Varying the Model Parameters

In various numerical experiments, we examined the impact of varying individual model parameters θ, β, ϕ, λ, ρ, b, and γ from the base scenario described above. In each case, we verified the convergence of the approximate value function and optimal controls to the actual solution as in Figure 1. For conciseness of exposition, we do not present all the details but rather summarize our results in Table 2. However, we do provide the details for two experiments that yielded counterintuitive results.

Impact of changing the loss proportion β

From (5.6), we see that if $\beta > 0.36$, partial insurance at 36% is optimal. However, if $\beta \leq 0.36$, no insurance is optimal; i.e., $d^*_t = \beta W^*_t$. The first graph in Figure 2 confirms this. It is surprising to note in the third graph that as the potential loss increases, the individual should consume more. A similar counterintuitive result was described and explained in Example 3.5 when $\gamma \in (0,1)$. For a similar choice of parameters with $\gamma = -1$, we find that c^* decreases as β increases, which is more consistent with our expectation of investor behavior.

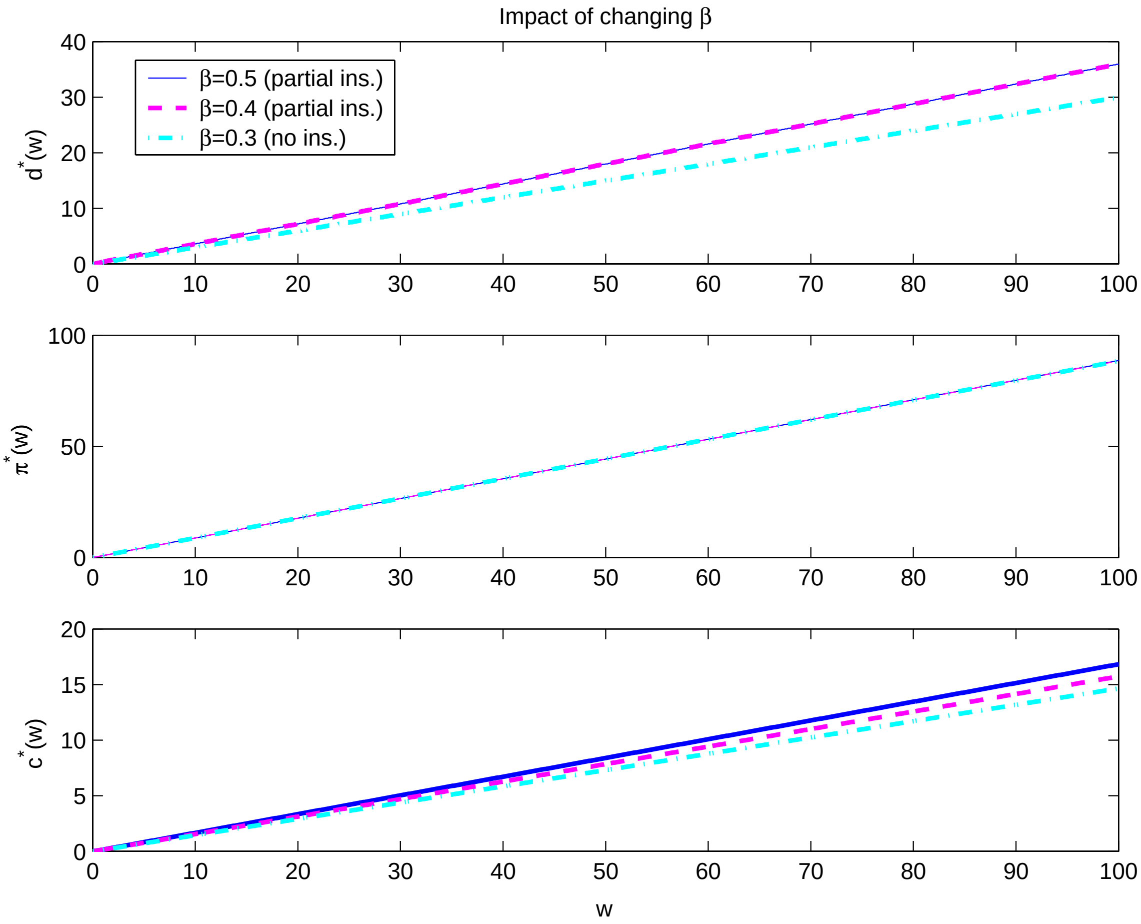

Figure 2: *Impact of changing the loss proportion*

Impact of changing the loss frequency ϕ

In (5.6) we see that the optimal deductible is independent of the Poisson parameter ϕ; the first graph in Figure 3 confirms this. The third graph shows that if the loss frequency increases, the individual should consume more. This counterintuitive result is similar to the one described in the previous paragraph and in Example 3.5.

