Optimal Design of a Perpetual Equity-Indexed Annuity

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Abstract

We find the participation rate, guaranteed death benefit, guaranteed surrender benefit, and initial and maintenance fees that most appeal to a buyer of a perpetual equity-indexed annuity (EIA) from the standpoint of maximizing the buyer's expected discounted utility of wealth at death, also called *bequest,* while still allowing the issuer of the EIA to (at least) break even on the basis of the expected discounted value of the issuer's payout. In calculating the buyer's expected utility, we use the physical probability faced by the buyer. However, in calculating the expected value of the issuer's payout, we use a type of risk-neutral probability by assuming that the issuer sells many independent policies. We demonstrate our method with an illustrative numerical example.

1. Introduction

Design of equity-indexed annuities (EIAs) is an important problem. Sales of EIAs have increased dramatically in recent years: sales have climbed to over $6.4 billion in 2001 since their introduction in 1995 (The Advantage Group, 31 December 2001). We focus on five aspects of the design of an EIA: (1) The proportion of the return on a risky asset (for example, a stock index) that is credited to the EIA account, also called the *participation rate;* (2) the proportion of the initial deposit in the EIA returned in the event of death, also called the *guaranteed death benefit;* (3) the proportion of the initial deposit that serves as a minimum guarantee of the value of the fund, also called the *(minimum) guaranteed surrender benefit;* (4) the initial fee that is charged for handling the account and covering the guarantees; and (5) the maintenance fee that is withdrawn continually for handling the account and covering the guarantees.

For the sake of analytic tractability, we assume that the annuity is perpetual in the sense that the individual can surrender the annuity at any time or can continue to hold the annuity and receive the death benefit when she dies. If she surrenders the EIA, then she obtains the larger of a guaranteed surrender benefit or the value in the equity-indexed account. In Moore and Young (2004) we consider a more typical EIA with a finite horizon, but in that case we are forced to rely entirely on numerical approximations. The actuarial literature contains precedence for perpetual products; see, for example, Gerber and Shiu (2003) and the references therein, as well as the comprehensive text on equity-indexed insurance by Hardy (2003). In this paper we find the participation rate, guaranteed death benefit, guaranteed surrender benefit, and initial and maintenance fees that most appeal to buyers of perpetual EIAs from the standpoint of maximizing their expected discounted utility of wealth at death, also called *bequest,* while still allowing the issuer of the EIA to at least break even on the basis of the expected discounted value of the issuer's payout. We consider only *power* utility, that is, utility for which the coefficient of relative risk aversion is constant (Pratt 1964).

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For literature related to EIAs and pricing options inherent in such products, refer to the work of Tiong (2000), Gerber and Pafumi (2000), Imai and Boyle (2001), Gerber and Shiu (2003), Fung and Li (2003), and Lin and Tan (2003) in this journal. These authors apply techniques from martingale theory to price various features of EIAs. We, on the other hand, are not pricing EIAs but are determining when (if ever) an individual surrenders a given EIA. On the basis of that knowledge, we show how the issuer of the EIA can design an EIA to appeal most to buyers of annuities, while ensuring that the issuer breaks even.

We essentially solve our problem by working backwards. In Section 2.1 we suppose that the individual has surrendered her EIA and thereafter invests in risky and riskless assets to maximize her expected discounted utility of bequest. Next, in Section 2.2 we back up a step and suppose the individual holds the EIA and decides when to surrender it optimally, again to maximize her expected discounted utility of bequest. In calculating the expected utility of the buyer, we use the physical probability faced by the buyer. We find the value functions that correspond to these two maximization problems in Sections 2.1 and 2.2, respectively.

In Section 3 we find the expected discounted payout of the issuer of the EIA by assuming that the holder of the EIA will surrender it optimally. In calculating the expected value of the issuer's payout, we use a type of risk-neutral probability by assuming that the issuer sells many independent policies. This expected discounted payout is a function of the participation rate, guaranteed death benefit, guaranteed surrender benefit, and initial and maintenance fees. Then we define the optimal participation rate, optimal death benefit, optimal surrender benefit, and optimal initial and maintenance fees as the numerical value of these parameters that maximize the value function of the individual at the time she buys the EIA, among those that allow the issuer to break even on the basis of its expected discounted payout, in which we use a risk-neutral probability to calculate the expectation.

In Section 4 we present numerical examples to illustrate our results. Section 5 concludes the paper.

2. FINANCIAL MODEL

A person aged x, denoted by (x), purchases a perpetual equity-indexed annuity at time 0. We assume that the value of the EIA is based on the price of a risky asset (for example, a stock index) whose price process S can be modeled as a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{2.1}$$

The process B is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and the coefficients μ and σ are given positive constants. Note that we assume the risky asset pays no dividends, or equivalently, that its price is constructed with all dividends reinvested.

More specifically, the issuer credits a proportion $\bar{p}$ of the risky return to the EIA account; this proportion is also called the participation rate. Also, the issuer charges a continuous maintenance fee rate g as a proportion of the wealth in the account. Therefore, the wealth in the EIA account W_t follows the process

$$\begin{cases} dW_t = (\bar{p}\mu - g)W_t dt + \bar{p}\sigma W_t dB_t, \\ W_0 = w > 0. \end{cases} \tag{2.2}$$

One can write $W_t = w \exp[(\bar{p}\mu - g - \bar{p}^2\sigma^2/2)t + \bar{p}\sigma B_t]$. Note that, excepting the maintenance fee, one can think of the EIA account as investing a proportion $\bar{p}$ of wealth in the risky asset and $1 - \bar{p}$ in a riskless asset with 0% return, with continual rebalancing of the portfolio to maintain this fixed ratio. Later in this paper we compare the proportion $\bar{p}$ with the optimal proportion that the individual invests in the risky asset in the absence of the EIA (or after surrendering the EIA). Note that our participation rate is not the usual participation rate found in the EIA market (Tiong 2000), but we use it for mathematical tractability.

We assume that the individual acts to maximize her expected discounted utility of wealth at her random time of death τ_d; the wealth at time of death is also called the bequest. We calculate expected utility of the buyer via the physical measure. We assume that the time of death is independent of the financial market. At any given time she has to decide whether to continue holding the EIA or to surrender it. If the individual decides to surrender the policy, we assume that she dynamically invests the proceeds in a riskless bond and in the risky stock index to maximize her expected discounted utility of bequest. Let τ_s denote the random time of surrendering the EIA. This time is random because we anticipate that it will depend upon the random value of the wealth in the EIA account.

