Math 676, Homework 5

Do 5 problems of set of 11 problems.

1. [Cassels, Chap. 1, Problem 2].
   Let $S_k(n) = 1^k + 2^k + \cdots + (n-1)^k$, and let $B_k$ denote the $k$-th Bernoulli number, defined by
   $$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$  
   (Note that aside from $B_1$ all $B_{2l+1} = 0$.) Let $p \geq 5$ and let $k = 2l \geq 4$. Show that
   $$|p^{-m} S_{2l}(p^{-m}) - B_{2l}|_p \leq p^{-2m+a}$$
   where $a = 1$ if $p - 1$ divides $(2l - 2)$ and $a = 0$ otherwise.
   [Hint: Read Cassels, Chapter 1.3, and use the von-Staudt Clausen Theorem and its proof.]

2. [Cassels, Chap 1, Problem 3]
   For each positive $m$ let $N_m = \lfloor (1 + \sqrt{3})^{2m+1} \rfloor$. Prove that the 2-adic valuation of the integer $N_m$ is
   $$|N_m|_2 = 2^{-m-1}.$$  
   [Hint: Show $N_m$ satisfies a linear recurrence.]

3. [Cassels, Chap. 2, problem 22] A function $f(x)$ on a topological space is locally constant if every $y$ has a neighborhood $U(y)$ on which $f(x) = f(y)$ for all $x \in U(y)$.
   (a) Give an example of a $\mathbb{Q}_p$-valued function $f(x)$ on domain $\mathbb{Z}_p$ which is both continuous and locally constant, but not constant.
   (b) If the $\mathbb{Q}-p$-valued function on domain $\mathbb{Z}_p$ is both continuous and locally constant, show that the set of $x$ with $f(x) = b$ is both open and closed. Deduce that $f$ takes only finitely many values on this domain.
   (c) Show that any continuous $\mathbb{Q}_p$-valued function on domain $\mathbb{Z}_p$ is the uniform limit of continuous, locally constant functions.

4. [Cassels, Chap 4, problem 2]
   For what $a \in \mathbb{Z}$ is $5x^2 = a$ solvable in $\mathbb{Z}_5$? For what $a$ in $\mathbb{Q}_5$ is it solvable?
5. [Cassels, Chap 4, problem 4] 
Show that each of the following functions has a zero in \( \mathbb{Z}_p \) for every prime \( p \).

(i) \((X^2 - 2)(X^2 - 17)(X^2 - 34)\).

(ii) \((X^3 - 37)(X^2 + 3)\).

[Note: For (ii) you may need to use the law of quadratic reciprocity.]

6. [Cassels, Chap. 4, Problem 5] 
Show that the cubic polynomial

\[ F(x) = 5x^3 - 7x^2 + 3x + 6 \]

has a root \( \alpha \) in the 7-adic integers \( \mathbb{Z}_7 \) with \( |\alpha - 1|_7 < 1 \). Find an \( a \in \mathbb{Z} \) such that

\[ |\alpha - a|_7 \leq \frac{1}{7^4}. \]

7. Let \( f(x) = x^3 - x + 1 \).

(a) Factorize \( f(x) \) modulo 5, and then factorize it modulo \( 5^3 \). What can you say about the factorization of \( f(x) \) over the 5-adic field \( \mathbb{Q}_5 \)?

(b) Factorize \( f(x) \) modulo 101, and then factorize it modulo \( (101)^3 \). What can you say about the factorization of \( f(x) \) over the 101-adic field \( \mathbb{Q}_{101} \)?

[A calculator may be needed for (b). Or else, use a computer package.]

8. [Cassels, Chap. 4, Problem 20] Let \( p \) be prime and let for a positive integer let its base \( p \) expansion be

\[ m = a_0 + a_1p + \cdots + a_Jp^J \quad (J \geq 0), \quad 0 \leq a_j \leq p - 1. \]

Also write

\[ m! = p^M N, \quad \gcd(N, p) + 1. \]

(a) Show that

\[ (p - 1)M = m - \sum_j a_j \]
(b) Show that

\[ N \equiv (-1)^M \prod_j (a_j)! \quad (\text{mod } p). \]

9. Let \( p \geq 3 \) and let \( \exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \), and let \( \log(1-x) = -\sum_{n=2}^{\infty} \frac{1}{n} x^n \). Let \( p \geq 3 \).

(a) Determine the radius of convergence of \( \exp(x) \) around \( x = 0 \) in the \( p \)-adic variable \( x \in \mathbb{Q}_p \), i.e. for which values \( |x|_p \) does the series converge?

(b) Determine the radius of convergence of \( \log(1-x) \) similarly in \( \mathbb{Q}_p \).

(c) For which values \( x \in \mathbb{Q}_p \) can you conclude \( \exp(\log(1-x)) = 1-x \), and for which values \( x \in \mathbb{Q}_p \) can you conclude \( \log(1-\exp(1-x)) = x? \)

[Hint: See problem 8. These functions are sometimes denoted \( \exp_p(x) \) and \( \log_p(x) \).]

10. [Cassels, Chap. 4, Problem 25]

(i) Let \( K \) be a field of characteristic \( p \) which is complete with respect to the valuation \( | \cdot |_\nu \) and suppose that its residue class field \( k \) is finite with \( q \) elements. Show that \( K \) contains the finite field \( \mathbb{F}_q \) having \( q \) elements.

(ii) If, in addition, \( | \cdot |_\nu \) is a discrete valuation, show that one can write \( K \) as \( K = \mathbb{F}_q((\pi)) \) as a Laurent series field in a (non-unique) uniformizer \( \pi \).

11. [Cassels, Chap. 4, Problem 12]

Let \( u(0) = u(1) = 1 \) and consider the recurrence

\[ u(n+2) = 5u(n+1) - 11u(n) \quad (n \geq 0) \]

(a) By working in \( \mathbb{Q}_5 \), or otherwise, show that \( u_n = 1 \) only for \( n = 0, 1 \).

(b) Is it possible that \( u(n) = 0 \) for some \( n \geq 2 \)?