Dilated Floor Functions and Their Commutators

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Topics Covered

- **Part I.** Dilated floor functions

- **Part II.** Dilated floor functions that commute

- **Part III.** Dilated floor functions with positive commutators

- **Part IV.** Concluding remarks
• Takumi Murayama, Jeffrey C. Lagarias and D. Harry Richman, 
*Dilated Floor Functions that Commute*,
(arXiv:1611.05513, v1)

• J. C. Lagarias and D. Harry Richman, 
*Dilated Floor Functions with Nonnegative Commutators*, preprint.

• J. C. Lagarias and D. Harry Richman, 
*Dilated Floor Functions with Nonnegative Commutators II: Third Quadrant Case*, in preparation.

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Part 1. Dilated Floor Functions

- We start with the floor function $\lfloor x \rfloor$.

- The floor function discretizes the real line, rounding a real number $x$ down to the nearest integer:

  $$x = \lfloor x \rfloor + \{x\}$$

  where $\{x\}$ is the fractional part function, i.e. $x$ (modulo one).

- The ceiling function which rounds up to the nearest integer is conjugate to the floor function:

  $$\lceil x \rceil = - \lfloor -x \rfloor \quad \begin{cases} \equiv R \circ \lfloor \cdot \rfloor \circ R(x) \end{cases}$$

  using the conjugacy function $R(x) = -x$. 
Dilated Floor Functions-1

- The dilations for $\alpha \in \mathbb{R}^*$ act on the real line as

$$D_\alpha(x) = \alpha x,$$

They act under composition as the multiplicative group $GL(1, \mathbb{R}) = \mathbb{R}^*$.

- A dilated floor function with real dilation factor $\alpha$ is defined by

$$f_\alpha(x) := \lfloor \alpha x \rfloor = \lfloor \cdot \rfloor \circ D_\alpha(x)$$

- We allow negative $\alpha$, so we are able to build ceiling functions.
Dilated Floor Functions-2

- Dilated floor functions encode information on the Riemann zeta function.

- Its Mellin transform is given when $\alpha > 0$, for $Re(s) > 1$, by

$$\int_0^\infty [\alpha x] x^{-s-1} dx = \alpha^s \frac{\zeta(s)}{s}$$

Also for $0 < Re(s) < 1$,

$$\int_0^\infty \{\alpha x\} x^{-s-1} dx = -\alpha^s \frac{\zeta(s)}{s}$$

These two integrals are related via a (renormalized) integral which converges nowhere: $\int_0^\infty \alpha x \cdot x^{s-1} dx := 0$.

- Dilations are compatible with the Mellin transform. The Mellin transform preserves the $GL(1)$ scaling since $\frac{dx}{x}$ is unchanged by dilations.
Dilated Floor Functions -3

- Dilated floor functions can be used in describing data about lattice points in lattice polytopes, in the recently introduced notion of intermediate Ehrhart quasi-polynomials of the polytope. (Work of Baldoni, Berline, Köppe and Vergne, Mathematika 59 (2013), 1–22, and sequel with de Loera, Mathematika 62 (2016), 653–684).

- Their definition 23: For integer $q \geq 1$, let $[n]_q := q \lfloor \frac{1}{q} n \rfloor$ and let $\{n\}_q := n \pmod q$ (least nonnegative residue), so that

$$n = [n]_q + \{n\}_q.$$  

- Their Table 2 gives examples of representing intermediate Ehrhart quasipolynomials in terms of such functions, in which the dilated functions $(\{4t\}_1)^2$ and $\{-5t\}_2$ appear.
Dilated Floor Functions -4

- Dilated functions without the discretization: linear functions

\[ \ell_\alpha(x) = \alpha x. \]

- Fact. Linear functions commute under composition, and satisfy for all \( \alpha, \beta \in \mathbb{R} \),

\[ \ell_\alpha \circ \ell_\beta(x) = \ell_\beta \circ \ell_\alpha(x) = \ell_{\alpha \beta}(x). \]

for all \( x \in \mathbb{R} \).