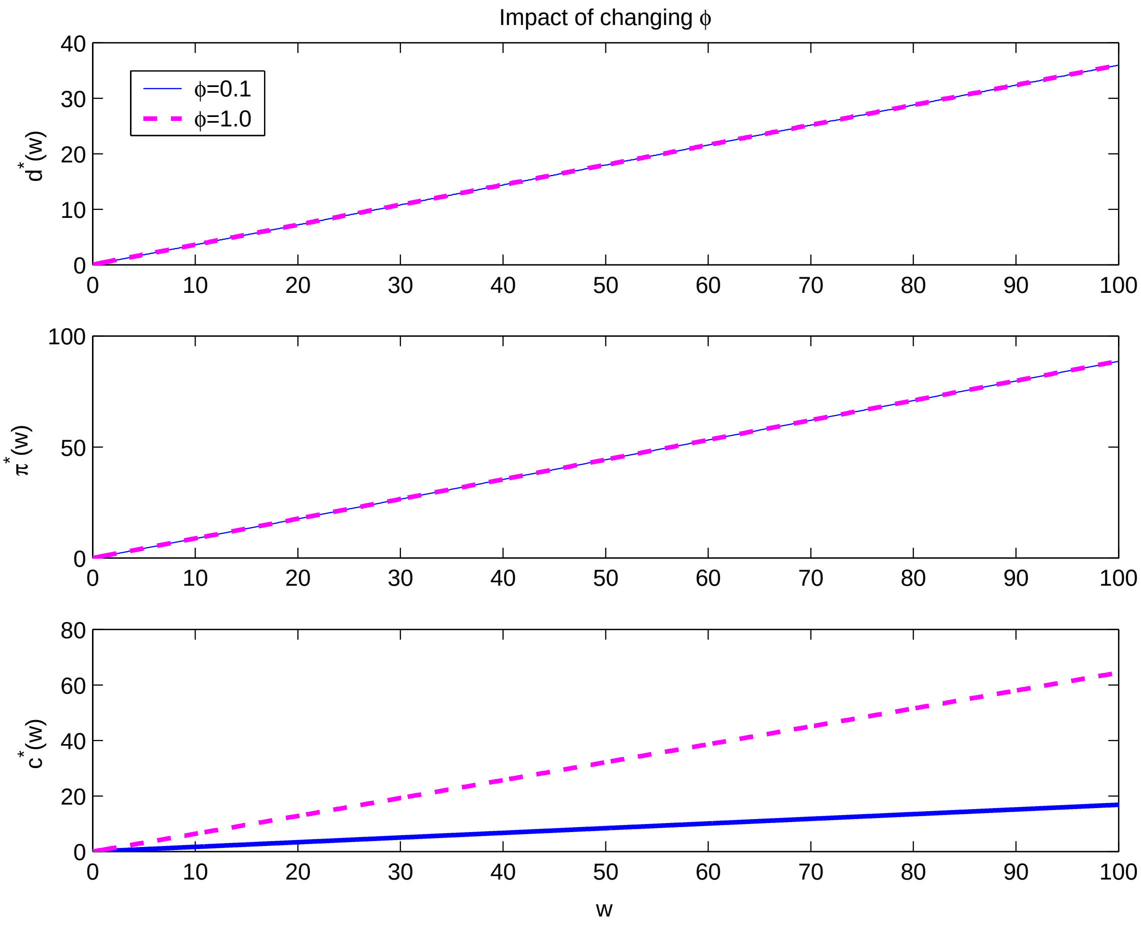

Figure 3*: Impact of changing the loss frequency*

The impact of varying the other model parameters is summarized in Table 2. ☐

5.1b Example *Impact of adding an exogenous wage*

In the previous example, the value function and optimal controls were given in closed form by (5.1)-(5.6), although one must determine the coefficient a of the value function by numerically solving the nonlinear equation (5.2). The purpose of that example was twofold: to demonstrate that the MCAM converges to the actual solution, and to examine the impact of varying the model parameters from the base scenario.

In this example, we modify the previous example slightly by adding a constant, non-zero exogenous wage ω to the wealth evolution. In this case, the HJB equation is not solvable in closed form. This is because the loss is not proportional to aggregate wealth, including the net present value of future wages. We use the MCAM to approximate the optimal controls and we contrast the optimal strategies with those from the previous example. Thus, we examine the qualitative impact of an exogenous wage on the optimal controls.

Impact of changing the wage

In Figure 4, we see that as we increase the wage ω from 0 to 1 to 20, the investor should purchase less insurance, invest more in the risky asset, and consume more. This is consistent with our intuition. Cocco, Gomes, and Maenhout (2005) considered a realistically-calibrated model for optimal investment during a fixed time period in the presence of wage income under CRRA utility. They found that as the wage income decreases, the amount invested in the riskless asset increases, which is consistent with our results.

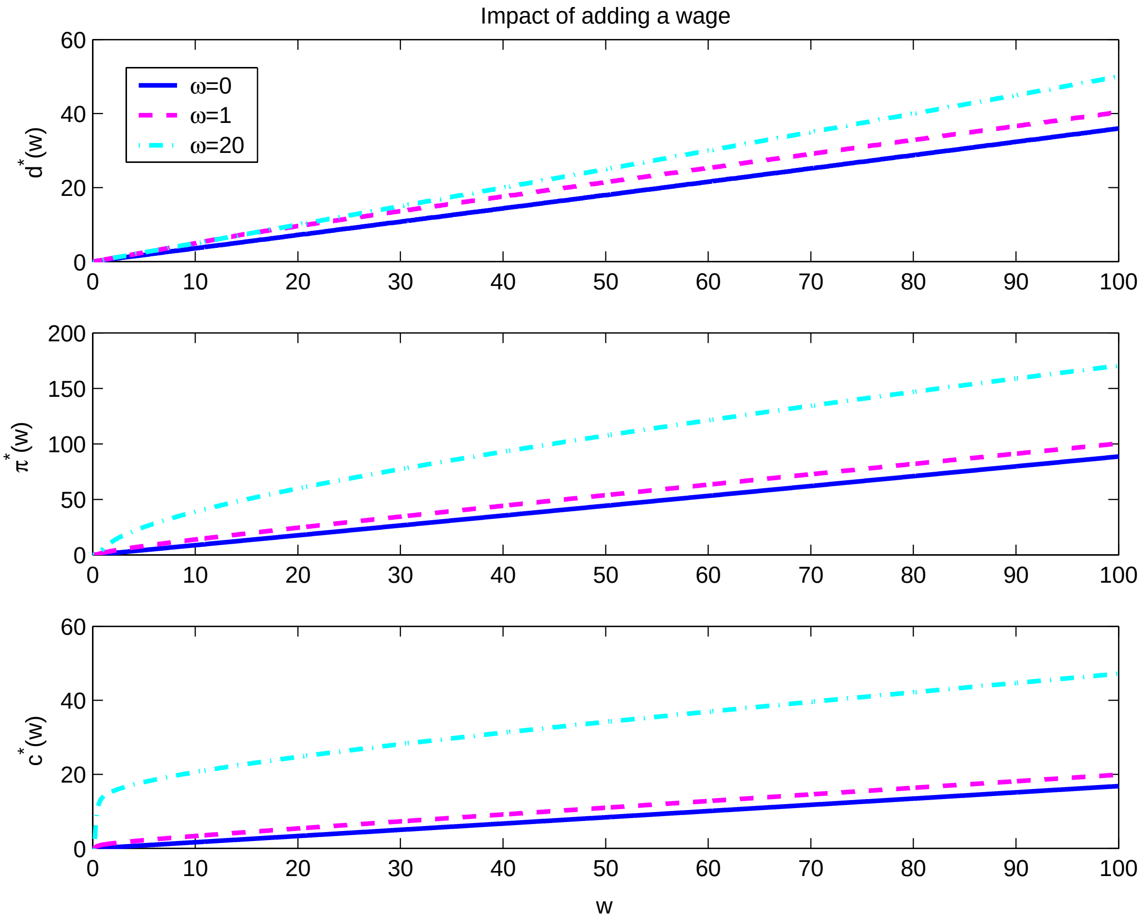

Figure 4: *Impact of adding a wage*

Impact of varying the model parameters

We repeated all of the experiments of Example 5.1a to examine the impact of changing the model parameters in the presence of an exogenous wage. We examined the base scenario described in Table 1 with $\omega = 1$ and changed the model parameters θ, β, ϕ, λ, ρ, b, and γ as in Example 5.1a. For conciseness of exposition, we will not include the graphs for the experiments.