2.1 Value Function after Surrendering the EIA

Before we define the value function of our individual *before* she surrenders the EIA (if at all), we first analyze the problem of what she does *after* surrendering the EIA. If the individual dies before she surrenders, then we do not need to consider the optimal policy after surrendering. Otherwise, given that the individual surrenders the EIA at time $\tau_s = t < \tau_d$ with wealth $W_t = w$, she is faced with a Merton problem of investment over a random horizon (Merton 1992). We assume that the individual invests in either the risky asset as described in equation (2.1) or a riskless asset that earns money at a constant rate (force of interest) r lying between 0 and μ. That is, at each point in time $s > t = \tau_s$, she chooses the amount π_s to invest in the risky asset. It follows that she will invest the remainder $\pi_s^0 = W_s - \pi_s$ in the riskless asset at time s. We deduce that the wealth of the individual W_s after surrendering the EIA follows the process

$$\begin{cases} dW_s &= \pi_s^0(rds) + \pi_s\left(\dfrac{dS_s}{S_s}\right) \\ &= (W_s - \pi_s)rds + \pi_s(\mu ds + \sigma dB_s) \\ &= (rW_s + (\mu - r)\pi_s)ds + \sigma\pi_s dB_s, \\ W_t &= w, \end{cases} \tag{2.3}$$

in which $\tau_s = t < s < \tau_d$.

Denote the value function of the Merton investment problem by V. Specifically V is given by

$$V(w, t) = \sup_{\{\pi_s\}\in\mathcal{A}} \mathrm{E}[e^{-\rho(\tau_d - t)}u(W_{\tau_d})|W_t = w]. \tag{2.4}$$

The set $\mathcal{A}$ is the set of admissible policies $\{\pi_s\}$ that are $\mathcal{F}_s$-progressively measurable, in which $\mathcal{F}_s$ is the augmentation of $\sigma(W_u : t \le u \le s)$, and that satisfy the integrability condition $\mathrm{E} \int_t^{\tau_d} \pi_s^2\, ds < \infty$. We assume that the utility function $u : \mathcal{R} \to \mathcal{R}$ is increasing, concave, and smooth. The parameter ρ is the individual's subjective discount rate, in which a large value of ρ corresponds to a more impatient individual and vice versa.

Throughout this paper we assume that preferences display constant relative risk aversion (CRRA); that is, $-wu''(w)/u'(w)$ is a constant (Pratt 1964). It follows that, up to an affine transformation, u is given by a power function: $u(w) = w^{1-\gamma}/(1 - \gamma)$, in which $\gamma > 0$, $\gamma \neq 1$, is the constant relative risk aversion. The limiting case for which $\gamma = 1$ corresponds to logarithmic utility, and that case can be handled easily. For simplicity of exposition, we restrict $\gamma \neq 1$.

We also assume that the force of mortality is a constant λ. Because the parameters of the model are constant, and because the EIA does not have a fixed maturity date, the value function V is independent of time (that is, $V(w, t) = V(w)$), and V solves the Hamilton-Jacobi-Bellman (HJB) equation (Björk 1998) on (τ_s, τ_d):

$$\begin{cases} rwV' + \max\limits_{\pi}[(\mu - r)\pi V' + \frac{1}{2}\sigma^2\pi^2 V''] + \lambda[u(w) - V] = \rho V, \\ \lim\limits_{s\to\infty} e^{-(\rho+\lambda)(s-t)}\mathrm{E}[V(W_s^*)|W_t = w] = 0, \end{cases} \tag{2.5}$$

in which W_s^* is the optimally controlled wealth process. The HJB equation is a result of combining dynamic programming and stochastic calculus. If one knows a priori that the value function is smooth, then the value function equals the unique smooth solution of the HJB equation. If it is not smooth, then one can work with viscosity solutions; see, for example, Young and Zariphopoulou (2000) and the references therein. For the case of CRRA preferences, the value function V in equation (2.4) is smooth.

Because the value function is smooth, the optimal investment policy can be specified via the first-order necessary condition from equation (2.5). The concavity of the utility function u and the linearity of the wealth equation (2.3) with respect to the wealth and portfolio processes imply that the value function itself inherits this property of concavity. Therefore, the maximum in equation (2.5) is well defined and achieved at

$$\pi^*(w) = -\frac{\mu - r}{\sigma^2}\frac{V'(w)}{V''(w)}. \tag{2.6}$$

Generally one assumes that $\mu > r$ so that $\pi^* > 0$; otherwise, if $r > \mu$, then $\pi^* < 0$, and we have short-selling of the stock.

It follows that the optimal investment process in the stock account is (with a slight abuse of notation)

$$\pi_s^* = \pi^*(W_s^*) = -\frac{\mu - r}{\sigma^2}\frac{V'(W_s^*)}{V''(W_s^*)}, \qquad t \le s \le \tau_d, \tag{2.7}$$

in which V solves equation (2.5) and W_s^* is the optimally controlled wealth process solving equation (2.3) with π_s^* used for π_s. These classical optimality results are known as the Verification Theorem (Fleming and Soner 1993, ch. 6).

For CRRA preferences one can show that the value function is given by

$$V(w) = A\,\frac{w^{1-\gamma}}{1-\gamma}, \tag{2.8}$$

in which

$$A = \frac{\lambda}{\rho + \lambda - (r + m/\gamma)(1-\gamma)}, \tag{2.9}$$

with $m = \frac{1}{2}((\mu - r)/\sigma)^2$. We assume that A is positive, specifically, that the denominator of A is positive. It turns out that this condition is equivalent to the terminal condition in equation (2.5), namely, $\lim_{s\to\infty} \mathrm{E}[e^{-(\rho+\lambda)(s-t)}V(W_s^*)|W_t = w] = 0$. See the Appendix for a proof of this statement. If $\gamma > 1$, which is generally believed to hold as a result of researchers fitting of CRRA preferences empirically, then A is automatically positive.

The optimal amount invested in the risky asset is

$$\pi_s^* = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}W_s^*, \tag{2.10}$$

a constant multiple of wealth. This phenomenon occurs because preferences exhibit constant relative risk aversion; equation (2.10) does not hold for other utility functions. In the next section we determine the value function of the holder of the EIA.