- General Question. Discretization destroys the convexity of linear functions. It generally destroys commutativity under composition. What properties remain?
Part II. Dilated Floor Functions that Commute

- **Question**: When do dilated floor functions commute under composition of functions?

- The question turns out to have an interesting answer.
Main Theorem

• **Theorem. (L-M-R (2016))** (Commuting Dilated Floor Functions)

The set of \((\alpha, \beta)\) for which the dilated floor functions commute

\[ \lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \text{for all} \quad x \in \mathbb{R}, \]

consists of:

(i) *Three one-parameter continuous families:* \(\alpha = 0\) or \(\beta = 0\), or \(\alpha = \beta\).

(ii) *A two-parameter discrete family:* \(\alpha = \frac{1}{m}\) and \(\beta = \frac{1}{n}\) for all integers \(m, n \geq 1\).

**Remark.** In case (ii), setting \(T_m(x) := \lfloor \frac{1}{m} x \rfloor\) we have

\[ T_m \circ T_n(x) = T_n \circ T_m(x) = T_{mn}(x) \quad \text{for all} \quad x \in \mathbb{R}, \]

for all \(m, n \geq 1\). (These are same relations as for linear functions.)
The Discrete Commuting Family

• **Claim:** Suppose $m, n \geq 1$ are integers. Then $\lfloor \frac{1}{m} \lfloor \frac{1}{n} x \rfloor \rfloor = \lfloor \frac{1}{mn} x \rfloor$.

• Exchanging $m$ and $n$, the claim implies $\lfloor \frac{1}{m} \lfloor \frac{1}{n} x \rfloor \rfloor = \lfloor \frac{1}{n} \lfloor \frac{1}{m} x \rfloor \rfloor$, which gives the commuting family.

• To prove the claim: The functions are step functions and agree at $x = 0$. We study where and how much the functions jump. The right side $\lfloor \frac{1}{mn} x \rfloor$ jumps exactly at $x$ an integer multiple of $mn$, and the jump is of size 1.

• For the left side $\lfloor \frac{1}{m} \lfloor \frac{1}{n} x \rfloor \rfloor$, the inner function $\lfloor \frac{1}{n} x \rfloor$ is always an integer, and it jumps by 1 at integer multiples of $n$. Now the outer function jumps exactly when the $k$-th integer multiple of $n$ (of the inner function) has $k$ divisible by $m$. So it jumps exactly at multiples of $mn$ and the jump is of size 1. QED.
Proof Method: Analyze Upper Level Sets

• **Definition.** The *upper level set* $S_f(y)$ of a function $f : \mathbb{R} \to \mathbb{R}$ is

$$S_f(y) := \{ x : f(x) \geq y \}.$$

• It is a closed set for the floor function (but not for the ceiling function).

**Example.** For the composition of dilated floor functions $f_\alpha \circ f_\beta(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor$ we use notation: $S_{\alpha,\beta}(y) := \{ x : \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq y \}$.

• **Key Lemma.** For $\alpha > 0$, $\beta > 0$ and $n$ an integer, the upper level set at level $y = n$ is the closed set

$$S_{\alpha,\beta}(n) = \left[ \frac{1}{\beta} \left\lfloor \frac{1}{\alpha} n \right\rfloor, +\infty \right).$$
Upper Level Sets-2