Below we summarize their results and highlight the ways in which the presence of the exogenous wage affects the investor's optimal strategies.

Table 2: Impact of Adding Exogenous Wage

Change	***Wage* $\omega = 0$**	***Wage* $\omega = 1$**
Increase premium load θ from 25% to 35%.	Buy partial insurance. π^* is unchanged. Change in c^* is nearly imperceptible.	Buy no insurance. π^* is unchanged. Change in c^* is nearly imperceptible.
Decrease loss proportion β from 50% to 40%.	Buy partial insurance. π^* is unchanged. c^* decreases. See Figure 2.	Buy no insurance. π^* is unchanged. c^* decreases.
Increase loss frequency ϕ from 0.1 to 1.0	d^* and π^* are unchanged. c^* increases. See Figure 3.	Deductible and investment in risky asset are higher than when $\omega = 0$, but when ϕ increases, ● lower deductible ● lower investment in risky asset ● higher consumption are optimal
Increase force of mortality λ from 0.05 to 0.15	d^* and π^* are unchanged. c^* increases.	Deductible and investment in risky asset are higher than when $\omega = 0$, but when λ increases, ● lower deductible ● lower investment in risky asset ● higher consumption are optimal
Increase the propensity to save b	d^* and π^* are unchanged. c^* decreases.	Similar results.
Increase the relative risk aversion $1 - \gamma$ by decreasing γ	● Lower deductible ● lower investment in risky asset ● lower consumption are optimal See Figure 7.	Similar results.

It is not surprising that, in the presence of a wage, the investor should purchase less insurance (i.e., opt for higher deductibles). More specifically, we found that when we increased the premium load to θ = 35%, with ω = 1, the individual should not purchase any insurance. (When ω = 0 and θ = 35%, the optimal strategy was partial insurance; see Figure 2.) Similarly, when we decreased the loss proportion to β = 40%, with ω = 1, the individual should not purchase any insurance. (When ω = 0 and β = 40%, the optimal strategy was partial insurance.)

We found that when ω = 1, if the loss frequency increases from ϕ = 0.1 to ϕ = 1.0, the individual should opt for a lower deductible and invest less in the risky asset. Thus, in the presence of a nonzero wage, the optimal allocation to the risky asset is affected by the parameters of the insurable loss; this is the first example in which this happens. (Recall that when ω = 0, the optimal insurance and investment strategies were not affected by increasing ϕ; see Figure 3 and equations (5.4) and (5.6).)

When ω = 1 and we increased the force of mortality from λ = 0.05 to λ = 0.15, we found that higher deductibles and greater investment in the risky asset are optimal when ω = 1 than when ω = 0. However, when ω = 1, the investor's insurance and investment strategies are affected by changes in λ; they were unaffected when ω = 0. Thus, mortality (or, alternatively, the investor's horizon) impacts the insurance and investment decisions in the presence of an exogenous wage. More specifically, an investor with a longer horizon (or smaller λ) should choose a higher deductible (i.e., purchase less insurance), invest more in the risky asset, and consume less. Thus, the investor assumes more risk, but decreases consumption. This is consistent with Gollier's (2002) observation that younger investors can be optimally time diversified by splitting risks on wealth into risks on consumption. □

5.2 Example: We assume that the individual faces a possible loss of L = 1 and that she seeks to maximize her expected utility of terminal wealth and consumption over her lifetime. We assume that the model parameters $\theta(t)$, $\phi(t)$, $\omega(t)$, and $\lambda_x(t)$ are constant. Moreover, we assume power utility of consumption and terminal wealth as in Examples 3.5 and 4.3; i.e., $u_1(c) = \frac{c^\gamma}{\gamma}$ and $u_2(w) = b\frac{w^\gamma}{\gamma}$ for some $\gamma < 1$, $\gamma \neq 0$ and $b \geq 0$. Unlike these earlier examples, we set the wage rate ω to be nonzero. Thus, the wealth process is given by

$$\begin{cases} dW_t = \left[rW_t + (\mu - r)\pi_t + \omega - c_t - (1+\theta)\phi(1-d_t) \right] dt + \sigma\pi_t dB_t - d_t dN_t \\ W_0 = w, \end{cases}$$

and the value function V is given by

$$V(w) = \sup_{\{c_t, \pi_t, d_t\}} E\left[\int_0^\tau e^{-\rho s} u_1(c_s) ds + e^{-\rho\tau} u_2(W_\tau) \,|\, W_0 = w \right].$$

It follows that V solves the HJB equation

$$\begin{cases} \max_\pi [(\mu - r)\pi V' + \frac{1}{2}\sigma^2\pi^2 V''] + \max_c [u_1(c) - cV'] + (rw + \omega)V' \\ \quad + \max_{0 \le d \le 1} [\phi\, V(w-d) - (1+\theta)\,\phi\,(1-d)V'] + \lambda\, u_2(w) = (\rho + \lambda + \phi)V, \\ \\ \lim_{t\to\infty} E\left[e^{-(\rho+\lambda)t} V(W^*_t) \middle| W^*_0 = w \right] = 0. \end{cases}$$

In this case, a closed form solution is not available. We set $b = 1$, $r = \lambda = \rho = 0.05$, $\theta = 0.01$, $\phi = 0.1$, $\mu = 0.07$, $\sigma = 0.3$, $\gamma = 0.5$, and $\omega = 1$ and use the MCAM to approximate the optimal strategies and to examine the impact on the optimal strategies of varying the model parameters. Because of the boundary error that we discussed in Example 5.1a, we restrict our attention to the domain [20,80].

Impact of changing the wage

We contrast the optimal strategies for $\omega = 1$ and $\omega = 20$. When the wage $\omega = 20$, a possible loss of $L = 1$ has a lower impact on the insured's financial health; thus, we expect the optimal deductible to be higher when $\omega = 20$ than when $\omega = 1$. As we see in Figure 5, this is indeed the case; when $\omega = 20$, zero insurance is optimal. This contradicts Gollier's (1994, pp 85ff) results because we consider conditions for which self-insurance is the optimal strategy. However, it is consistent with Gollier's (2003) observation that the optimal deductible is more sensitive to the wealth level in the dynamic framework because the individual is able to time-diversify her risk. He argues that this type of phenomenon could be driven by liquidity constraints at lower wealth levels, which inhibit investors' ability to allocate risk on wealth over future periods.

