Math 675, Homework 6-part 1

(Due Monday, November 23, 2015, in class, or by 3pm in the box outside EH3086)

The assigned problems on this part are 46, 47, although 44 is interesting.

44. (Some Additive Character Sum Bounds)

Let \( \chi : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathbb{C}^* \) be a nontrivial multiplicative character (mod \( p \)), with \( p \) a prime. Also let \( \psi : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{C}^* \) be a nontrivial additive character, so \( \psi(j) = e^{2\pi ijb/p} \) for some \( b \) with \((b,p) = 1\).

(a) Let \( G(\chi, \psi) = \sum_{a=1}^{p-1} \chi(a)\psi(a) \). Show that \( |G(\chi, \psi)| = \sqrt{p} \). (Here \( G(\chi, \psi) \) is one version of a Gauss sum).

(b) Let \( S_k(\psi) := \sum_{j=1}^{q-1} \psi(j^k) \).

Determine conditions on \( k \) under which \( S_k(\psi) = 0 \).

(c) Prove that, for all \( k \geq 1 \),

\[ |S_k(\psi)| \leq (k - 1)\sqrt{p}. \]

45. (Distributional Poisson summation formula) [This problem presumes some knowledge of distributions.]

A tempered distribution \( T(x) \) is a continuous linear functional \( T : \mathcal{S} \rightarrow \mathbb{C} \), where \( \mathcal{S} = \mathcal{S}(\mathbb{R}) \) is the Schwartz space of rapidly decreasing \( C^\infty \)-functions on \( \mathbb{R} \), i.e. they are all their derivatives go to zero faster than any inverse polynomial at \( \infty \). We write \( \langle T, \varphi \rangle \) for the value of \( T \) on the test function \( \varphi(x) \in \mathcal{S} \). By convention the Fourier transform \( \mathcal{F}(T) = \hat{T} \) is defined by \( \langle \mathcal{F}(T), \varphi \rangle := \langle T, \mathcal{F}(\varphi) \rangle \). The Schwartz space \( \mathcal{S} \) is invariant under the Fourier transform and its inverse, as is the space \( \mathcal{S}' \) of tempered distributions.

(a) Show that \( T = \delta_a \), the Dirac delta function defined by \( \langle \delta_a, \varphi \rangle = \varphi(a) \), is a tempered distribution. Show that \( \mathcal{F}(\delta_a) = e^{-2\pi ia\xi} \), as a tempered distribution.

(b) Show that, in the sense of tempered distributions, the Fourier transform of

\[ \mathcal{F} \left( \sum_{n=-\infty}^{\infty} \delta_n(x) \right) = \sum_{n=-\infty}^{\infty} \delta_n(x). \] (particles)

Equivalently

\[ \mathcal{F} \left( \sum_{n=-\infty}^{\infty} e^{-2\pi inx} \right) = \sum_{n=-\infty}^{\infty} e^{2\pi in\xi} \] (waves)

(Some authors define the Fourier transform as \( \int_{-\infty}^{\infty} \varphi(x)e^{-ix\xi}dx \), and then one side of the Poisson formula will replace \( n \) with \( 2\pi n \), multiplied by a factor of \( 2\pi \) as well.)

46. (Gamma Function) The gamma function \( \Gamma(s) \) is defined by Euler as

\[ \Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt \]

This defines it as an analytic function on the half-plane \( Re(s) > 0 \).
(a) Deduce the functional equation \( s \Gamma(s) = \Gamma(s + 1) \) by integration by parts.

(b) Use the functional equation to meromorphically continue \( \Gamma(s) \) to the entire plane. Show that its only singularities are simple poles located at \( s = 0, -1, -2, \ldots \). Determine the residues at these poles.

(c) Prove the reflection formula
\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.
\]
Deduce from it (or vice versa) that \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \).

(d) Prove Legendre’s duplication formula
\[
\Gamma(s)\Gamma(s + \frac{1}{2}) = 2^{1-2s}\sqrt{\pi} \Gamma(2s).
\]
(For part (d) you may assume as known the functional equation for the Riemann zeta function.)

47. (Inverse Gamma Function)

(a) Method 1. Prove Euler’s infinite product formula
\[
\frac{1}{\Gamma(s)} := \lim_{n \to \infty} \frac{s(s+1) \cdots (s+n)}{n!n^s} = s \prod_{n=1}^{\infty} \frac{1 + \frac{s}{n}}{(1 + \frac{1}{n})^s}
\]
is valid on \( \mathbb{C} \), and defines the left side as an entire function. (This problem does not directly identify the left side with the Gamma function of problem 46. Can you show this?)

(b) Method 2. Prove that \( \frac{1}{\Gamma(s)} \) is an entire function, using problem 46 (b), and by showing \( \Gamma(s) \) has no zeros in the strip \( 0 \leq \Re(s) \leq 1 \).