The $3x + 1$ Problem: An Annotated Bibliography

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ABSTRACT. The $3x + 1$ problem concerns iteration of the map $T : \mathbb{Z} \to \mathbb{Z}$ given by

$$T(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \ . \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2} \ . \end{cases}$$

The $3x + 1$ Conjecture asserts that each $m \geq 1$ has some iterate $T^{(k)}(m) = 1$. This is an annotated bibliography of work done on the $3x + 1$ problem and related problems. At present the $3x + 1$ Conjecture remains unsolved.

1. Terminology

The $3x + 1$ *function* is

$$T(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2} \ . \end{cases}$$

The $3x + 1$ *Conjecture* states that every $m \geq 1$ has some iterate $T^{(k)}(m) = 1$.

The *trajectory* or *foward orbit* of an integer m is the set

$$\{m, \ T(m) \ , \ \ T^{(2)}(m), \ldots\} \ .$$

The *stopping time* $\sigma(m)$ of m is the least k such that $T^{(k)}(m) < m$, and is ∞ if no such k exists. The *total stopping time* $\sigma_\infty(m)$ is the least k such that m iterates to 1 under k applications of the function T i.e.

$$\sigma_\infty(m) := \inf \ \{k \ : \ T^{(k)}(m) = 1\}.$$

The *scaled total stopping time* or *gamma value* $\gamma(m)$ is

$$\gamma(m) := \frac{\sigma_\infty(m)}{log \ m}$$

The *height*

$$h(m) = \sigma_\infty(m) + d(m),$$

where $d(m)$ counts the number of iterates $T^{(k)}(m) \equiv 1 \pmod 2$ for $0 \le k < \sigma_\infty(m)$. The height $h(m)$ is also the least k for which the *Collatz function*

$$C(x) = \begin{cases} 3x+1 & \text{if } x \equiv 1 \pmod 2 \ , \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2 \ , \end{cases}$$

has $C^{(k)}(m) = 1$. Finally, the function $\pi_a(x)$ counts the number of n with $|n| \le x$ whose forward orbit under T includes a.

The $3x+1$ problem remains unsolved. It has been verified up to $235 \times 2^{50} \approx 2.64 \times 10^{17}$ by an ongoing distributed computation run by E. Roosendaal (2003+). Surveys on results on the $3x+1$ problem can be found in Lagarias (1985), Müller (1991), the first chapter of Wirsching (1998a), and Chamberland (2003+).

2. Bibliography

This bibliography mainly covers research articles and survey articles on the $3x+1$ problem, with a few references to earlier history.

1. E. Akin (2003^+), *Why is the $3x+1$ Problem Hard?,* In: *Conference on Ergodic Theory* (I. Assani, Ed.), Contemp. Math., to appear.

   This paper analyzes the $3x+1$ problem by viewing the map T as acting on the domain $\mathbb{Z}_2$ of 2-adic integers. The map T is topologically conjugate over $\mathbb{Z}_2$ to the 2-adic shift map

$$S(x) = \begin{cases} \dfrac{x-1}{2} & \text{if } x \equiv 1 \pmod 2 \ , \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2 \ , \end{cases}$$

   by a conjugacy map $Q_3 : \mathbb{Z}_2 \to \mathbb{Z}_2$, i.e. $Q_3 \circ \circ T = S \circ Q_3$. The $3x+1$ Conjecture can be reformulated in terms of the behavior of h, namely that Q_3 maps $\mathbb{Z}^+$ into $\frac{1}{3}\mathbb{Z}$. Consider more generally for any odd rational a the map $T_a(x)$ which sends $x \mapsto \frac{ax+1}{2}$ or $\frac{x}{2}$, according as x is an odd or even 2-adic integer. The author observes there is an associated conjugacy map $Q_a : \mathbb{Z}_2 \to \mathbb{Z}_2$ with the same property as above, and formulates the Rationality Conjecture for Q_a, which asserts that Q_a maps the rationals with odd denominators to rationals. He shows that the Rationality conjecture is true for $a = \pm 1$ and is false for any odd rational a that is not an integer. For the remaining cases of odd integer a, where the Rationality Conjecture remains unsolved, he presents a heuristic argument suggesting that it should be true for $a = \pm 3$ and false for odd integer $|a| \ge 5$.

2. S. Albeverto, D. Merlini and R. Tartini (1989), *Una breve introduzione a diffusioni su insiemi frattali e ad alcuni essempi di sistemi dinamici semplici,* Note di matematica e fisica, Edizioni Cerfim Locarno **3** (1989), 1–39.