Note also that the optimal consumption and investment in the risky asset are higher for $\omega = 20$ than for $\omega = 1$; this is consistent with our financial intuition.

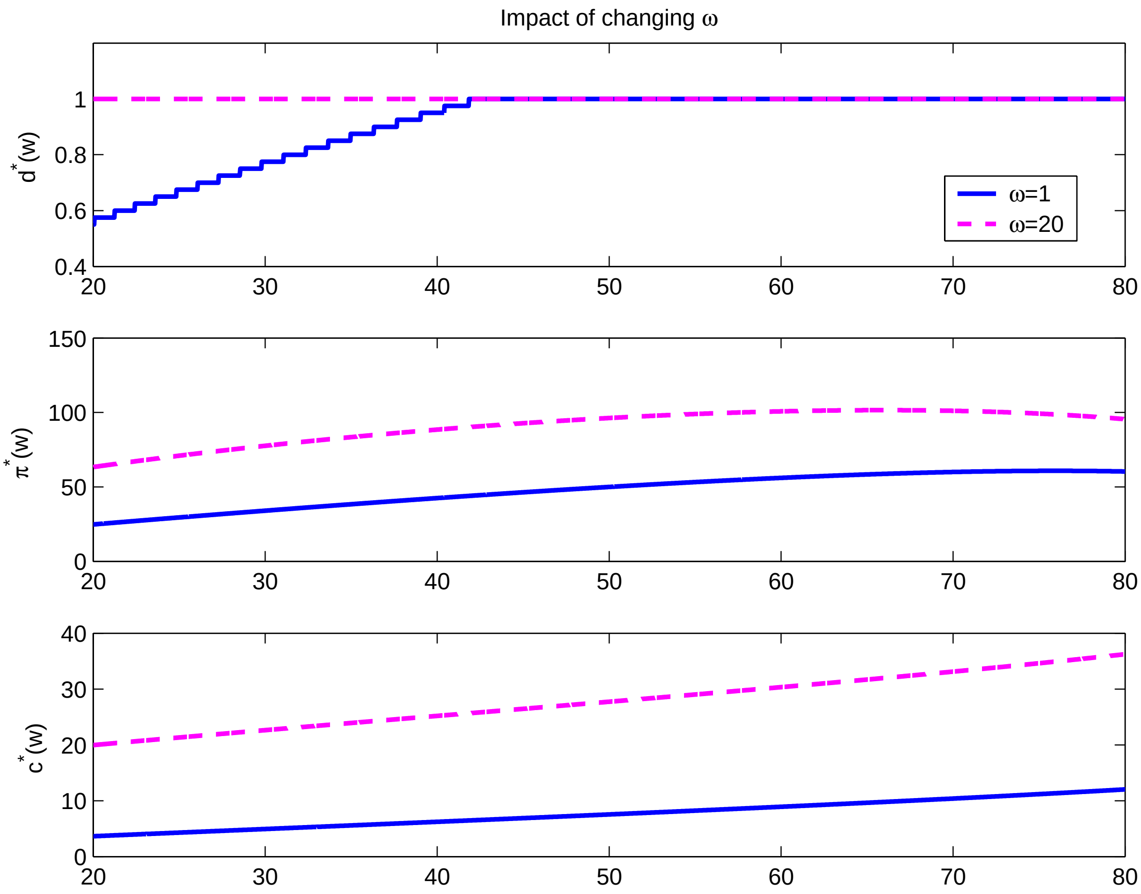

Figure 5: *Impact of changing the wage*

Impact of changing the cost of insurance

We see in Figure 6 that when we increase the premium load from $\theta = 0.01$ to $\theta = 0.25$, the demand for insurance decreases. When insurance is expensive (θ=0.25), zero insurance is optimal for $w \in [20, 80]$, but when insurance is inexpensive ($\theta = 0.01$), $d^* \in [0.55, 0.95]$ for the lower wealth levels. This contradicts Gollier's (1994, pp 85ff) result of constant deductible because he did not consider the fact that the deductible will be higher at lower wealth levels because negative wealth is precluded under power utility. Again, however, it is consistent with Gollier's (2003) observation that wealthier individuals should self-insure. Note that the optimal consumption and investment strategies are not affected by this change in θ.