2.2 Value Function before Surrendering the EIA

As we mentioned in Section 2.1, while the individual still holds the EIA, at any given time $t < \tau_d$, she has to decide whether to continue holding the EIA or to surrender it. If the individual decides to surrender the policy, then she invests the proceeds optimally in risky and riskless assets as prescribed in the previous section. On the face of it, if the participation rate $\bar{p}$ in the risky stock is not optimal in the sense of equation (2.10), then the individual has no incentive to continue with the EIA, especially if she stands to lose upon her death. Therefore, to provide incentives for the individual to continue, we assume that there is a guaranteed surrender value equal to a multiple $\bar{s}$ of the initial deposit w_0 in the EIA, something that the risky asset alone cannot provide. Here the initial deposit is *net* of the initial fee rate of f: that is, if the individual spends wealth w_1 to purchase the EIA, then $w_0 = (1 - f)w_1$ is credited to the EIA account.

We also assume that if she dies before she surrenders the policy, then her estate receives the maximum of a proportion $\bar{d}$ of her initial deposit w_0 or the value of the account at time of death. We allow $\bar{s}$ and $\bar{d}$ to be any positive real numbers, while we restrict $\bar{p}$ and f to lie between 0 and 1; however, in practice, we expect that $\bar{s}$ will lie between 0 and 1 and that $\bar{s} \le \bar{d}$.

Thus, if the individual surrenders the EIA, then she receives the larger of the guaranteed amount $\bar{s}w_0$ and the value of the account at that time. Afterwards, she invests this surrender value and derives utility V from that action. However, if she dies before she surrenders, then she receives the death benefit $\max\{\bar{d}w_0, W_{\tau_d}\}$ and derives utility u from that death benefit. Recall that τ_s denotes the time of surrendering the EIA; it is a random variable because it will depend on the random value of the account. It follows that the value function of the individual before surrendering the EIA is given by

$$U(w, t) = \sup_{\tau_s} \mathrm{E}[e^{-\rho(\tau_s - t)}V(\max\{\bar{s}w_0, W_{\tau_s}\})1_{\{\tau_s < \tau_d\}} + e^{-\rho(\tau_d - t)}u(\max\{\bar{d}w_0, W_{\tau_d}\})1_{\{\tau_d \le \tau_s\}}|W_t = w], \qquad (2.11)$$

in which w is the amount in the account at time t, $1_{\{\tau_s < \tau_d\}}$ is the indicator function for the event that the individual surrenders the EIA before she dies, and $1_{\{\tau_d \le \tau_s\}}$ is defined similarly. In other words, the *only* decision that the individual has to make is when to surrender the EIA because V embodies the optimal investment in the financial market after surrender.

As it was for V, the value function U is independent of time (that is, $U(w, t) = U(w)$). By following the arguments of Krylov (1980) or Øksendal (1998), one can show that U solves the variational inequality:

$$\rho U \ge (\bar{p}\mu - g)wU' + \tfrac{1}{2}\bar{p}^2\sigma^2 w^2 U'' + \lambda[u(\max\{\bar{d}w_0, w\}) - U], \qquad (2.12)$$

and

$$U(w) \ge V(\max\{\bar{s}w_0, w\}), \qquad (2.13)$$

with equality holding in at least one of equations (2.12) and (2.13). We omit the details of this derivation, but interested readers can see Young and Zariphopoulou (2002), Moore and Young (2003), and Young (2003) for derivations of similar equations. See Wilmott, Dewynne, and Howison (1993) for related work concerning American options.

The optimal time to surrender one's EIA is the first time for which there is equality in equation (2.13). If it is optimal to continue to hold the EIA, then there is strict inequality in equation (2.13) and equality in the HJB equation (2.12). Note that because of the initial fee rate of f, one needs to compare $U(w_0)$ with $V(w_1)$ to see if the individual will buy the EIA in the first place.

For CRRA preferences, in the region of "no surrender," U solves

$$(\rho + \lambda)U = (\bar{p}\mu - g)wU' + \frac{1}{2}\bar{p}^2\sigma^2 w^2 U'' + \lambda \frac{(\max\{\bar{d}w_0, w\})^{1-\gamma}}{1-\gamma}, \qquad (2.14)$$

from which it follows that U is given by

$$U(w) = \begin{cases} C_1 w^{\alpha_1} + C_2 w^{\alpha_2} + \dfrac{\lambda}{\rho + \lambda}\dfrac{(\bar{d}w_0)^{1-\gamma}}{1-\gamma}, & 0 < w < \bar{d}w_0, \\ \tilde{C}_1 w^{\alpha_1} + \tilde{C}_2 w^{\alpha_2} + \tilde{A}\dfrac{w^{1-\gamma}}{1-\gamma}, & w \ge \bar{d}w_0, \end{cases} \qquad (2.15)$$

in which a_1 and a_2 are the positive and negative roots, respectively, of

$$\frac{1}{2}\bar{p}^2\sigma^2a^2 + (\bar{p}\mu - g - \frac{1}{2}\bar{p}^2\sigma^2)a - (\rho + \lambda) = 0, \tag{2.16}$$

and $\tilde{A}$ is given by

$$\tilde{A} = \frac{\lambda}{\rho + \lambda - (\bar{p}\mu - g)(1 - \gamma) + \bar{p}^2\sigma^2\gamma(1 - \gamma)/2}. \tag{2.17}$$

C_1, C_2, $\tilde{C}_1$, and $\tilde{C}_2$ are constants to be determined by the free boundary conditions between the regions of "no surrender" and "surrender" and by continuity and smoothness at $w = \bar{d}w_0$. Note that U'' is continuous for the following reason: One can solve equation (2.14) for U'' in terms of U, U', and $\max\{\bar{d}w_0, w\}$. Because U, U', and $\max\{\bar{d}w_0, w\}$ are continuous for all $w > 0$, so is U''.

We have $U(w) = V(\max\{\bar{s}w_0, w\})$ on the boundary of the region of "no surrender." We hypothesize that the region of "no surrender" is an interval (w_l, w_u) because if the wealth in the EIA account is low enough, it will be optimal for the individual to take the guarantee $\bar{s}w_0$ and invest on her own. On the other hand, if the wealth in the EIA account is high enough, the individual will want to invest on her own, especially if r is significantly greater than 0 and if $\bar{p}$ is very different from the proportion of wealth given in equation (2.10). Note that the time homogeneity of this optimal stopping problem leads to time independence of the boundary $\{w_l, w_u\}$. This is a nice feature of perpetual American options, in general; otherwise, for a finite horizon, the boundary $\{w_l, w_u\}$ would be time dependent.