   This paper gives a short description of the $3x+1$ problem, along with other iteration problems.

3. J. P. Allouche (1979), *Sur la conjecture de "Syracuse-Kakutani-Collatz"*, Séminaire de Théorie des Nombres 1978–1979, Expose No. 9, 15pp., CNRS Talence (France), 1979. (MR 81g:10014).

This paper studies generalized $3x + 1$ functions of the form proposed by Hasse. These have the form

$$T(n) = T_{m,d,R}(n) := \begin{cases} \dfrac{mn + r_j}{d} & \text{if } n \equiv j \pmod{d},\ 1 \le j \le d-1 \\ \\ \dfrac{x}{d} & \text{if } n \equiv 0 \pmod{d}\ . \end{cases}$$

in which the parameters (d, m) satisfy $d \ge 2$, $gcd(m, d) = 1$, and the set $R = \{r_j : 1 \le j \le d-1\}$ has each $r_j \equiv -mj \pmod{d}$. The author notes that it is easy to show that all maps in Hasse's class with $1 \le m < d$ have a finite number of cycles, and for these maps all orbits eventually enter one of these cycles. Thus we may assume that $m > d$.

The paper proves two theorems. Theorem 1 improves on the results of Heppner (1978). Let $T(\cdot)$ be a function in Hasse's class with parameters d, m. Let $a > 1$ be fixed and set $k = \lfloor \frac{\log x}{a \log m} \rfloor$. Let A, B be two rationals with $A < B$ that are not of the form $\frac{m^i}{d^j}$ for integers $i \ge 0, j \ge 1$, and consider the counting function

$$F_{a,A,B}(x) := \sum_{n=1}^{x} \chi_{(A,B)} \left(\frac{T^{(k)}(n)}{n} \right),$$

in which $\chi_{(A,B)}(u) = 1$ if $A < u < B$ and 0 otherwise. Let C be the maximum of the denominators of A and B. Then:

(i) If $A < B < 0$ then

$$F_{a,A,B}(x) = O\left(C x^{\frac{1}{a}} \right),$$

where the constant implied by the O-symbol is absolute.

(ii) If $B > 0$ and there exists $\epsilon > 0$ such that

$$\frac{d-1}{d} \ge \frac{\log d}{\log m} + \frac{\log B}{k \log m} + \epsilon,$$

then for $k > k_0(\epsilon)$, there holds

$$F_{a,A,B}(x) = O\left(C x^{\frac{1}{a}} + x^{1 - \frac{|\log \eta|}{a \log m}} \right)$$

for some η with $0 < \eta < 1$ which depends on ϵ. This is true in particular when $m > d^{\frac{d}{d-1}}$ with B fixed and $x \to \infty$.

(iii) If $B > A > 0$ and if there exists ϵ such that

$$\frac{d-1}{d} \le \frac{\log d}{\log m} + \frac{\log A}{k \log m} - \epsilon,$$

then for $k > k_0(\epsilon)$, there holds

$$F_{a,A,B}(x) = O\left(C x^{\frac{1}{a}} + x^{1 - \frac{|\log \eta|}{a \log m}} \right)$$

for some η with $0 < \eta < 1$ which depends on ϵ. This is true in particular when $m < d^{\frac{d}{d-1}}$, with A fixed and $x \to \infty$.

Theorem 2 constructs for given values d, m with $gcd(m, d) = 1$ with $m > d \geq 2$ two functions $F(\cdot)$ and $G(\cdot)$ which fall slightly outside Hasse's class and have the following properties:

(i) The function $F(\cdot)$ has a finite number of periodic orbits, and every n when iterated under $F(\cdot)$ eventually enters one of these orbits.

(ii) Each orbit of the function $G(\cdot)$ is divergent, i.e. $|G^{(k)}(n)| \to \infty$ as $k \to \infty$, for all but a finite number of initial values n.

To define the first function $F(\cdot)$ pick an integer u such that $d < m < d^u$, and set

$$F(x) = \begin{cases} \dfrac{mx + r_j}{d} & \text{if } n \equiv j (\text{mod } d^u),\ gcd(j, d) = 1, \\[2ex] \dfrac{x}{d} + s_j & \text{if } x \equiv j \ (\text{mod } d^u),\ \ j \equiv 0 (\text{mod } d). \end{cases}$$

in which $0 < r_j < d^u$ is determined by $mj + r_j \equiv 0(\text{mod } d^u)$, and $0 \leq s_j < d^{u-1}$ is determined by $\frac{j}{d} + r_j \equiv 0(\text{mod } d^{u-1})$. The second function $G(x)$ is defined by $G(x) = F(x) + 1$.