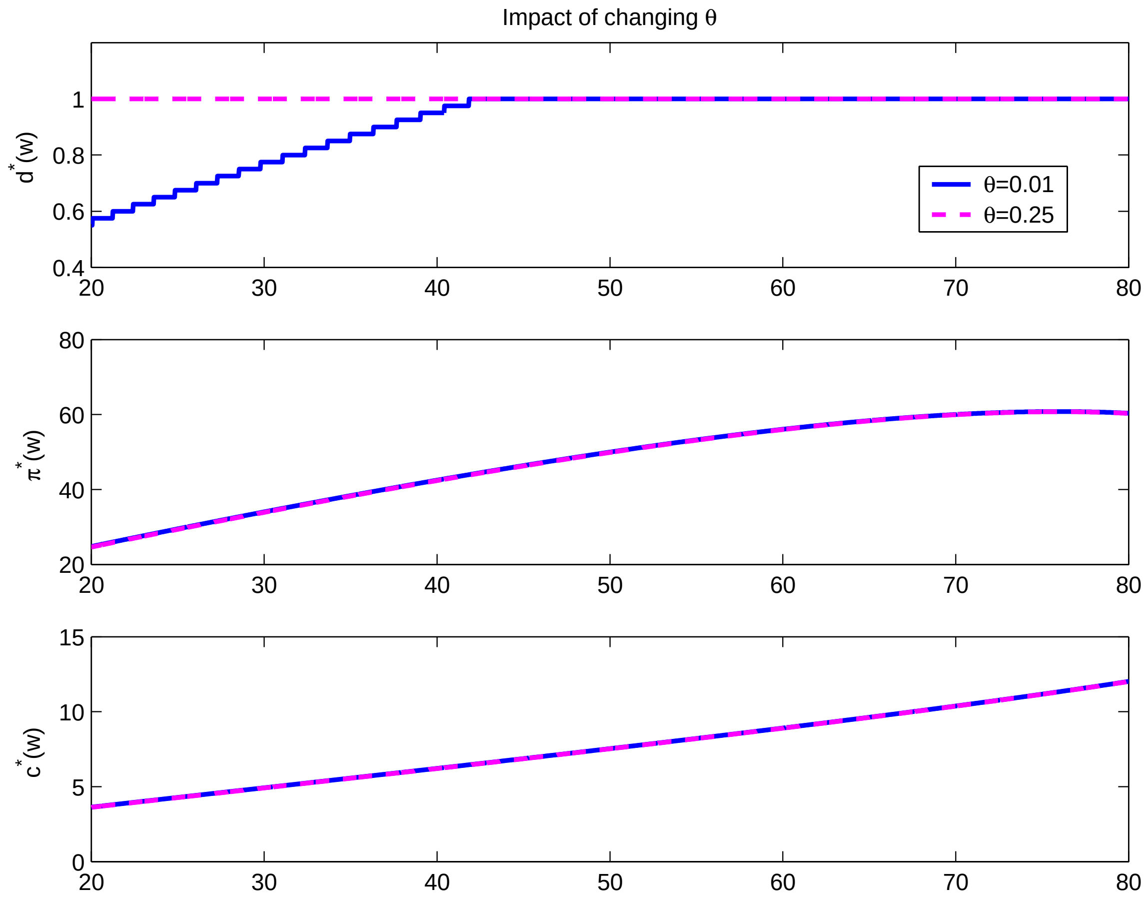

Figure 6: *Impact of changing the cost of insurance*

Impact of changing the frequency of loss

We contrast the optimal strategies as we change the loss frequency from $\phi = 0.1$ to $\phi = 1.0$. Figure 7 shows that when the loss frequency is higher, at the lower wealth levels, the insured should opt for a lower deductible. However, at the higher wealth levels, zero insurance is optimal for both choices of ϕ. We see also that the optimal investment in the risky asset and the optimal consumption are lower when the loss frequency is higher. This interaction between the insurable loss and the risky asset is consistent with our financial intuition and the result of Example 5.1b. Note that increasing the loss frequency had the opposite effect on the consumption strategy when $L = \beta w$; see Figure 3 and Table 2 from Examples 5.1a and 5.1b.

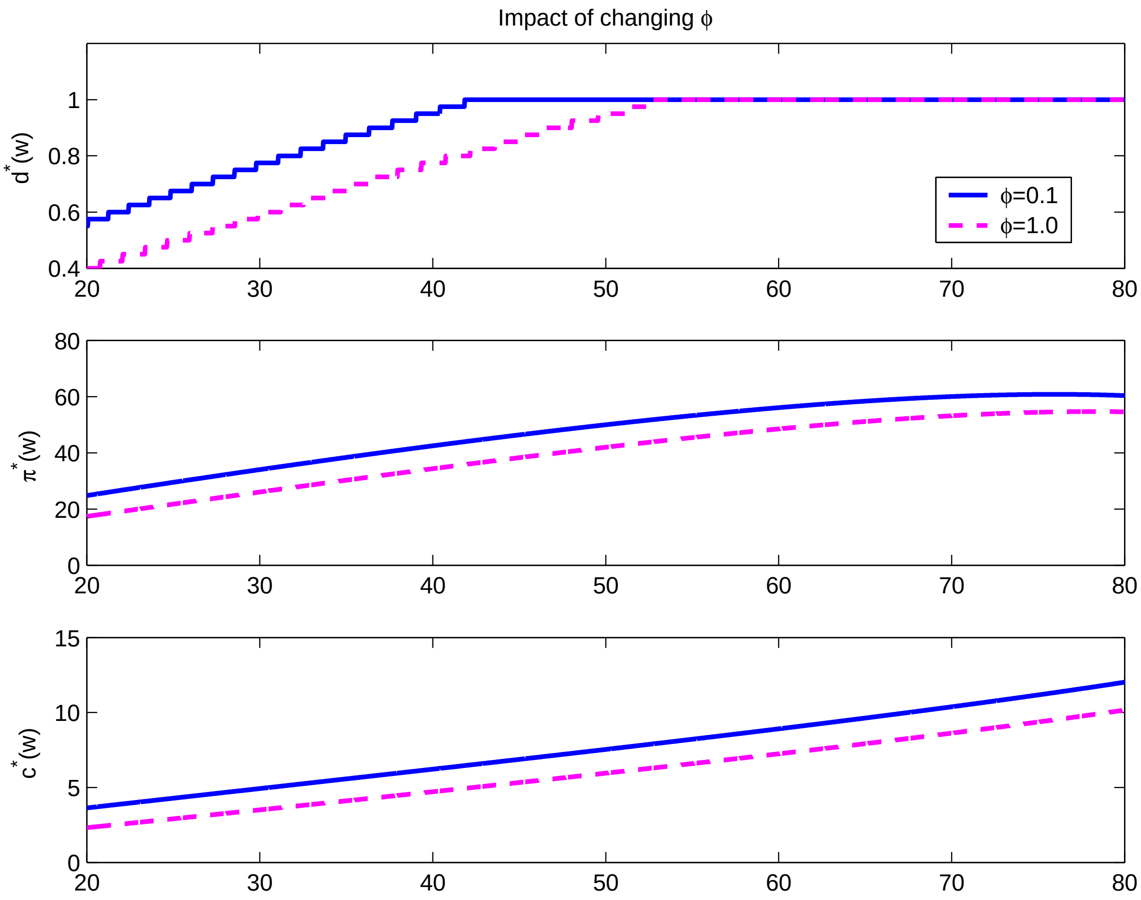

Figure 7: *Impact of changing the loss frequency*

Impact of changing the emphasis on consumption versus bequest

In Figure 8, we see that if we increase the individual's propensity to save b from 1 to 4 (so that the individual values bequest over consumption), the optimal consumption level decreases. Similarly, if we decrease b from 1 to 0.25, the optimal consumption level increases. This is consistent with our intuition and with the result from Example 5.1a; see Table 2. Recall that in Examples 5.1a and 5.1b, the optimal insurance and investment strategies were unaffected by the value of b; see (5.4), (5.6), and Table 2. However, in this case, if the individual values bequest over consumption, she should choose a lower deductible and invest less in the risky asset. Similarly, if she values consumption over bequest, she should choose a higher deductible and invest more in the risky asset.