**Key Equivalence:** For $y$ equal to an *integer* $n$ the upper level set is

$$x \in S_{\alpha, \beta}(n) \iff \left\lceil \alpha \lceil \beta(x) \rceil \right\rceil \geq n \quad \text{(the definition)}$$

$$\iff \alpha \lceil \beta x \rceil \geq n \quad \text{(the right side is in } \mathbb{Z} \text{)}$$

$$\iff \left\lfloor \beta x \right\rfloor \geq \frac{1}{\alpha} n \quad \text{(since } \alpha > 0 \text{)}$$

$$\iff \left\lfloor \beta x \right\rfloor \geq \left\lfloor \frac{1}{\alpha} n \right\rfloor \quad \text{(the left side is in } \mathbb{Z} \text{)}$$

$$\iff \beta x \geq \left\lceil \frac{1}{\alpha} n \right\rceil \quad \text{(the right side is in } \mathbb{Z} \text{)}$$

$$\iff x \geq \frac{1}{\beta} \left\lceil \frac{1}{\alpha} n \right\rceil \quad \text{(since } \beta > 0 \text{)}.$$
Proof Ideas-1

• First quadrant case $\alpha > 0, \beta > 0$. Now change variables to $1/\alpha, 1/\beta$.

• Using **Key Lemma**, for commutativity to hold for (new variables) $\alpha, \beta > 0$ we need the ceiling function identities:
  \[ \beta \lceil n\alpha \rceil = \alpha \lceil n\beta \rceil \]  holds for all integer $n$.

For $n \neq 0$ rewrite this as:

\[ \frac{\alpha}{\beta} = \frac{\lceil n\alpha \rceil}{\lceil n\beta \rceil}. \]

If $\alpha, \beta$ integers this relation clearly holds for all nonzero $n$, the floor functions have no effect.

• We have to check that if $\alpha \neq \beta$ and if they are not both integers, then commutativity fails. All we have to do is pick a good $n$ to create a problem, if one is not integer. (Not too hard.)
Proof Ideas-2

• There is a **Key Lemma** for upper level sets of each of the other three sign patterns of $\alpha$ and $\beta$. (Other three quadrants). Sometimes the upper level set obtained is an open set, the finite endpoint is omitted.

• **Remark.** The discrete commuting family was used by J.-P. Cardinal (Lin. Alg. Appl. 2010) to relate the Riemann hypothesis to some interesting algebras of matrices with rational entries.
Part III. Dilated Floor Functions with Nonnegative Commutator

- The commutator function of two functions \( f(x), g(x) \) is the difference of compositions

\[
[f, g](x) := f(g(x)) - g(f(x)).
\]

- **Question:** Which dilated floor function pairs \((\alpha, \beta)\) have nonnegative commutator

\[
[f_\alpha, f_\beta] = \lceil \alpha \lfloor \beta x \rfloor \rceil - \lfloor \beta \lceil \alpha x \rceil \rfloor \geq 0 \tag{1}
\]

for all real \( x \)?

- We let \( S \) denote the set of all solutions \((\alpha, \beta)\) to (1).
\( \alpha = \frac{13}{14}, \beta = \frac{12}{13} \) (table by Jon Bober)
\[ \alpha = \frac{1}{13}, \beta = \frac{13}{14} \]  

(table by Jon Bober)
Commutator Function-2

• Reasons to study dilated floor commutators:

1. They measure deviation from commutativity, and are “quadratic" functions.

2. Non-negative commutator parameters might shed light on commuting function parameters, which are the intersection of $S$ with its reflection under the map $(\alpha, \beta) \mapsto (\beta, \alpha)$.

• For dilated floor functions the commutator function is a bounded function. It is an example of a bounded generalized polynomial in the sense of Bergelson and Leibman (Acta Math 2007). These arose in distribution modulo one, and in ergodic number theory.
Theorem. (Partial commutator inequality classification) The set $S_0$ of parameters $(\alpha, \beta) \in \mathbb{R}^2$ that satisfy the inequality

$$\alpha [\beta x] \geq \beta [\alpha x] \quad \text{for all } x \in \mathbb{R}$$

are the two coordinate axes, all points in the open second quadrant, no points in the open fourth quadrant, and:

(i) (First Quadrant) For each integer $m_1 \geq 1$ $S_0$ contains all points with $\alpha > 0$ that lie on the oblique line $\beta = m_1 \alpha$ of slope $m_1$ through the origin, i.e. $\{(\alpha, m_1 \alpha) : \alpha > 0\}$.

(iii) (Third quadrant) For each integer $m_1 \geq 1$ $S_0$ contains all points with $\alpha < 0$ that lie on the oblique line $\alpha = m_1 \beta$ of slope $\frac{1}{m_1}$ through the origin, i.e. $\{(\alpha, \frac{1}{m_1} \alpha) : \alpha < 0\}$.
“Partial Commutator" Set $S_0$
Features of “Partial Commutator" Set $S_0$

- The partial commutator solution set $S_0$ has various symmetries.

1. The set $S_0$ is reflection-symmetric around the line $\alpha + \beta = 0$.

2. The set $S_0$ is invariant under positive dilations: If $(\alpha, \beta) \in S_0$ then $(\lambda \alpha, \lambda \beta) \in S_0$ for each real $\lambda > 0$.

- **Feature.** The partial commutator solution set $S_0$ lies above or on the anti-diagonal line $\alpha = \beta$ except for parts of the two coordinate axes.

- In particular, the only solutions $(\alpha, \beta) \in S_0$ that commute are the three “trivial" continuous families: $\alpha = 0$, $\beta = 0$ and $\alpha = \beta$. 
Main Result: Classification of Nonnegative Commutator Parameters

• The main result classifies the structure of the set $S$ of $(\alpha, \beta)$ for which
  \[ [\alpha \lfloor \beta x \rfloor] - [\beta \lfloor \alpha x \rfloor] \geq 0 \quad \text{for all} \quad x \in \mathbb{R}. \]

• The set $S$ contains 2-dimensional, 1-dimensional and 0-dimensional components. These components are real semi-algebraic sets.

• The set $S$ contains the set $S_0$.

• The set $S$ has some discrete internal symmetries and also some “broken” symmetries that hold for most components but not all.

• The existence of the discrete family of solutions to the commuting dilated floor functions requires “broken” symmetries.
Classification Theorem: The Set $S$
Main Theorem-1: Second and Fourth Quadrant

**Theorem.** (Classification Theorem-1) (L & Harry Richman (2017+))

The set $S$ of all parameters $(\alpha, \beta) \in \mathbb{R}^2$ that satisfy the inequality

$$[\alpha \lfloor \beta x \rfloor] - [\beta \lfloor \alpha x \rfloor] \geq 0 \quad \text{for all} \quad x \in \mathbb{R}$$

consists of the coordinate axes $\{(\alpha, 0) : \alpha \in \mathbb{R}\}$ and $\{(0, \beta) : \beta \in \mathbb{R}\}$ together with

1. **(ii)** All points in the open second quadrant,

2. **(iv)** No points in the open fourth quadrant,

and the following points in the open first quadrant and third quadrant:
Main Theorem-2: First Quadrant

- **Theorem** (Classification Theorem-2)
  
  (i) (First Quadrant Case) Here $\alpha > 0$ and $\beta > 0$. Points in $S$ fall into three collections of one-parameter continuous families.

  (i-a) For each integer $m_1 \geq 1$ all points with $\alpha > 0$ on the **oblique line** 
  $\beta = m_1 \alpha$ of slope $m_1$ through the origin, i.e. $\{ (\alpha, m_1 \alpha) : \alpha > 0 \}$.

  (i-b) For each integer $m_2 \geq 1$ all points with $\beta > 0$ on the **vertical line** 
  $\alpha = \frac{1}{m_2}$ i.e. $\{ (\frac{1}{m_2}, \beta) : \beta > 0 \}$.