These functions fall outside Hasse's class because each is linear on residue classes $n(\text{mod } d^u)$ for some $u \geq 2$, rather than linear on residue classes (mod d). However both these functions exhibit behavior qualitatively like functions in Hasse's class: There is a constant C such that

$$|F(n) - \frac{n}{d}| \leq C \quad \text{if} \quad n \equiv 0 \ (\text{mod } d).$$

$$|F(n) - \frac{mn}{d}| \leq C \quad \text{if} \quad n \not\equiv 0 \ (\text{mod } d).$$

and similarly for $G(n)$, taking $C = d^{u-1} + 1$. The important difference is that functions in Hasse's class are mixing on residue classes (mod d^k) for all powers of k, while the functions $F(\cdot)$ and $G(\cdot)$ are not mixing in this fashion. The nature of the non-mixing behaviors of these functions underlies the proofs of properties (i), resp. (ii) for $F(\cdot)$, resp. $G(\cdot)$.

4. P. Andaloro (2000), *On total stopping times under* $3X + 1$ *iteration,* Fibonacci Quarterly **38** (2000), 73–78. (MR 2000m:11024).

   This paper shows various results on the minimal elements having a given stopping time, where the "stopping time" is defined to be the number of odd elements in the trajectory up to and including 1. It also obtains a new congruential "sufficient set" criterion to verify the $3x + 1$ Conjecture. It shows that knowing that the $3x + 1$ Conjecture is true for all $n \equiv 1$ (mod 16) implies that it is true in general.

5. P. Andaloro (2002), *The* $3X + 1$ *problem and directed graphs,* Fibonacci Quarterly **40** (2002), 43–54. (MR 2003a:11018).

   This paper considers various "compressed" versions of the $3x + 1$ graph, in which only a subset of the vertices are retained with certain directed paths in original $3x + 1$

graph iterates of $T(\cdot)$ replaced by single directed edges. The initial "compressed" graph corresponds to odd integers, and the paper introduces two further "compressed" graphs with fewer allowed vertices. In each case, the $3x+1$ Conjecture is equivalent to the graph being weakly connected, i.e. being connected when viewed as an undirected graph. The paper shows that certain kinds of vertex pairs in such graphs are weakly connected, typically for allowed vertices in certain congruence classes (mod 2^k) for small k.

6. S. Anderson (1987), *Struggling with the* $3x{+}1$ *problem,* Math. Gazette **71** (1987), 271–274.

This paper studies simple analogues of the $3x+1$ function such as

$$g(x)=\begin{cases} x+k & \text{if } x\equiv 1 \pmod 2 ,\\ \dfrac{x}{2} & \text{if } x\equiv 0 \pmod 2 .\end{cases}$$

For $k=1$ when iterated this map gives the binary expansion of x. The paper also reformulates the $3x+1$ Conjecture using the function:

$$f(x)=\begin{cases} \dfrac{x}{3} & \text{if } x\equiv 0 \pmod 3 ,\\ \dfrac{x}{2} & \text{if } x\equiv 2 \text{ or } 4 \pmod 6 ,\\ 3x+1 & \text{if } x\equiv 1 \text{ or } \pmod 6 .\end{cases}$$

7. St. Andrei, M. Kudlek, R. St. Niculescu (2000), *Some results on the Collatz problem,* Acta Informatica **37** (2000), 145–160. (MR 2002c:11022).

This paper studies the internal structure of $3x+1$-trees. Among other things, it observes the that integers of the form $2^m k-1$ eventually iterate to the integers $3^m k-1$, respectively. and that integers of the form $2^{3m}k-5$ iterate to the integers $3^{2m}k-5$, and integers of the form $2^{11m}k-17$ iterate to the integers $3^{7m}k-17$. These facts are related to the cycles associated to -1, -5 and -17, respectively.

[This paper appears to be a revised version of the preprint by the same authors: *Chains in Collatz's Tree,* Fachbereich Informatik, U. of Hamburg, Bericht 217 (1999).]

8. St. Andrei and C. Maslagiu (1998), *About the Collatz Conjecture,* Acta Informatica **35** (1998), 167–179. (MR 99d:68097).