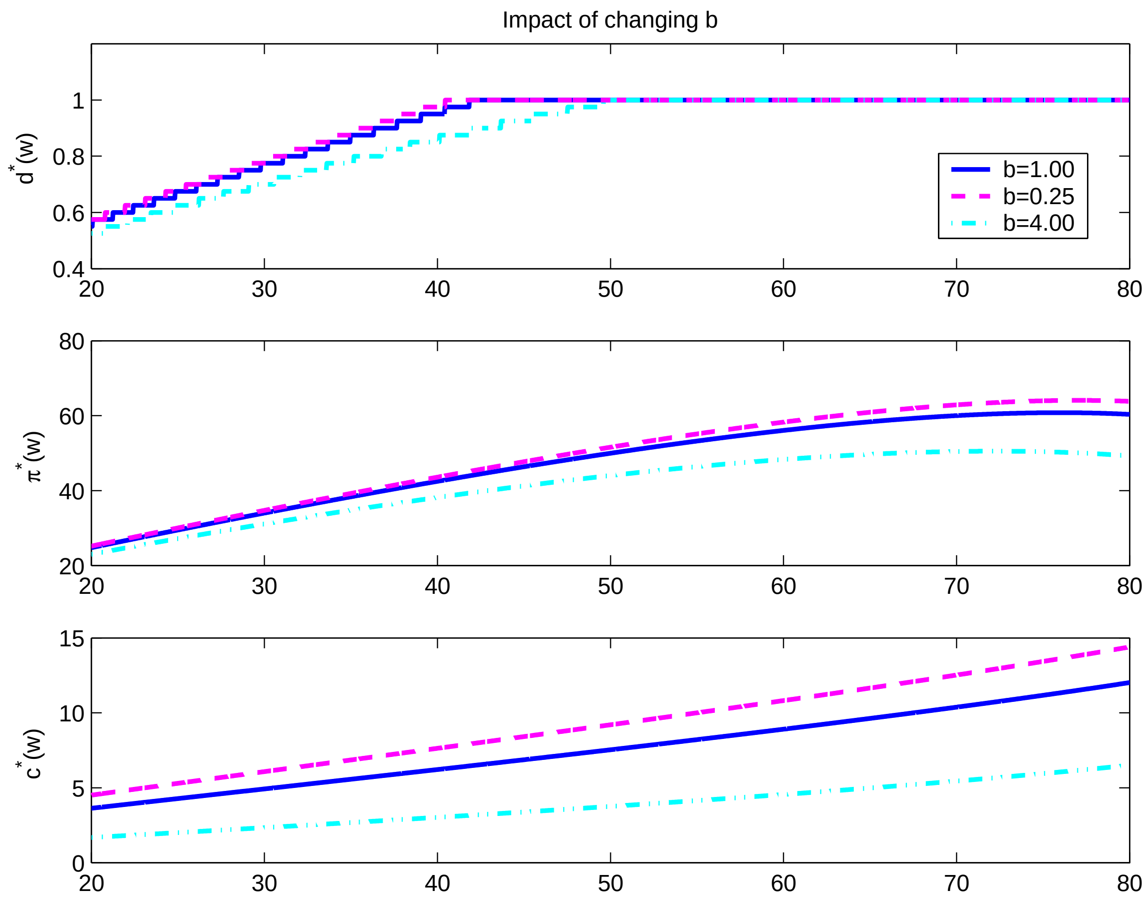

Figure 8*: Impact of changing the propensity to save*

Impact of changing the relative risk aversion

We examine the impact of changing the utility parameter γ from 0.5 to 0.1; these values of γ correspond to relative risk aversion of $1-\gamma = 0.5$ and $1-\gamma = 0.9$, respectively. Figure 9 shows that when the relative risk aversion is higher ($\gamma = 0.1$), the individual should choose a lower deductible, invest less in the risky asset, and consume less. This is consistent with our intuition and the result from Examples 5.1a and 5.1b; see Table 2.

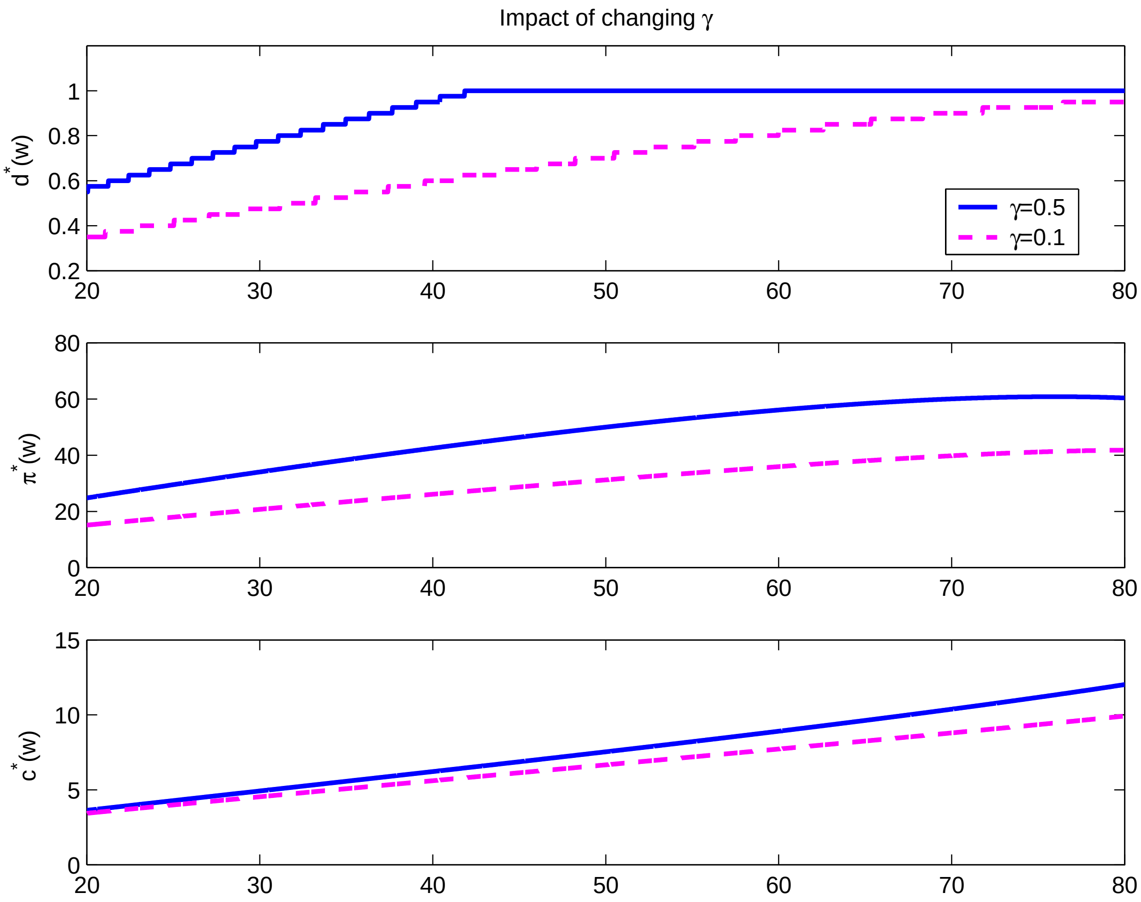

Figure 9*: Impact of changing the relative risk aversion*

Impact of changing the force of mortality and subjective discount rate

Figure 10 shows that as we increase the force of mortality λ from 0.05 to 0.15, at lower wealth levels, the optimal deductible decreases; thus, an investor with a shorter horizon purchases more insurance. At higher wealth levels, self-insurance is optimal. In the third graph in Figure 10, we see that an investor with a shorter horizon consumes more. These results are consistent with the results of Example 5.1b and Gollier (2002). However, the impact of the horizon on the optimal investment strategy is more complicated. At lower wealth levels, an investor with shorter horizon assumes less financial risk; the reverse is true at higher wealth levels. The impact of varying the subjective discount rate is qualitatively similar.

