1. **Orders of number fields** Let the monic polynomial \( f(x) = x^n = a_n x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x] \) be irreducible, and set \( K = \mathbb{Q}(\theta) \) where \( \theta \) is a root, so \([K : \mathbb{Q}] = n\). An order is a subring with unit \( R \) which contains 1, and is of full rank \( n \) as a \( \mathbb{Z} \)-module. (For example \( R = \mathbb{Z}[1, \theta, \ldots, \theta^{n-1}] \).)

   The ring of integers \( O_K \) is (a) maximal order, i.e. it is integrally closed in \( K \), and all other orders are contained in \( O_K \) and are non-maximal. For a non-maximal order \([O_K : R] = f > 1\).

   (i) Prove the a non-maximal order is not a Dedekind domain. [Read the proof in Lang, Chap. 1, Theorem 2, and find out where the proof of Dedekind domain fails for an order. Which axioms of a Dedekind domain hold, and which fail?]

   (ii) Define (integral) ideals and fractional ideals in the usual way for \( R \) They are \( J = (1/\alpha)A \) for some element \( \alpha \) of \( K \) and some integral ideal \( A \) of \( R \). For an integral \( R \)-ideal define the norm \( N.J = \#(R/A)|\frac{1}{N_{K/\mathbb{Q}}}(\alpha)| \). Let \( I_m \) denote the set of fractional ideals that are quotients of integral ideals of norm relatively prime to the integer \( m \), and let \( P_m \) denote the set of principal fractional ideals contained in \( I_m \). Show that \( I_m, P_m \) are abelian groups, so that the group \( C_m = I_m / P_m \) is defined. (This is a ring class group.)

   (iii) Define a homomorphism \( j_m : I_m \rightarrow I_1 \) taking \( P_m \) to \( P_1 \). Show that this induces a well-defined homomorphism \( j_m : C_m \rightarrow C_1 \) where \( C_1 \) is the ideal class group. (What is the norm of \( AO_K \) in \( I_1 ? \))

   (iv) Show that \( j_m \) is surjective. (May assume as known that there are infinitely many prime ideals in each class of \( I_m \).)

   **Remark.** Neukirch, Chap. 1, Sect. 13 (read this!), views the property of being integrally closed as “smoothness” in algebraic geometry terms. Thus non-maximal orders are a kind of “singular’ object.

2. **Integral Closure of Non-maximal orders** Let \( L/K \) be a number field and let \( R \) be a (possibly non-maximal) order \( R \) in the number field \( K \).

   (a) Prove that the integral closure \( R' \) of \( R \) in \( L \) is a Dedekind domain. [This result is in Neukirch, Chap. 1, Prop. 12.8.]

   (b) What happens in the case \( L = K \)? What can you say about the integral closure of \( R \).

3. **Irreducible Trinomial** Let \( f(x) = x^n - x - 1 \).

   (i) Show that \( f(x) \) has exactly one real root \( \theta_n > 1 \).

   (ii)(*) Show that the discriminant of the polynomial \( f(x) \) is

   \[
   D_n = (-1)^{(n-1)(n-2)/2}(n^n + (-1)^n(n-1)^{n-1}).
   \]
(iii) (*) Show that \( f(x) \) is irreducible over \( \mathbb{Q} \). (For this see E. S. Selmer, *On the irreducibility of certain trinomials*, Math. Scand. 4 (1956), 287–302. Download and read this paper, the relevant section, for an amazing proof.)

(iv) Set \( K = \mathbb{Q}(\theta_n) \), which is a number field of degree \( n \) for all \( n \geq 3 \) (by (iii)). Prove that \( K \) is not a normal extension of \( \mathbb{Q} \) for all \( n \geq 3 \).

(v) (**)[May be unsolved] For which \( n \) is \( \mathbb{Z}[1, \theta, \ldots, \theta^{n-1}] \) the maximal order \( O_K \). List some \( n \) for which this is the case. (A computer package like PARI or SAGE will be useful here.)

(vi) (*) Show the normal closure of \( K \) has Galois group \( S_n \).

3. In this problem, assume as known the Chebotarev (or Cebotarev) density theorem.

Let \( K = \mathbb{Q}(\theta) \) be a number field with \( [K : \mathbb{Q}] = n \). Let \( d(K/\mathbb{Q}) \) be the density of the set of rational primes \( p \) that have at least one degree one prime ideal of \( K \) lying over \( L \). That is,

\[
d(K) = \lim_{x \to \infty} \frac{1}{\pi(x)} \# \{ p \leq x : \text{there is a prime ideal in } K \text{ of norm } p' \}.
\]

(It is a deep fact that the limit exists.) Equivalently if \( K = \mathbb{Q}(\theta) \), where \( \theta \) satisfies an irreducible monic degree \( f \) polynomial \( f(x) = x^n = a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x] \). then there is a degree one prime ideal over \( (p) \) in \( K \) if and only if \( f(x) \) has a linear factor \( \text{(mod } p) \).

(i) Prove that

\[
d(K) \geq \frac{1}{n}.
\]

(ii) Prove that strict inequality holds in (i) if the normal closure of \( K \) is a number field with Galois group \( S_n \). [*Hint. One will need to count the number of permutations containing at least one 1-cycle.*]

(ii) Prove that equality holds in (i) if and only if \( K/\mathbb{Q} \) is a normal extension of \( \mathbb{Q} \). [*Neukirch, Chap. 7, Corollary 13.6.*]


4. Consider the pure cubic field \( K = \mathbb{Q}(10^{1/3}) \).

(a) Determine which primes ramify in this field.

(b) Determine which primes ramify tamely, and which (if any) ramify wildly.

(c) Determine the primes \( p < 50 \) which have a degree one prime factor in \( K \). Which of these split completely? [*Hint: You might use the factorizations of \( f(X) = x^3 - 10(\text{mod } p) \). Determine \( Disc(f) \). A computer will be necessary.*]

(d)(*) Find the fundamental unit of this field. [Again, use a computer package.]