Furthermore, we hypothesize that $\bar{s}w_0$ and $\bar{d}w_0$ lie between w_l and w_u. Thus,

$$U(w_l) = V(\bar{s}w_0) \text{ and } U(w_u) = V(w_u). \tag{2.18}$$

Smooth pasting at the free boundaries implies that

$$U'(w_l) = 0 \text{ and } U'(w_u) = V'(w_u). \tag{2.19}$$

Finally, we assume that U is continuous and has continuous first derivatives at $w = \bar{d}w_0$:

$$U(\bar{d}w_0-) = U(\bar{d}w_0+) \text{ and } U'(\bar{d}w_0-) = U'(\bar{d}w_0+). \tag{2.20}$$

Thus, we have six equations for the six unknowns w_l, w_u, C_1, C_2, $\tilde{C}_1$, and $\tilde{C}_2$. If we show that our hypothesized solution solves equations (2.12)–(2.13), then it is the value function U by the Verification Theorem (Øksendal, 1998, ch. 10).

The first equation in expression (2.18) implies that

$$C_1w_l^{a_1} + C_2w_l^{a_2} + \frac{\lambda}{\rho + \lambda}\frac{(\bar{d}w_0)^{1-\gamma}}{1 - \gamma} = A\frac{(\bar{s}w_0)^{1-\gamma}}{1 - \gamma}, \tag{2.21}$$

and the first equation in expression (2.19) implies that

$$C_1a_1w_l^{a_1} + C_2a_2w_l^{a_2} = 0. \tag{2.22}$$

Solve equations (2.21) and (2.22) for C_1 and C_2 to obtain

$$C_1 = -\frac{a_2}{a_1 - a_2}\frac{w_0^{1-\gamma}}{1 - \gamma}\left(A\bar{s}^{1-\gamma} - \frac{\lambda}{\rho + \lambda}\bar{d}^{1-\gamma}\right)w_l^{-a_1}, \tag{2.23}$$

and

$$C_2 = \frac{a_1}{a_1 - a_2} \frac{w_0^{1-\gamma}}{1-\gamma} \left(A\bar{s}^{1-\gamma} - \frac{\lambda}{\rho + \lambda} \bar{d}^{1-\gamma} \right) w_l^{-a_2}. \tag{2.24}$$

Next, the second equation in expression (2.18) implies that

$$\tilde{C}_1 w_u^{a_1} + \tilde{C}_2 w_u^{a_2} + \tilde{A} \frac{w_u^{1-\gamma}}{1-\gamma} = A \frac{w_u^{1-\gamma}}{1-\gamma}, \tag{2.25}$$

while the second equation in expression (2.19) implies that

$$\tilde{C}_1 a_1 w_u^{a_1} + \tilde{C}_2 a_2 w_u^{a_2} + \tilde{A} w_u^{1-\gamma} = A w_u^{1-\gamma}. \tag{2.26}$$

Solve equations (2.25) and (2.26) for $\tilde{C}_1$ and $\tilde{C}_2$ to obtain

$$\tilde{C}_1 = -\frac{A - \tilde{A}}{a_1 - a_2} \left(\frac{a_2}{1-\gamma} - 1 \right) w_u^{1-\gamma-a_1}, \tag{2.27}$$

and

$$\tilde{C}_2 = \frac{A - \tilde{A}}{a_1 - a_2} \left(\frac{a_1}{1-\gamma} - 1 \right) w_u^{1-\gamma-a_2}. \tag{2.28}$$

Next, we can use expression (2.20) to write $\tilde{C}_1$ and $\tilde{C}_2$ in terms of C_1 and C_2 as follows:

$$\tilde{C}_1 = C_1 - \left[\frac{a_2}{1-\gamma} \left(\frac{\lambda}{\rho + \lambda} - \tilde{A} \right) + \tilde{A} \right] \frac{(\bar{d} w_0)^{1-\gamma-a_1}}{a_1 - a_2} \tag{2.29}$$

and

$$\tilde{C}_2 = C_2 + \left[\frac{a_1}{1-\gamma} \left(\frac{\lambda}{\rho + \lambda} - \tilde{A} \right) + \tilde{A} \right] \frac{(\bar{d} w_0)^{1-\gamma-a_2}}{a_1 - a_2}. \tag{2.30}$$

By substituting C_1 from equation (2.23) and $\tilde{C}_1$ from equation (2.27) into equation (2.29), we obtain

$$\left(\frac{w_u}{w_l}\right)^{a_1} = \frac{1}{a_2} \frac{1}{A\bar{s}^{1-\gamma} - \frac{\lambda}{\rho+\lambda}\bar{d}^{1-\gamma}} \times \left\{ (A - \tilde{A})(a_2 - (1-\gamma)) \left(\frac{w_u}{w_0}\right)^{1-\gamma} - \left[a_2 \left(\frac{\lambda}{\rho+\lambda} - \tilde{A} \right) + \tilde{A}(1-\gamma) \right] \bar{d}^{1-\gamma-a_1} \left(\frac{w_u}{w_0}\right)^{a_1} \right\}. \tag{2.31}$$

Similarly, by substituting C_2 from equation (2.24) and $\tilde{C}_2$ from equation (2.28) into equation (2.30), we obtain

$$\left(\frac{w_u}{w_l}\right)^{a_2} = \frac{1}{a_1} \frac{1}{A\bar{s}^{1-\gamma} - \frac{\lambda}{\rho+\lambda}\bar{d}^{1-\gamma}} \times \left\{ (A - \tilde{A})(a_1 - (1-\gamma)) \left(\frac{w_u}{w_0}\right)^{1-\gamma} - \left[a_1 \left(\frac{\lambda}{\rho+\lambda} - \tilde{A} \right) + \tilde{A}(1-\gamma) \right] \bar{d}^{1-\gamma-a_2} \left(\frac{w_u}{w_0}\right)^{a_2} \right\}. \tag{2.32}$$

Thus, we can recover the optimal exercise strategy and compute the investor's value function as follows:

- Eliminate w_u/w_l from equations (2.31) and (2.32).
- Solve for w_u/w_0; thus, we can get w_u.
- Substitute w_u into the equations for $\tilde{C}_1$ and $\tilde{C}_2$, namely, equations (2.27) and (2.28).