  (i-c) For each pair of integers $m_1 \geq 1$ and $m_2 \geq 1$, all points with $\beta > 0$ on the **rectangular hyperbola** 
  
  $$m_1 \alpha \beta + m_2 \alpha - \beta = 0.$$
Main Theorem-3-Third Quadrant

- **Theorem** (Classification Theorem-3)
  
  
  (iii) (Third Quadrant Case) Here $\alpha, \beta < 0$. All solutions have $|\alpha| \geq |\beta|$. They include three collections of one parameter continuous families.

(iii-a) For each integer $m_1 \geq 1$ all points with $\alpha < 0$ on the oblique line $\alpha = m_1 \beta$ of slope $\frac{1}{m_1}$ through the origin, i.e. $\{ (\alpha, \frac{1}{m_1} \alpha) : \alpha < 0 \}$.

(iii-b) For each positive rational $\frac{m_1}{m_2}$ given in lowest terms, all points $(-\frac{m_1}{m_2}, -\beta)$ on the vertical line segment $0 < \beta \leq \frac{1}{m_2}$.

(iii-c) For each pair of integers $m_1 \geq 1$ and $m_2 \geq 1$, all points having $\alpha < 0$ on the rectangular hyperbola

$$m_1 \alpha \beta + \alpha - m_2 \beta = 0.$$
Main Theorem-4-Third Quadrant

- **Theorem** (Classification Theorem-4)
  
  *In addition there are* sporadic rational solutions *in the third quadrant.*

(iii-d) For each positive rational \( \frac{m_1}{m_2} \) in lowest terms satisfying \( m_1 \geq 2 \), there are infinitely many sporadic rational solutions \((-\frac{m_1}{m_2}, -\beta)\). All such sporadic solutions have \( \frac{1}{m_2} < -\beta < \frac{2}{m_2} \), and the only limit point of such solutions is \((-\frac{m_1}{m_2}, -\frac{1}{m_2})\). There are no sporadic rational solutions having \( m_1 = 1 \).

- **The set of all sporadic rational solutions** having \( m_2 = 1 \) consists of all \((\alpha, \beta) = (-m_1, -\frac{m_1r}{m_1r-j})\), with integer parameters \((m_1, j, r)\) having \( 1 \leq j \leq m_1 - 1 \), with \( m_1 \geq 2 \) and with \( r \geq 1 \). These solutions comprise all sporadic solutions having \( \beta < -1 \).
Classification Theorem-Discussion-1

- Compared to the “partial commutator case” or the “commuting dilations case”, the first and third quadrant solutions include new continuous families of solutions. These families are parts of straight lines and parts of rectangular hyperbolas.

- **rectangular hyperbola** means: its asymptotes are parallel to the coordinate axes.

- The rectangular hyperbolas are related to *Beatty sequences*. The non-existence of first quadrant sporadic rational solutions is related to *two-dimensional Diophantine Frobenius problem*. 
Interlude: Beatty Sequences

• Given a positive real number $u > 1$, its associated Beatty sequence is

$$\mathcal{B}(u) := \{\lfloor nu \rfloor : n \geq 1\}. $$

It is a set of positive integers.

• “Beatty’s Theorem.” Two Beatty sequences $\mathcal{B}(u)$ and $\mathcal{B}(v)$ partition the positive integers, i.e.

$$\mathcal{B}(u) \cup \mathcal{B}(v) = \mathbb{N}^+, \quad \mathcal{B}(u) \cap \mathcal{B}(v) = \emptyset,$$

if and only if $u$ and $v$ lie on the rectangular hyperbola

$$\frac{1}{u} + \frac{1}{v} = 1.$$

and $u$ is irrational (whence $v$ is also irrational.)
Classification Theorem-Discussion-2

- **Scaling Symmetries.** The set $S$ is mapped into itself by some discrete families of linear maps, scaling symmetries, restricted to each quadrant. In the first quadrant, for integers $m, n \geq 1$:

\[(\alpha, \beta) \in S \implies \left(\frac{1}{n} \alpha, \frac{m}{n} \beta\right) \in S.\]

