Math 675, Homework 1-part 2
(Due Monday, Sept. 21, 2015, in class, or by 3pm in the box outside EH3086)

9. (Chebyshev-Type Estimates) Let \( \{ h(m) : 1 \leq m \leq M \} \) be a finite sequence of real numbers having the properties that:

(P1) \( h(1) = 1 \)

(P2) \( \sum_{m=1}^{M} \frac{h(m)}{m} = 0 \).

(P3) The periodic function \( f(x) := \sum_{m=1}^{M} h(m) \lfloor \frac{x}{m} \rfloor \) has \( 0 \leq f(x) \leq 1 \), for all \( x > 0 \).

Associate to this sequence the numbers:

\[
H = H_f := - \sum_{m=1}^{M} h(m) \log \frac{m}{m}.
\]

and

\[
a = a_f := \sup \{ k : f(x) = 1, \hspace{1em} \text{for} \hspace{1em} 1 \leq x < k \}.
\]

(1) Show that \( f(x) \) is a piecewise constant function, and that \( a_f \) is an integer with \( a_f \geq 2 \).

Next suppose that \( \{ a_n : n \geq 1 \} \) is a sequence of nonnegative numbers, and set \( A(x) = \sum_{n \geq x} a_n \). Suppose that

\[
B(x) := \sum_{n \leq x} \frac{a_n}{n} = x \log x + Cx + o(x)
\]

as \( x \to \infty \).

(2) Show that, for any \( \epsilon > 0 \), there holds

\[
(H - \epsilon) x < A(x) < \left( \frac{a}{a - 1} H + \epsilon \right) x
\]

for all sufficiently large \( x \geq x_0(\epsilon) \).

(3) Show that the choice \( M = 6 \) and \( h(1) = 1, h(2) = -1, h(3) = -2, h(4) = h(5) = 0, h(6) = 1 \) satisfies the hypotheses. Determine \( a_f \) and \( H \). Explain why this choice suffices to prove Bertrand’s postulate.

(4) Recover Chebyshev’s original estimate with the choice \( M = 30 \) and \( h(1) = h(30) = 1, h(2) = h(3) = h(5) = -1, h(m) = 0 \) for all other values of \( 4 \leq m \leq 29 \), not given above.

Remark. Erdős and Diamond [Enseign. Math. 26 (1980), 313–321] show that, in principle, an infinite set of better and better choices of functions \( \{ h(m) \} \) exist, with \( M \to \infty \), such that \( H \to 1 \) and \( a_f \to \infty \), and the Prime Number Theorem follows. Unfortunately their proof of existence of these functions requires the use of the Prime Number Theorem, so it does not give a (new) elementary proof of the PNT. (I am not sure if their method fits exactly in the framework of this problem, since condition (P3) is allowed to be less restrictive in their argument.)