Math 675: 2. Evaluations and an Estimate

1 Arithmetico-Geometric Series

Problem. Sum $\sum_{j=1}^{\infty} j^r x^j$ in closed form when $r$ is a nonnegative integer and $|x| < 1$ is a complex number.

Solution. For $r \geq 1$ we have, when $|x| < 1$,
\[
\left( x \frac{d}{dx} \right)^r \left( \frac{1}{1-x} \right) = \sum_{j=0}^{\infty} j^r x^j.
\]
For $r = 0$ we have $\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$. The result now follows by repeatedly applying the operator $x \frac{d}{dx}$, since $(x \frac{d}{dx}) x^n = nx^n$, for $n \geq 1$. Thus
\[
\left( x \frac{d}{dx} \right) \left( \frac{1}{1-x} \right) = \frac{x}{(1-x)^2} = \sum_{j=0}^{\infty} j x^j,
\]
\[
\left( x \frac{d}{dx} \right)^2 \left( \frac{1}{1-x} \right) = \frac{x^2 + x}{(1-x)^3} = \sum_{j=0}^{\infty} j^2 x^j,
\]
and so on. $\square$

Remark. One can also evaluate, for $|x| < 1$ a finite sum $\sum_{j=1}^{n} j^r x^j$ in closed form as a rational function of two variables, $n$ and $x$. Note that for $x = 1$ the answer will be related to Bernoulli polynomials (in the variable $n$).

2 Bounding the Exponential

Problem. Estimate $e^x$ for real or complex $x$ near zero.

Lemma. For real $x$ with $|x| \leq 1$,
\[
1 + x \leq e^x \leq 1 + x + x^2.
\]
For complex $z$ with $|z| \leq 1$,
\[
1 - |z| \leq |e^z| \leq 1 + |z| + |z|^2.
\]

Proof. For $-1 \leq x \leq 0$, the expansion of $e^x$ has terms of alternating sign, decreasing monotonically. This gives both inequalities in this case. For $0 \leq x \leq 1$, the left inequality is immediate. The right inequality holds since one may estimate that
\[
\sum_{j=3}^{\infty} \frac{x^j}{j!} \leq \frac{1}{2} x^2 \left( \frac{x}{3} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{4 \cdot 5} + \cdots \right) \leq \frac{1}{2} x^2.
\]
for $0 \leq x \leq 1$ since
\[
\frac{1}{3} + \frac{1}{3 \cdot 4} + \cdots = \frac{1}{3} + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \cdots \leq \frac{2}{3} \leq 1.
\]
(The complex case can be inferred from the above.) $\square$