Math 675, Homework 6-Part 2

(Due Monday, November 23, 2015, in class, or by 3pm in the box outside EH3086)

Problems with two stars (**) are not assigned.

48. (A Dirichlet series)

Let $\chi : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{C}^*$ be any Dirichlet character (mod $m$). (The answer to convergence regions below may depend on the particular character, please specify the dependence.)

(a) Show that 
$$F(s, \chi) := \sum_{n=1}^{\infty} d(n)^2 \chi(n)n^{-s} = \frac{L(s, \chi)^4}{L(2s, \chi^2)}.$$ 

(b) Determine $\sigma_a$ for $F(s, \chi)$.

(c) Determine $\sigma_c$ for $F(s, \chi)$. (If you cannot determine it completely, find an upper bound.)

49. Alternate proof of $L(1, \chi) \neq 0$ for real characters (Mertens 1895)

Let $r_\chi(n) = \sum_{d|n} \chi(d)$.

(a) Show that if $\chi$ is non-principal character (mod $m$) then there is a positive constant $C_1(\chi)$ such that, for $2 \leq x < \infty$,
$$|\sum_{n>x} \frac{\chi(n)}{\sqrt{n}}| \leq C_1(\chi) \frac{1}{\sqrt{x}}.$$

(b) Show that if $\chi$ is a non-principal character (mod $m$) then, for $2 \leq x < \infty$,
$$\sum_{n \leq x} r_\chi(n) \frac{1}{\sqrt{n}} = 2\sqrt{x}L(1, \chi) + O_\chi(1)$$

(The implied constant in the $O$-symbol depends on $\chi$.)

(c) Suppose that $\chi$ is a (nontrivial) real character. Show that $r_\chi(n) \geq 0$ for all $n \geq 1$ and that $r_\chi(n^2) \geq 1$ for all $n$ with $(n, m) = 1$. Deduce that
$$\sum_{n \leq x} r_\chi(n) \frac{1}{\sqrt{n}} \geq C_2(\chi) \log x,$$

for a positive constant $C_2(\chi)$. Choosing $x$ sufficiently large, conclude using (b) that $L(1, \chi) > 0$.

(d)(**) Can one determine an explicit lower bound for $L(1, \chi)$ (as a function of $m$) where $\chi$ is a real character? [This requires determining the constants in (a)-(c).]
50. (*Goblins and Gaussians Revisited*)

Consider the complex Gaussian \( f_\alpha(t) = e^{-\pi \alpha t^2} \), allowing \( t \in \mathbb{C} \).

(a) Show that

\[
\int_0^\infty e^{-\pi t^2} dt = \lim_{Y \to \infty} \sqrt{i} \int_0^Y e^{\pi it^2} dt,
\]

by using a limit of appropriate (closed) contour integrals in the complex \( t \)-plane, and bounding integral on parts of the contour. (Also answer: which branch of the square root of \( i \) must be chosen here, in terms of the sign of its imaginary part? )

(b) Explain why the method fails to establish that

\[
\int_0^\infty e^{-\pi t^2} dt = (?) \lim_{Y \to \infty} \int_0^Y e^{\pi it^2} dt,
\]

Which contour would you use in this case.

51. (*A Throwback*)

Prove or disprove the following theorems.

(a) **Theorem A.** Suppose \( f(x) \) and \( g(x) \) are continuous on \([0, \infty)\), \( g(x) > 0 \) if \( x > a \), \( f(x) = O(g(x)) \) as \( x \to \infty \). Then, for all sufficiently large \( x \),

\[
\int_a^x f(t) dt = O\left( \int_a^x g(t) dt \right).
\]

(b) **Theorem B.** Suppose for \( m \geq 1 \), \( f_m(x) \) and \( g_m(x) \) are continuous on \([0, \infty)\), \( g_m(x) > 0 \) if \( x > a \), \( f_m(x) = O(g_m(x)) \) as \( x \to \infty \). Then, for all sufficiently large \( x \),

\[
\sum_{m=1}^\infty f_m(x) = O\left( \sum_{m=1}^\infty g_m(x) \right).
\]

52. (*Imprimitive Gauss Sums*)

For fixed \( m \) take the additive character \( \psi(k) = e(k/m) = e^{2\pi ik/m} \). Define for a Dirichlet character \( \chi \) (mod \( m \)) the Gaussian sum

\[
G(\chi) := G(\chi, \psi) = \sum_{k=1}^m \chi(k)\psi(k) = \sum_{k=1}^m \chi(k)e^{2\pi ik/m}.
\]

(a) Show that if \( \chi \) is a primitive character then

\[
|G(\chi)| = \sqrt{m}.
\]

(b) Show that if \( \chi \) is a nontrivial character then

\[
|G(\chi)| \leq \sqrt{m}.
\]

(c) If \( m = p^j \) is an odd prime power, with \( j \geq 2 \), and \( \chi \) is imprimitive, with associated primitive character \( \chi_1 \) (mod \( p \)), find a formula relating \( G(\chi, \psi) \) to \( G(\chi_1, \psi_1) \) where \( \psi_1(k) = e^{2\pi ik/p} \).