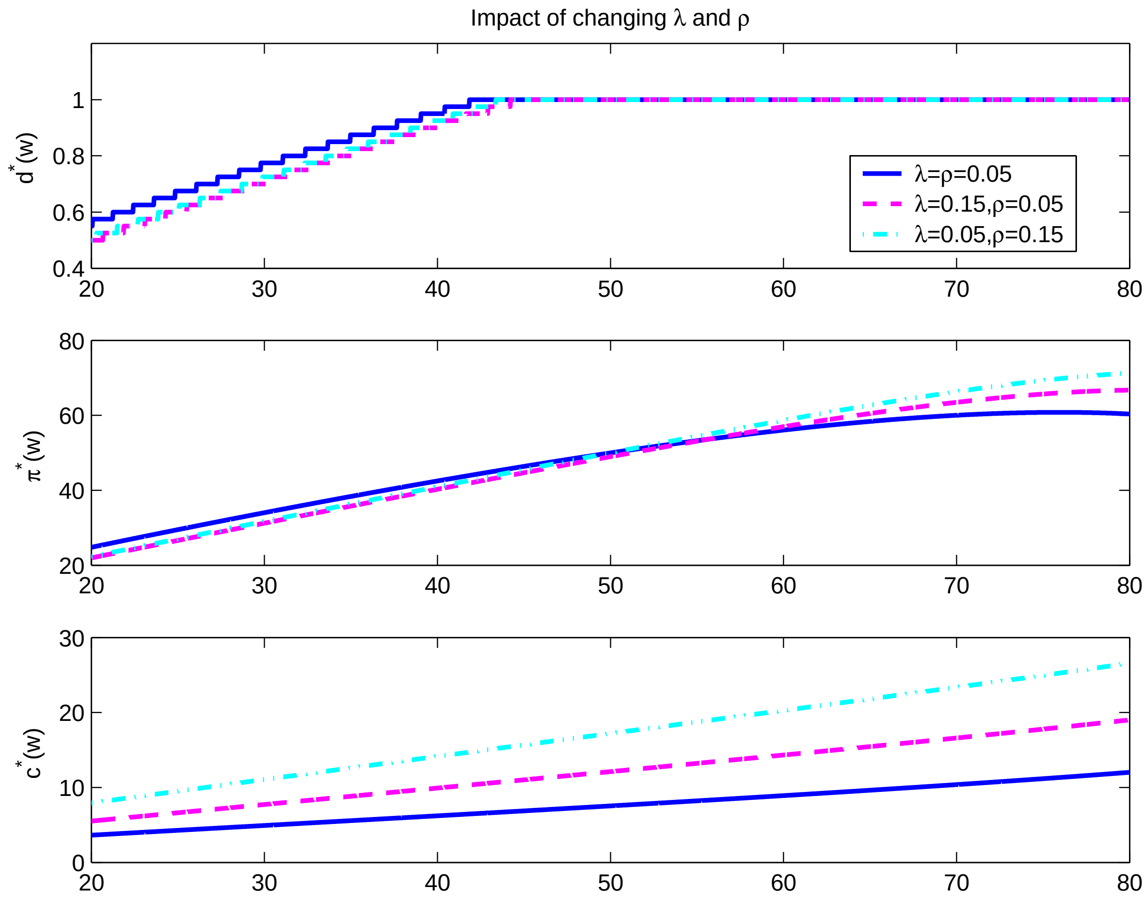

***Figure 0**: Impact of changing the force of mortality and subjective discount rate*

6. Conclusions

We showed that in our dynamic setting, for “per-claim” indemnity coverage, optimal insurance is deductible insurance. This is consistent with the static results of Mossin (1968), van Heerwaarden (1991, Theorem 9.4.1), and Gollier and Schlesinger (1995). We considered a fixed horizon and a horizon that terminates at the time of death of the insured.

We analyzed many examples, both analytically and numerically. In each example, we examined the impact of the investor's horizon (or mortality) on the optimal strategies and the interaction between investment, consumption, and the insurable loss. Some of our results are summarized in the table in Appendix 2.

In many examples, the optimal investment and insurance strategies are independent of each other and the demand for the risky asset is independent of the parameters of the loss process. However, in the presence of an exogenous wage, the frequency of the loss affects the individual's demand for the risky asset. Moreover, when premiums are paid as a lump-sum, under exponential utility, the parameters of the stock price affect the demand for insurance.

In most of the examples, the optimal consumption strategy depends on the investor's horizon. However, in many of the examples, the optimal insurance and investment strategies are independent of the horizon except via the effect on the optimally-controlled wealth. By contrast, in the presence of an exogenous wage, the impact of the horizon on the optimal controls is pronounced; an individual with a longer horizon assumes more risk by purchasing less insurance and investing more in the risky asset, but consumes less. This strategy is consistent with "time diversification" argument of Gollier (2002).