- Solve equation (2.31) for w_u/w_l, and calculate $w_l = w_u/(w_u/w_l)$
- Finally, substitute w_l into the equations for C_1 and C_2, namely, equations (2.23) and (2.24).

Note that to ensure that the computed strategy is valid, we must verify that $\bar{s}w_0$ and $\bar{d}w_0$ lie between w_l and w_u.

3. A Risk-Neutral Valuation of the EIA

In this section we calculate the expected discounted payout of the issuer of the EIA as a function of the participation rate $\bar{p}$ in the risky asset, guaranteed death benefit rate $\bar{d}$, guaranteed surrender benefit rate $\bar{s}$, initial fee rate f, and maintenance fee rate g. Thereby, one can find feasible values for the parameters from the standpoint of the issuer, in which we define *feasible* to mean that the issuer at least breaks even on the basis of expected value. Then we define the optimal participation rate, guaranteed death benefit, guaranteed surrender benefit, and initial and maintenance fees as those that maximize the value function of the buyer of the EIA, U from Section 2.2, over the set of feasible values of the parameters at the wealth level w_0, the amount initially deposited in the EIA account. We have no closed-form solution for the optimal participation rate, guaranteed death benefit, guaranteed surrender benefit, and initial and maintenance fees, so we calculate them for an illustrative example in Section 4.

The actuarial literature contains much discussion of actuarial versus financial valuation of cash flows; see, for example, Brennan and Schwartz (1976), Aase and Persson (1994), Schweizer (2001), and Møller (2001) and the references therein. Wendt (1999) uses the example of Bodie (1995) to illustrate that the techniques of traditional actuarial valuation are not applicable to financial insurance (such as equity-linked products) and that traditional actuarial analysis may significantly understate the cost. This understatement occurs because the financial risks are not independent; if the underlying index performs poorly, every policy is a loser. Therefore, in this section we rely on financial valuation to determine the expected discounted payout of the issuer.

We assume that the issuer values the expected discounted payout via a product probability measure, in which the first "factor" equals the physical measure associated with the random time of death and the second equals the risk-neutral financial measure. We use the physical measure because we assume that the issuer sells a large number of independent policies. If the issuer does so, then the mortality risk is diversified. Thus, the annuity market changes from an incomplete market to a complete one. The risk-neutral financial measure Q is such that the discounted price process of the risky asset is a martingale with respect to Q, in which we discount the price by r, the continuously compounded rate of return of the riskless asset (Panjer 1998). If B is a Brownian motion under the physical measure, then $\tilde{B}$ is a Brownian motion under Q, in which $\tilde{B}$ is defined by

$$\tilde{B}_t = B_t + \frac{\mu - r}{\sigma} t. \tag{3.1}$$

The expected discounted payout of the issuer of the EIA (as a function of the value w of the EIA account) is therefore given by

$$H(w) = \tilde{\mathrm{E}}[e^{-r\tau}P(W_\tau, \tau)|W_0 = w], \tag{3.2}$$

in which P is the payout at the time of surrender ($\tau = \tau_s$) or at the time of death of the buyer ($\tau = \tau_d$), and $\tilde{\mathrm{E}}$ is expectation with respect to the risk-neutral product measure. Specifically the payout P is given by

$$P(W_\tau, \tau) = \begin{cases} \bar{s}w_0, & \tau = \tau_s,\ W_\tau = w_l \\ w_u, & \tau = \tau_s,\ W_\tau = w_u \\ \bar{d}w_0, & \tau = \tau_d,\ W_\tau < \bar{d}w_0 \\ W_\tau, & \tau = \tau_d,\ W_\tau \geq \bar{d}w_0. \end{cases} \tag{3.3}$$

By applying equation (3.1) to equation (2.2), we learn that under the risk-neutral measure, wealth in the EIA account follows the process

$$\begin{cases} dW_t = (\bar{p}\mu - g)W_t dt + \bar{p}\sigma W_t \left(d\tilde{B}_t - \frac{\mu - r}{\sigma} dt \right) \\ \quad\;\; = (\bar{p}r - g)W_t dt + \bar{p}\sigma W_t d\tilde{B}_t, \\ W_0 = w. \end{cases} \tag{3.4}$$

It follows that H solves the following ordinary differential equation for w between w_l and w_u:

$$\begin{cases} (\bar{p}r - g)wH' + \frac{1}{2}\bar{p}^2\sigma^2 w^2 H'' + \lambda(\max\{\bar{d}w_0, w\} - H) = rH, \\ H(w_l) = \bar{s}w_0, \qquad H(w_u) = w_u. \end{cases} \tag{3.5}$$

It is easy to show that H is given by

$$H(w) = \begin{cases} D_1 w^{b_1} + D_2 w^{b_2} + \frac{\lambda}{r + \lambda} \bar{d}w_0, & 0 < w < \bar{d}w_0 \\ \tilde{D}_1 w^{b_1} + \tilde{D}_2 w^{b_2} + \frac{\lambda}{(1 - \bar{p})r + g + \lambda} w, & w \geq \bar{d}w_0, \end{cases} \tag{3.6}$$

in which b_1 and b_2 are the positive and negative roots, respectively, of the quadratic equation

$$\tfrac{1}{2}\bar{p}^2\sigma^2 b^2 + (\bar{p}r - g - \tfrac{1}{2}\bar{p}^2\sigma^2)b - (\lambda + r) = 0, \tag{3.7}$$

and D_1, D_2, $\tilde{D}_1$, and $\tilde{D}_2$ are constants determined by the boundary conditions in equation (3.3) and by continuity of H and its first derivative at $w = \bar{d}w_0$. We spare the reader the algebraic details of that calculation and simply remark that because we already know w_u and w_l, we need only solve a 4×4 linear system. Thus, the calculation is easier than the corresponding one in Section 2.2. As for U, H'' is continuous because H and H' are continuous and because H solves equation (3.5).

