

**Math 575, Homework 7 (due Wednesday, November 28).**

You should look Niven, Zuckerman and Montgomery, Chap. 4.1-4.3, and 8.3, and satisfy yourself that you understand the material. Then hand in the following.

1. (a) Prove that for every pair of real numbers  $(\alpha, \beta)$  the greatest integer function satisfies

$$\lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor \geq \lfloor \alpha \rfloor + \lfloor \beta \rfloor + \lfloor \alpha + \beta \rfloor.$$

Prove that the inequality

$$\lfloor 3\alpha \rfloor + \lfloor 3\beta \rfloor \geq \lfloor \alpha \rfloor + \lfloor \beta \rfloor + \lfloor 2\alpha + 2\beta \rfloor.$$

is not valid for all real  $(\alpha, \beta)$ .

(b) Use the results of (a) or otherwise, to show that for positive integers  $a, b$  the fraction

$$\frac{(2a)!(2b)!}{a!b!(a+b)!}$$

is an integer. Similarly show that

$$\frac{(3a)!(3b)!}{a!b!(2a+2b)!}$$

is not always an integer.

2. Define the function

$$f(n) = \frac{1}{n} \left( \sum_{\substack{a=1 \\ (a,n)=1}}^n a \right)$$

(a) Show that  $f(n)$  satisfies the equation

$$\sum_{d|n} f(d) = \frac{1}{2}(n+1).$$

(b) Deduce that  $f(n) = \frac{1}{2}\phi(n)$  for all  $n \geq 2$ .

3. Define the function

$$f(n) = \frac{1}{n^3} \left( \sum_{\substack{a=1 \\ (a,n)=1}}^n a^3 \right)$$

Prove that

$$f(n) = \frac{1}{4}\phi(n) \left( 1 + \frac{(-1)^k p_1 p_2 \cdots p_k}{n^2} \right).$$

4. (a) Prove that

$$\sum_{1 \leq n \leq x} \frac{\sigma(n)}{n^2} = \frac{1}{6} \pi^2 \log x + O(1).$$

(b) Write down a relation between the Euler function  $\phi(n)$  and the Möbius function  $\mu(n)$ . Hence, or otherwise, obtain an expression, analogous to that given above, for  $\sum_{1 \leq n \leq x} \frac{\phi(n)}{n^2}$ .

5. Show that

$$\sigma(n) = \sum_{k|n} \phi\left(\frac{n}{k}\right) d(k),$$

where  $\sigma(n)$  is the sum of divisors function,  $\phi(n)$  is Euler's totient function, and  $d(n)$  counts the number of divisors of  $n$ .

[Hint: Recall that the convolution of multiplicative functions is multiplicative.]

6. Let  $r_n := r_{a,b}(n)$  count the number of representations of the integer  $n$  by the positive definite quadratic form  $ax^2 + by^2$ , where  $a$  and  $b$  are given positive integers.

(a) Estimate  $\sum_{1 \leq n \leq x} r(n)$ .

(b) Estimate  $\sum_{1 \leq n \leq x} (r(n))^2$ .

7. Let  $\alpha(n)$  denote the arithmetical function defined by

$$\alpha(n) = \prod_{\substack{1 \leq a \leq n \\ (a,n)=1}} a.$$

Prove that

$$\alpha_n = n^{\phi(n)} \prod_{d|n} \left( \frac{d!}{d^d} \right)^{\mu\left(\frac{n}{d}\right)}.$$

8.(Bonus) (a) Show that if  $\alpha > 1$  is irrational and  $\beta$  is defined by  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ , so that  $\beta = 1 + \frac{1}{\alpha}$ , then the two sequences  $\{\lfloor n\alpha \rfloor : n \geq 1\} \cup \{\lfloor n\beta \rfloor : n \geq 1\}$ , between them include every positive integer exactly once.

(b) Show the converse: If  $\alpha, \beta > 1$  have the property that the two sequences  $\{\lfloor n\alpha \rfloor : n \geq 1\}$ ,  $\{\lfloor n\beta \rfloor : n \geq 1\}$ , include every positive integer exactly once, then  $\alpha$  is irrational and  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ .