Problems with two stars may be difficult and are (very) optional.

26. Deduce from the hypothesis \( \psi(x) \sim x \) that \( m(x) = \sum_{n \leq x} \frac{\mu(n)}{n} = o(1) \), as \( x \to \infty \).

(a) Show the identity of arithmetic functions

\[ \Lambda(n) = -(\mu \log) \ast 1, \]

where \( \mu \log(n) = \mu(n) \log n \) and \( 1_n = 1 \).

(b) Derive the identity \( \sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1 \) and use it to show that \( xm(x) = 1 + O(x) \) and deduce \( m(x) = O(1) \).

(c) Introduce \( G(x) = \sum_{m \leq x} \frac{\mu(m) \log m}{m} \). Show the equivalence

\[ m(x) = o(1) \text{ as } x \to \infty \iff G(x) = o(\log x) \text{ as } x \to \infty, \]

as follows. Use the expansion \( \frac{1}{n} \sum_{n=md} \mu(d) = \delta(n) \) to deduce

\[ 1 = \sum_{m \leq x} \frac{\mu(m)}{m} (\sum_{d \leq \frac{x}{m}} \frac{1}{d}). \]

Now apply the asymptotic formula for \( \sum_{d \leq x} \frac{1}{d} \) to deduce

\[ 1 = m(x)(\log x + \gamma) - G(x) + O(1), \]

and use this to obtain the equivalence.

(d) Aim to show \( G(x) = o(\log x) \). Using (a) verify the convolution identity

\[ \mu \log = -\Lambda \ast \mu = (1 - \Lambda) \ast \mu - \delta \]

From this obtain

\[ G(x) = \sum_{j \leq x} \left( 1 - \frac{\Lambda(j)}{j} \right) \frac{\mu(k)}{k} \]

and express this as a Stieltjes integral

\[ G(x) = -1 + \int_{1^+}^{x} m\left( \frac{x}{t} \right) \frac{1}{t} dR(t) \]

with \( R(t) := |t| - \psi(t) \) for \( t \geq 1 \).

(e) Use the hypothesis \( R(t) = o(t) \) to deduce by Stieltjes integration by parts

\[ G(x) = -1 + \int_{1^+}^{x} m\left( \frac{x}{t} \right) \frac{1}{t^2} dt - \int_{1^+}^{x} R(t) dm\left( \frac{x}{t} \right). \]

Explain why the first integral is \( o(\log x) \). For the second integral bound it using the inequality of measures \( |dm(y)| \leq d\{\sum_{n \leq y} \frac{1}{n}\} \), to conclude \( G(x) = o(\log x) \) as \( x \to \infty \).
27. *(Selberg’s identity)*

Define an arithmetic function \( \Lambda_2 = (\log)^2 \ast \mu \).

(a) Show that, for distinct primes \( p, q \) and all integers \( j \geq 1, k \geq 1 \), there holds
\[
\Lambda_2(p^j) = (2j - 1)(\log p)^2, \quad \Lambda_2(p^j q^k) = 2(\log p)(\log q)
\]
and \( \Lambda_2(n) = 0 \) if \( n \) has three or more distinct prime factors.

(b) Show that
\[
\sum_{n \leq x} \Lambda_2(n) = \sum_{p \leq x} (\log p)^2 + \sum_{pq \leq x} (\log p)(\log q) + O(x).
\]

(c) Show that
\[
\sum_{n \leq x} (d \ast 1)(n) = \frac{1}{2} x(\log x)^2 + ax \log x + bx + O\left(x^{2/3}\log(2x)\right),
\]
where \( a \) and \( b \) are real constants.

(d) Deduce that for suitable constants \( A \) and \( B \), the function
\[
h(n) := 2(d \ast 1)(n) + Ad(n) + B1(n) - (\log n)^2
\]
has a summatory function \( H(x) = \sum_{n \leq x} h(n) \) that is \( O(x^{3/4}) \).

(e) Prove that
\[
\sum_{n \leq x} \Lambda_2(n) = 2x \log x + O\left(x\right).
\]

28. Let \( f(x) \) be a complex-valued multiplicative function satisfying
\[
\sum_{n=1}^{\infty} \left| \frac{f \ast \mu(n)}{n} \right| < \infty. \tag{1}
\]

Show
\[
\frac{1}{x} \sum_{n \leq x} f(n) = M(f) + o(1)
\]
where we define
\[
M(f) := \prod_p \left[ (1 - \frac{1}{p}) \sum_{k=0}^{\infty} f(p^k) p^{-k} \right],
\]
and this product is absolutely convergent.

*[Hint.]* Generalize the proof of Theorem 3.6.1. Start from \( h := f \ast \mu \) and show that for \( x \geq 1 \),
\[
\frac{1}{x} (\sum_{n \leq x} f(n)) = \frac{1}{x} \sum_{d \leq x} h(d) \left\lfloor \frac{x}{d} \right\rfloor
\]
and that
\[
\frac{1}{x} \sum_{d \leq x} h(d) \left\lfloor \frac{x}{d} \right\rfloor \rightarrow \sum_{d=1}^{\infty} \frac{h(d)}{d}, \quad \text{as} \quad x \rightarrow \infty,
\]
using dominated convergence theorem. To complete the argument show that \((??)\) implies that \( \sum_{d=1}^{\infty} \frac{h(d)}{d} = M(f) \), with absolute convergence.