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Appendix 1: Markov Chain Approximation Method (MCAM)

We describe the intuition behind the MCAM in the context of Example 5.1a. First, discretize the state (wealth) space into a finite set of values $\{\xi_0, \xi_1, \xi_2, ..., \xi_N\}$ and assume that in a small time interval Δt, wealth changes from ξ_i to ξ_j with transition probability p_{ij}. We employ the Dynamic Programming Principle to write the equation for the approximate value function recursively as

$$V(\xi_i) \approx \sup_{\{c_i,\pi_i,d_i\}} \left\{ \Delta t\, \lambda\, u_2(\xi_i) + \Delta t u_1(c_i) + e^{-(\lambda+\rho)\Delta t} \left(\phi\, \Delta t V(\xi_i - d) + (1-\phi\,\Delta t) E\left[V(\xi) \,|\, W_t = \xi_i \right] \right) \right\},$$

in which $E[V(\xi)\,|\,W_t = \xi_i] = \sum_{j=o}^{N} p_{ij} V(\xi_j)$. We refer the reader to sections 2.1.1, 2.1.2, and 5.6.2 of Kushner and Dupuis (2001) for the derivation, but the intuition is as follows. Over the time interval $[0,\Delta t]$, the approximate utility from consumption is $\Delta t u_1(c_i)$. If death occurs in the small time interval $[0,\Delta t]$, which happens with probability $1 - e^{-\lambda\Delta t} \approx \lambda\,\Delta t,$ the individual has approximate utility of terminal wealth $u_2(\xi_i)$. If death does not occur in $[0,\Delta t]$, which happens with probability $e^{-\lambda\,\Delta t}$, we examine the outcome on the interval $[\Delta t, \tau]$ and discount it back to time zero (hence the factor of $e^{-\rho\,\Delta t}$). If a Poisson loss occurred in $[0,\Delta t]$, which happens with probability $\phi\Delta t$, the individual pays the deductible d and continues maximizing her expected utility. If a Poisson loss did not occur on $[0, \Delta t]$, which happens with probability $1 - \phi\Delta t$, again, the individual continues maximizing her expected utility.

We wish to compute the approximate value function $V_i = V(\xi_i)$ and to recover the optimal controls $c^*_i = c^*(\xi_i)$, $\pi^*_i = \pi^*(\xi_i)$, and $d^*_i = d^*(\xi_i)$ for $i = 0, 1, \ldots, N$. We sketch the algorithm below.

1. Fix the controls c_i, π_i and d_i.
2. Compute the value function for these fixed controls; i.e., compute

$$J_i := \left\{ \Delta t\; \lambda\, u_2(\xi_i) + \Delta t u_1(c_i) + e^{-(\lambda+\rho)\,\Delta t} \left(\phi\;\Delta t\; J(\xi_i - d_i) + (1-\phi\;\Delta t) \sum_j p_{ij} J_j \right) \right\}. \tag{A.1}$$

The transition probabilities p_{ij} depend on the model parameters μ, r, σ, ω, θ, and ϕ, the mesh size dw, and the strategies c_i, π_i and d_i; we briefly describe the choice of p_{ij} below. We remark that this computation amounts simply to solving the $(N+1)\times(N+1)$ linear system above for the unknown vector $\boldsymbol{J}$.

3. Given ***J***, compute new policies π_i, d_i that maximize ***J***; i.e., compute the argument of the maximum of ***J***. The expression that results for the updated π_i is simply a discretized version of the first order necessary condition $\pi^* = -\frac{(\mu - r)}{\sigma^2}\frac{V'}{V''}$. Note that since the wealth space is discretized with mesh *dw*, each d_i must be an integral multiple of *dw*.
4. Repeat, starting at step 2 with the updated controls, until the change in the vector ***J*** is sufficiently small. (We used a tolerance of 10^{-6}.)

We remark that if we choose the transition probabilities p_{ij} so that the approximating chain is *locally consistent* with the continuous wealth process (i.e., so that the first two moments are close; see Section 4.1 of Kushner and Dupuis (2001) for details), then the value function and optimal controls for the discretized problem converge to the value function and optimal controls for the continuous problem. Moreover, under our choice of transition probabilities, solving the linear system (A.1) is equivalent to solving the HJB equation under fixed controls, i.e., to solving

$$\rho J = \left[\left\{ (\mu - r)\pi + \omega + rw - c - (1+\theta)\phi(\beta w - d)_+ \right\} J' + \frac{1}{2}\sigma^2\pi^2 J'' + \phi\left\{ J(w-d) - J(w) \right\} \right] \\ + \lambda\left[u_2(w) - J \right] + u_1(c)$$

via a modified finite difference scheme.

Finally, we remark that it is tempting to apply a straightforward finite difference method directly to the HJB equation, but as Kushner (1995) points out, for many problems in stochastic optimal control, the associated HJB equations have only a formal meaning and standard methods of numerical analysis are not usable to prove convergence; moreover, the classical methods might not converge. However, as Kushner and Dupuis (Chapter 5, 2001) and Hanson (Section IVB, 1996) point out, when a carefully-chosen, “renormalized” finite difference is used, the coefficients of the resulting discrete equation serve as the transition probabilities that satisfy the local consistency conditions in the discretized dynamic programming equation (A.1). Indeed, this is the case for our example.

Appendix 2: Summary of Examples

Ex.	$u_1(c)$	$u_2(w)$	Other	Loss process affects		Horizon (or mortality) affects		
				π^*	c^*	d^*	π^*	c^*
3.4	0	Exp.	$r \neq 0$ $L(w,t)=L(t)$ $\omega \neq 0$	No	N/A	Yes. Decreases with T.	Yes. Decreases with T.	N/A
3.5	Power and Log	Power and Log	$r \neq 0$ $L(w,t)=\beta(t)w$ $\omega=0$	No	Power: yes Log: no	Implicitly	Implicitly	Yes
4.1	0	Exp.	$r=0$ $L(w,t)=L(t)$ $\omega \neq 0$	No	N/A	No	No	N/A
4.2	0	Exp.	$r=0$ $L(w,t)=L$ $\omega \neq 0$ Lump sum prem.	No	N/A	Yes. Impact is ambiguous.	No	N/A
4.3	Power and Log	Power and Log	$r \neq 0$ $L(w,t)=\beta(t)w$ $\omega=0$	No	Power: yes Log: no	Implicitly	Implicitly	Yes
5.1a	Power	Power	$r \neq 0$ $L(w,t)=\beta(t)w$ $\omega=0$ All parameters constant	No	Yes. c^* increases with β and ϕ	Implicitly	Implicitly	Yes. c^* increases with λ
5.1b	Power	Power	Same as 5.1a, but $\omega \neq 0$	Yes. π^* decreases with ϕ	Yes. c^* increases with β and ϕ	Yes. d^* decreases with λ.	Yes. π^* decreases with λ.	Yes. c^* increases with λ.
5.2	Power	Power	$r \neq 0$ $L(w,t)=1$ $\omega \neq 0$	Yes. π^* decreases with ϕ.	Yes. c^* decreases with ϕ.	Yes. High wealth: no ins. Low wealth: d^* decr. with λ.	Yes. High wealth: π^* incr. with λ. Low wealth: π^* decr. with λ.	Yes. c^* increases with λ.