4. NUMERICAL RESULTS

In this section we begin by calculating the free boundary (w_l, w_u) for a given set of parameter values. In Section 4.1 we examine how the free boundary changes as we change those values. Finally, in Section 4.2 we calculate optimal values of the parameters that constitute the design of the EIA. For our base case, suppose that

- The relative risk aversion $\gamma = 2$;
- The rate of return on the riskless asset $r = 0.04$;
- The drift on the risky asset $\mu = 0.08$;
- The volatility on the risky asset $\sigma = 0.20$, so that from equation (2.10), the optimal proportion of wealth invested in the risky stock after surrendering the EIA is 50%;
- The participation rate in the EIA $\bar{p} = 0.9$;
- The proportion of the initial deposit returned upon death $\bar{d} = 1.4$;
- The proportion of the initial deposit guaranteed upon surrender $\bar{s} = 0.9$;
- The force of mortality $\lambda = 0.04$, so that the expected future lifetime of $(x + t)$ is 25 years for all $t > 0$;
- The personal discount rate $\rho = 0.04$;
- The initial purchase price of the EIA $w_1 = 1.0$;
- The initial fee rate $f = 0.05$, so that $w_0 = 0.95$, and the free boundary values are relative to $\bar{s}w_0 = 0.855$ and $\bar{d}w_0 = 1.330$; and
- The maintenance fee rate $g = 0.02$.

For this set of parameters, $a_1 = 1.377$, $a_2 = -3.587$, $w_u = 1.424$, and $w_l = 0.212$. Note that $w_l \leq \bar{s}w_0 \leq \bar{d}w_0 \leq w_u$. Thus, if the wealth in the EIA account drops below 0.212 or rises above 1.424, the individual will surrender the EIA and invest optimally on her own thereafter.

The value function of the buyer of the EIA before buying is $V(w_1) = -0.3077$, while the value function of the buyer of the EIA immediately after buying is $U(w_0) = -0.2843 > -0.3077$. Thus, the buyer will actually purchase the EIA. The expected discounted payout of the issuer is $H(w_0) = 0.9785 < w_1 = 1.0$; therefore, this policy is feasible from the standpoint of the issuer.

4.1 Comparative Statics

In this section we change one parameter at a time and investigate how the free boundary (w_l, w_u) changes as a result. In each case we test whether purchasing the EIA increases the individual's utility by comparing $U(w_0)$ with $V(w_1)$. If $U(w_0) > V(w_1)$, then the individual will invest in the EIA; otherwise, she will not. We also compare the participation rate $\bar{p}$ with the optimal proportion of wealth π^*/W_t that the individual invests in the risky asset in the absence of the EIA (or after surrendering the EIA).

In Table 1 note that $U(w_0)$ dominates $V(w_1)$ for γ between 1.25 and 2.25, inclusive; thus, the individual will invest in the EIA if her relative risk aversion lies in that range. For γ less than about 1.1, $w_u < \bar{d}w_0$, so the solution in Section 2.2 does not apply in that case. We see that as the relative risk aversion γ increases, the lower free boundary w_l, below which the individual will surrender her EIA, decreases with increasing γ. Also, as the relative risk aversion increases, the upper free boundary w_u, above which the individual will surrender her EIA, increases. It appears that as the individual become more risk averse, she values the guarantees more highly, even when the wealth in her account exceeds the guarantees.

In Table 2 we see that as the rate of return on the riskless asset r increases, the region of "no surrender" narrows both from below and from above. Because the individual essentially earns zero return from the riskless asset in the EIA and 90% of the return from the risky asset, as the riskless rate of return increases, investing in the financial market becomes more attractive relative to the EIA. This phenomenon might also be driven by the divergence of π^*/W_t^* from $\bar{p} = 0.9$.

In Table 3 note that the lower boundary w_l increases with increasing μ, and the effect of μ on w_u is rather slight compared with the effect of r on w_u. We remark that in complete markets μ has *no* effect on prices and exercise times of American options.

In Table 4 we see that as the volatility on the risky asset σ increases, the region of "no surrender" widens both from above and from below. Because the guarantee is more valuable for more volatile risky assets, the individual is less likely to surrender her EIA if the volatility increases, despite the difference between π^*/W_t^* and $\bar{p} = 0.9$.

In Table 5 we see that the region of "no surrender" widens (at both ends) as the force of mortality λ increases. We observe a similar phenomenon when we change the personal discount rate ρ. Thus, as the individual becomes more likely to die, the death benefit becomes more important, and she is less likely to surrender her EIA. In another example (not shown here) with a lower death benefit, we observed the opposite behavior concerning the region of "no surrender"; in that case, the death benefit was less important.

Table 1

Free Boundary (w_l, w_u) as Function of Relative Risk Aversion γ

γ	w_l	w_u	π^*/W_t^*	$U(w_0)$	$V(w_1)$
1.25	0.473	1.368	0.80	−1.6860	−1.7021
1.50	0.392	1.398	0.67	−0.7298	−0.7500
1.75	0.307	1.415	0.57	−0.4274	−0.4498
2.00	0.212	1.424	0.50	−0.2843	−0.3077
2.25	0.091	1.428	0.44	−0.2031	−0.2268

Table 2

Free Boundary (w_l, w_u) as Function of Riskless Rate of Return r

r	w_l	w_u	π^*/W_t^*	$U(w_0)$	$V(w_1)$
0.030	0.073	1.492	0.63	−0.2884	−0.3184
0.035	0.147	1.458	0.56	−0.2865	−0.3133
0.040	0.212	1.424	0.50	−0.2843	−0.3077
0.045	0.272	1.389	0.44	−0.2817	−0.3015
0.050	0.327	1.356	0.38	−0.2787	−0.2949

Table 3

Free Boundary (w_l, w_u) as Function of Risky Drift μ

μ	w_l	w_u	π^*/W_t^*	$U(w_0)$	$V(w_1)$
0.070	0.096	1.401	0.38	−0.2943	−0.3184
0.075	0.163	1.414	0.44	−0.2894	−0.3133
0.080	0.212	1.424	0.50	−0.2843	−0.3077
0.085	0.253	1.431	0.56	−0.2789	−0.3015
0.090	0.290	1.435	0.63	−0.2734	−0.2949
0.095	0.323	1.436	0.69	−0.2677	−0.2880
0.100	0.355	1.433	0.75	−0.2618	−0.2807
0.105	0.384	1.427	0.81	−0.2559	−0.2732
0.110	0.412	1.419	0.88	−0.2499	−0.2656
0.115	0.439	1.408	0.94	−0.2438	−0.2578
0.120	0.464	1.394	1.00	−0.2376	−0.2500

