Math 575, Homework 1 (Due Friday, Sept. 14)

This homework covers material in Chapter 1.2, 1.3. Look at the problems of these sections to satisfy yourself that you understand the material. Then hand in the following.

1. Prove that the product of three consecutive integers is divisible by 6, and that the product of four consecutive integers is divisible by 24.

2. Find positive integers a and b satisfying the equations $(a, b) = 10$ and $[a, b] = 100$. Find all solutions.

3. Call an integer *square-free* if it is not divisible by any integer of the a^2 with $a > 1$. Show that each positive integer n can be written uniquely in the form $n = ab$ where a is square-free and b is a square. Show that b is then the largest square dividing n.

4. (i) Find integers x, y such that $95x + 432y = 1$.

(ii) Find integers x, y, z such that $35x + 55y + 77z = 1$.

5. (a) Show that at least one of the following statements is true:

(i) There exist infinitely many primes of the form $10n + 3$.

(ii) There exist infinitely many primes of the form $10n + 7$.

(iii) There exist infinitely many primes of the form $10n + 9$.

(b) Derive a lower bound for the total number of integers k up to x which are prime of the form $10n + 3$, $10n + 7$ or $10n + 9$.

[Hint: Consider carefully Euclid's proof of the infinitude of primes. For bonus credit, prove at least one of statements (i) and (ii) above is true.]

6.(*) Let n be given, and let $1 < a_1 < a_2 < \cdots < a_k < 2n$ be integers *not* dividing each other, i.e. no a_i divides any a_j $(j \neq i)$.

(a) Prove that $k \leq n$.

(b) Exhibit a solution with $k = n$.

(c) Prove that if $k = n$, then a_1 cannot be too small. Namely, show $a_1 \geq 2^m$, where m is the largest power of 3 smaller than $2n$, i.e. $3^m < 2n < 3^{m+1}$.

[Hint: Write each a_i in the form $(2b+1)2^c$. In the last part write $a_1 = (2m_1+1)2^{r_1}$ and investigate how many numbers a_i must be of the form $(2m_1 + 1)2^c3^d$.]