Homework 7–Part 2

Read Niven, Zuckerman and Montgomery Chap. 7, concentrating on theorem statements, not proofs, especially 7.4, and 7.7 (or use material from lectures.) Then answer the following problems.

The following problems concern simple continued fractions.

8. (Sect. 7.4) Find the simple continued fraction expansions of:

(a) $\sqrt{3}$; (b) $\frac{1}{7}(24 + \sqrt{15})$; (c) $\sqrt{54}$; (d) $\sqrt{69}$.

9. Let $a$ be a positive integer. Determine the simple continued fraction expansions of:

(a) $\sqrt{a^2 + 1}$
(b) $\sqrt{a^2 + 2}$

[Hint. You may test on examples of small $a$ to see what is going on.]

10. (Sect. 7.5) Let $\theta$ be a real number having simple continued fraction expansion $[a_0, a_1, a_2, \cdots]$ with convergents $\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n]$. (The book writes $\frac{h_n}{k_n}$ for convergents.) Suppose that the continued fraction expansion satisfies:

Hypothesis (i). $a_0 = 1$; so that $\frac{p_0}{q_0} = \frac{1}{1}$.

Hypothesis (ii). For each $n \geq 1$ there holds

$$a_n \geq (q_{n-1})^n.$$  

By using Liouville’s theorem, or otherwise, prove that $\theta$ is a transcendental number. (That is, the real number $\theta$ is not a root of any polynomial with integer coefficients.)

Liouville’s theorem. Let $\alpha$ be an irrational real algebraic number of degree $n$. Then there exists a positive constant $c = c(\alpha)$ depending on $\alpha$ such that

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^n},$$

holds for all integer pairs $(p, q)$ with $q \geq 1$. 

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B2-1. (Bonus, Section 7.7) Using continued fractions, or otherwise, determine (the pattern of) all solutions in integers of:

(a) $x^2 - 19y^2 = 1$.

(b) $x^2 - 31y^2 = 1$.

B2-2. (Bonus, Section 7.8) Let $d$ be a positive integer which is not a perfect square. Prove that if $(x_n, y_n)$ is the sequence of positive solutions of the equation $x^2 - dy^2 = 1$ written in order of increasing values of $x$ with $(x_0, y_0) = (1, 0)$, then the sequences $\{x_n : n \geq 0\}$ and $\{y_n : n \geq 0\}$ each separately satisfy the same recurrence relation,

$$u_{n+2} = au_{n+1} - u_n$$

in which $a$ is a positive integer, which depends on $d$. 