Table 4

Free Boundary (w_l, w_u) as Function of Risky Volatility σ

σ	w_l	w_u	π^*/W_t^*	$U(w_0)$	$V(w_1)$
0.16	0.365	1.337	0.78	−0.2803	−0.2949
0.18	0.281	1.383	0.62	−0.2828	−0.3022
0.20	0.212	1.424	0.50	−0.2843	−0.3077
0.22	0.154	1.457	0.41	−0.2852	−0.3119
0.24	0.107	1.486	0.34	−0.2857	−0.3151
0.26	0.067	1.510	0.30	−0.2858	−0.3177
0.28	0.034	1.530	0.26	−0.2859	−0.3197

In Table 6 we see that as the participation rate in the EIA $\bar{p}$ increases, the region of "no surrender" widens (in both directions). This is a bit surprising because the optimal proportion that the individual would invest in the risky asset if she were investing on her own is 50%. However, within the EIA the return on the riskless asset is effectively 0%; thus, the EIA becomes more attractive as the participation rate increases *despite* the fact that the participation rate diverges from 50%. We hypothesize that this widening occurs because as the participation rate increases, the guarantees become more important. Plus, as the participation rate increases, the volatility effectively increases, and we saw the same widening in Table 4.

In Table 7 we see that as the death benefit rate $\bar{d}$ increases, the region of "no surrender" widens (in both directions). This makes sense because as $\bar{d}$ increases, the benefit of holding the EIA increases, as witnessed by the corresponding increase in U. Also, note that until $\bar{d}$ is at least about 1.2, the individual will not invest in the EIA because for $\bar{d}$ less than that, we have $U(w_0) < V(w_1) = -0.3077$.

In Table 8 we see that both the lower and upper boundaries increase as the surrender benefit rate $\bar{s}$ increases. It makes sense that w_u increases with $\bar{s}$ because the individual can "afford" to stay in the EIA longer at the high end because the guaranteed surrender value increases. Note that the

Table 5
Free Boundary (w_l, w_u) as Function of Force of Mortality λ

λ	w_l	w_u	$U(w_0)$	$V(w_1)$
0.000	0.486	1.335	0.0000	0.0000
0.005	0.464	1.347	−0.0506	−0.0526
0.010	0.439	1.360	−0.0956	−0.1000
0.015	0.413	1.371	−0.1358	−0.1429
0.020	0.383	1.383	−0.1718	−0.1818
0.025	0.349	1.394	−0.2043	−0.2174
0.030	0.311	1.404	−0.2336	−0.2500
0.035	0.266	1.414	−0.2601	−0.2800
0.040	0.212	1.424	−0.2843	−0.3077
0.045	0.141	1.433	−0.3064	−0.3333
0.049	0.051	1.439	−0.3227	−0.3525

Table 6
Free Boundary (w_l, w_u) as Function of Participation Rate $\bar{p}$

$\bar{p}$	w_l	w_u	$U(w_0)$
0.80	0.264	1.343	−0.2904
0.85	0.236	1.382	−0.2873
0.90	0.212	1.424	−0.2843
0.95	0.190	1.468	−0.2814
1.00	0.170	1.516	−0.2786

Table 7
Free Boundary (w_l, w_u) as Function of Death Benefit Rate $\bar{d}$

$\bar{d}$	w_l	w_u	$U(w_0)$
1.00	0.666	1.101	−0.3171
1.05	0.638	1.126	−0.3150
1.10	0.604	1.156	−0.3121
1.15	0.564	1.192	−0.3087
1.20	0.517	1.233	−0.3047
1.25	0.461	1.277	−0.3001
1.30	0.394	1.324	−0.2952
1.35	0.313	1.373	−0.2898
1.40	0.212	1.424	−0.2843
1.45	0.068	1.474	−0.2787

effect of $\bar{s}$ on w_u is much less than on w_l because at w_l, the individual receives $\bar{s}w_0$ when surrendering, while at w_u, the individual receives w_u and the guarantee acts only as a safety net for wealth near w_u.

In Table 9 we see the effect on the free boundary if we change the initial fee rate f. It is interesting that both w_l and w_u decrease with f. The individual will not invest in the EIA if f is greater than about 12%. Although it is not shown here, the issuer will not break even if f is less than about 3%.

Finally, in Table 10 we see that the effect of increasing g is that the region of "no surrender" narrows (from both directions). This makes sense because the individual is less likely to be willing to continue paying the maintenance fee if it were to increase.

4.2 Expected Discounted Payout of the Issuer of the EIA

In this section we numerically calculate the expected discounted payout of the issuer, H from equation (3.2), as a function of the participation rate, death benefit, surrender value, and initial and

Table 8
Free Boundary (w_l, w_u) as Function of Surrender Benefit Rate $\bar{s}$

$\bar{s}$	w_l	w_u	$U(w_0)$
0.865	0.038	1.424	−0.2843
0.90	0.212	1.424	−0.2843
0.95	0.373	1.425	−0.2840
1.00	0.498	1.429	−0.2831

Table 9
Free Boundary (w_l, w_u) as Function of Initial Fee Rate f

f	w_l	w_u	$U(w_0)$
0.00	0.223	1.499	−0.2701
0.02	0.219	1.469	−0.2756
0.04	0.214	1.439	−0.2813
0.06	0.210	1.409	−0.2873
0.08	0.205	1.379	−0.2936
0.10	0.201	1.349	−0.3001
0.12	0.196	1.319	−0.3069
0.14	0.192	1.289	−0.3141

Table 10
Free Boundary (w_l, w_u) as Function of Maintenance Fee Rate g

g	w_l	w_u	$U(w_0)$
0.000	0.139	1.772	−0.2706
0.005	0.156	1.634	−0.2743
0.010	0.174	1.544	−0.2778
0.015	0.192	1.477	−0.2811
0.020	0.212	1.424	−0.2843
0.025	0.232	1.380	−0.2874
0.030	0.252	1.342	−0.2904

maintenance fees by varying the values from the base case given at the beginning of Section 4. Recall that the initial investment is $w_1 = 1$, so for the issuer (at least) to break even, the expected discounted payout should be 1 or less. Among those parameter values that yield $H(w_0) \leq 1$, we find the ones that maximize $U(w_0)$.

From the results in Tables 6–10, we restrict our attention to $\bar{p}$ and $\bar{s}$ equal to 0.90, 0.95, and 1.00; restrict $\bar{d}$ equal to 1.20, 1.30, 1.40, and 1.50; and restrict f appropriately. In Table 11 we report only for those values of $\bar{p}$, $\bar{d}$, $\bar{s}$, and f for which $H(w_0) \leq 1$ and $U(w_0) > V(w_1) = -0.3077$; that is, we consider parameter values that yield a product that is feasible for the issuer and desirable for the investor. Generally U decreases with respect to f, so we consider the smallest f for which these two constraints hold. Finally, we set $g = 0.02$ to keep the table from becoming too large.