- **Birational Symmetries.** The set $S$ is mapped into itself by certain birational maps, restricted to each quadrant. In the first quadrant,

\[(\alpha, \beta) \in S \iff \left(\frac{\alpha}{\beta}, \frac{1}{\beta}\right) \in S.\]

- There are additional partially broken symmetries.
Partially Broken Symmetries

• **Broken Symmetry I.** The “Partial commutator” solutions have nothing below the anti-diagonal line $\alpha = \beta$ except the $\alpha$-axis and $\beta$-axis. The Classification Theorem breaks this restriction in first quadrant case (i-b). It adds some vertical lines which extend into region $\alpha > \beta$. *These extra solutions in $S$ were necessary to get the two-parameter discrete family where dilated floor functions commute.*

• **Broken Symmetry II.** There is a partial reflection symmetry around the diagonal line $\alpha + \beta = 0$. This was perfect for the “partial commutator” case which covers oblique line cases (i-a) matching (iii-a). It also has all the hyperbolas in case (i-c) matching hyperbolas in case (iii-c). However it is broken for straight lines in case (i-b) not matching (iii-b), and the sporadic rational solutions have no counterpart at all.
$S$ is a Closed Set

- **Corollary of Classification Theorem.** Let $S$ denote the set of all solutions $(\alpha, \beta)$ to (1). Then $S$ is a closed set in $\mathbb{R}^2$.

- This fact is not obvious a priori because the functions $f_\alpha(x)$ are discontinuous in the $x$-variable. It was proved using the classification.
Proof Ideas-1

- Symmetries of $S$ suggested birational changes of variables that simplify analysis. For example, using the change of variable $(X, Y) = (\alpha, \frac{\alpha}{\beta})$ makes all 1-dimensional solution curves linear. (See the next slide)

- Many equivalent conditions to (1) were found which helped analyze different parameter domains. (See two later slides)

- The connection with Beatty sequences (in a suitable coordinate system) allowed known machinery to analyze them to be used, going back to Thoralf Skolem (1957). Used topological dynamics of iterates of a point $\{k(\gamma, \delta) : k \geq 1\}$ on the unit square (modulo one), a torus.
$X - Y$ Coordinates: First Quadrant Solutions

$Y = \alpha / \beta$

$X = \alpha$

$1$

$1$
First Quadrant Equivalences-1

The following conditions on \((\alpha, \beta)\) in the first quadrant, are successively shown to be equivalent:

(1) original inequality: \([\alpha \lfloor \beta x \rfloor \geq \lfloor \beta \alpha x \rfloor\) for all \(x \in \mathbb{R}\)

(2) upper level set inclusions: \(S_{\alpha, \beta}(n) \supseteq S_{\beta, \alpha}(n)\) for all \(n \in \mathbb{Z}\)

(3) rounding function inequalities \(r_{\alpha}(n) \leq r_{\beta}(n)\) for all \(n \in \mathbb{Z}\)

(4) rescaled rounding function inequality \(r_1(x) \leq r_v(x)\) for all \(x \in u\mathbb{Z}\).
First Quadrant Equivalences-2

(5) disjoint residual set intersection conditions:

\[ u\mathbb{Z} \cap R_v^\pm = \emptyset \iff R_u^\pm \cap R_v^\pm = \emptyset \]

(6) reduced Beatty sequence empty intersection condition.

(7) dual Beatty sequence empty intersection condition.

(8) All solutions with real vectors \((X, Y)\) with \(0 < X, Y < 1\) satisfy some integer dependence of form \(m_1X + m_2Y = 1\) with nonnegative integers \(m_1, m_2\). Solutions with \(X \geq 1\) or \(Y \geq 1\) are accounted for separately.
Part IV. Concluding Remarks

- Why care about this problem?

- This result found is structural about fundamental functions. The answer could not be guessed in advance.

- One-sided inequalities are potentially valuable in number theory estimates.
Thank you

*Thank you for your attention!*