In Table 11 we see that the EIA that allows the issuer to (at least) break even and that maximizes the value function of the buyer of the policy is the one for which the participation rate $\bar{p} = 0.90$, guaranteed death benefit rate $\bar{d} = 1.50$, guaranteed surrender benefit rate $\bar{s} = 0.95$, initial fee rate $f = 0.050$, with maintenance fee rate $g = 0.02$. Thus, for this particular example, the buyer of the EIA seems to value a higher guaranteed death benefit over a higher participation rate.

Table 11
Issuer's Expected Discounted Payout with $g = 0.02$

$\bar{p}$	$\bar{d}$	$\bar{s}$	f	$H(w_0)$	$U(w_0)$
0.90	1.20	0.90	0.027	0.9990	−0.2975
0.90	1.20	0.95	0.043	1.0000	−0.3003
0.90	1.20	1.00	0.063	0.9993	−0.3032
0.90	1.30	0.90	0.027	0.9993	−0.2882
0.90	1.30	0.95	0.045	0.9992	−0.2925
0.90	1.30	1.00	0.065	0.9997	−0.2966
0.90	1.40	0.90	0.030	0.9991	−0.2784
0.90	1.40	0.95	0.045	0.9992	−0.2825
0.90	1.40	1.00	0.065	0.9995	−0.2876
0.90	**1.50**	**0.95**	**0.050**	**0.9990**	**−0.2731**
0.90	1.50	1.00	0.065	0.9992	−0.2773
0.95	1.20	0.90	0.035	0.9997	−0.2974
0.95	1.20	0.95	0.052	0.9993	−0.3006
0.95	1.20	1.00	0.071	0.9991	−0.3034
0.95	1.30	0.90	0.037	0.9998	−0.2884
0.95	1.30	0.95	0.054	0.9999	−0.2925
0.95	1.30	1.00	0.074	0.9994	−0.2968
0.95	1.40	0.90	0.042	0.9996	−0.2790
0.95	1.40	0.95	0.056	0.9998	−0.2829
0.95	1.40	1.00	0.075	0.9997	−0.2878
0.95	1.50	0.95	0.063	0.9995	−0.2740
0.95	1.50	1.00	0.077	0.9993	−0.2780
1.00	1.20	0.90	0.044	0.9999	−0.2976
1.00	1.20	0.95	0.061	0.9991	−0.3010
1.00	1.20	1.00	0.079	0.9994	−0.3036
1.00	1.30	0.90	0.048	0.9996	−0.2890
1.00	1.30	0.95	0.064	1.0000	−0.2929
1.00	1.30	1.00	0.083	0.9996	−0.2970
1.00	1.40	0.90	0.055	0.9992	−0.2800
1.00	1.40	0.95	0.068	0.9994	−0.2837
1.00	1.40	1.00	0.086	0.9992	−0.2885
1.00	1.50	0.95	0.076	0.9997	−0.2750
1.00	1.50	1.00	0.089	0.9995	−0.2787

5. CONCLUSION

We have presented a method by which issuers of equity-indexed annuities can evaluate the benefit of various policy features. We have focused on five such features: the participation rate, guaranteed death benefit, guaranteed surrender benefit, and initial and maintenance fees. Because the EIA does not have a fixed maturity date, we were able to obtain an "implicit analytical" solution to the problem of when it is optimal for the individual to surrender the EIA. Based on the optimal action of the buyer of the EIA, we then calculated the expected payout of the issuer. Among those policy features that allow the issuer to break even, the EIA that will be most attractive to the buyer is the one for which the value function of the buyer is maximum. We calculated these parameter values for a specific example in Section 4.2 to demonstrate our method.

In future work (Moore and Young 2004), we will tackle the more realistic problem for which there is a finite horizon (that is, the individual must surrender the policy on or before a specified date, such as when she turns $70\frac{1}{2}$) and for which the force of mortality of the buyer is not necessarily constant. In that case we employ numerical techniques to solve the problem because no "implicit analytical" solution exists.

APPENDIX

In this Appendix we show that $A > 0$ given in equation (2.9) is equivalent to the terminal condition in equation (2.5), namely, $\lim_{s\to\infty} e^{-(\rho+\lambda)(s-t)}\mathrm{E}[V(W_s^*)|W_t = w] = 0$. If we substitute the optimal investment strategy (2.10) into the wealth process (2.3), we obtain

$$dW_s^* = \left(r + \frac{2m}{\gamma}\right)W_s^* + \frac{\sqrt{2m}}{\gamma} W_s^* dB_s. \tag{A.1}$$

Note that W_s^* follows geometric Brownian motion, so we can express it explicitly. Indeed, by conditioning on $W_t = w$, we get

$$W_s^* = w \exp\left[\left(r + \frac{2m}{\gamma} - \frac{m}{\gamma^2}\right)(s - t) + \frac{\sqrt{2m}}{\gamma}(B_s - B_t)\right]. \tag{A.2}$$

Thus, conditional on $W_t = w$,

$$V(W_s^*) = A\,\frac{w^{1-\gamma}}{1-\gamma} \exp\left[\left(r + \frac{2m}{\gamma} - \frac{m}{\gamma^2}\right)(1-\gamma)(s - t) + \frac{\sqrt{2m}}{\gamma}(1-\gamma)(B_s - B_t)\right]. \tag{A.3}$$

It follows that

$$\lim_{s\to\infty} e^{-(\rho+\lambda)(s-t)} \mathrm{E}[V(W_s^*)|W_t = w]$$

$$\propto \lim_{s\to\infty} e^{-(\rho+\lambda)(s-t)} \frac{A}{1-\gamma} \exp\left[\left(r + \frac{2m}{\gamma} - \frac{m}{\gamma^2}\right)(1-\gamma)(s - t) + \frac{m}{\gamma^2}(1-\gamma)^2(s-t)\right]$$

$$= \lim_{s\to\infty} \frac{A}{1-\gamma} \exp\left[-\frac{\lambda}{A}(s-t)\right]. \tag{A.4}$$

Thus, the terminal condition in equation (2.5) holds if and only if $A > 0$.

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