This paper describes two recursive algorithms for computing $3x+1$-trees, starting from a given base node. A $3x+1$-tree is a tree of inverse iterates of the function $T(.)$. The second algorithm shows a speedup of a factor of about three over the "naive' first algorithm.

9. D. Applegate and J. C. Lagarias (1995a), *Density Bounds for the* $3x+1$ *Problem I. Tree-Search Method,* Math. Comp., **64** (1995), 411–426. (MR 95c:11024)

Let $n_k(a)$ count the number of integers n having $T^{(k)}(n)=a$. Then for any $a\not\equiv 0 \pmod 3$ and sufficiently large k, $(1.299)^k\le n_k(a)\le(1.361)^k$. Let $\pi_k(a)$ count the number of $|n| \le x$ which eventually reach a under iteration by T. If $a \not\equiv 0 \pmod 3$ then $\pi_a(x) > x^{.643}$ for all sufficiently large x. The extremal distribution of number of leaves in $3x+1$ trees with root a and depth k (under iteration of T^{-1}) as a varies are computed for $k \le 30$. The proofs are computer-intensive.

10. D. Applegate and J. C. Lagarias (1995b), *Density Bounds for the $3x+1$ Problem II. Krasikov Inequalities,* Math. Comp., **64** (1995). 427–438. (MR 95c:11025)

    Let $\pi_a(x)$ count the number of $|n| \le x$ which eventually reach a under iteration by T. If $a \not\equiv 0 \pmod 3$, then $\pi_a(x) > x^{.809}$ for all sufficiently large x. It is shown that the inequalities of Krasikov (1989) can be used to construct nonlinear programming problems which yield lower bounds for the exponent γ in $\pi_a(x) > x^{\gamma}$. The exponent above was derived by computer for such a nonlinear program having about 20000 variables.

11. D. Applegate and J. C. Lagarias (1995c), *On the distribution of $3x+1$ trees,* Experimental Mathematics **4** (1995), 101–117. (MR 97e:11033).

    The extremal distribution of the number of leaves in $3x+1$ trees with root a and depth k (under iteration of T^{-1}) as a varies were computed for $k \le 30$ in Applegate and Lagarias (1995a). These data appear to have a much narrower spread around the mean value $(\frac{4}{3})^k$ of leaves in a $3x+1$ tree of depth k than is predicted by (repeated draws from) the branching process models of Lagarias and Weiss (1992). Rigorous asymptotic results are given for the branching process models.

    The paper also derives formulas for the expected number of leaves in a $3x+1$ tree of depth k whose root node is $a \pmod{3^{\ell}}$. A 3-adic limit is proved to exist almost everywhere as $k \to \infty$, the expected number of leaves being $W_\infty(a)\left(\frac{4}{3}\right)^k$ where the function $W_\infty : \mathbb{Z}_3^\times \to \mathbb{R}$ almost everywhere satisies the 3-adic functional equation

    $$W_\infty(\alpha) = \frac{3}{4}\left(W_\infty(2\alpha) + \psi(\alpha \bmod 9) W_\infty\left(\frac{2\alpha - 1}{3}\right)\right) , \qquad (*)$$

    in which $\psi(\alpha) = 1$ if $\alpha \equiv 2$ or $8 \pmod 9$ and is 0 otherwise. (Here $\mathbb{Z}_3^* = \{\alpha \in \mathbb{Z}_3 : \alpha \not\equiv 0 \pmod 3\}$). It is conjectured that W_∞ is continuous and everywhere nonzero. It is an open problem to characterize solutions of the functional equation $(*)$.

12. D. Applegate and J. C. Lagarias (2003), *Lower bounds for the total stopping time of $3x+1$ iterates,* Math. Comp. **72** (2003), 1035–1049. (eprint: `math.NT/0103054`).

    This paper proves there are infinitely many positive n which have a finite total stopping time $\sigma_\infty(n) > 6.14316 \log n$. It also shows that there is a positive c such that at least $cx^{1/60}$ of all integers $1 < n \le x$ have a finite total stopping time $\sigma_\infty(n) > 5.9 \log n$. The proofs are computer-intensive, and produce a "certificate" encoding a proof, which is based on a search of $3x+1$ trees to depth 60. The "certificates" are quite large, involving about 350 million trees for the lower bound $6.14316 \log n$, which corresponds to a density of odd integers in a trajectory (the" ones-ratio") of $\frac{14}{29} \approx 0.483$.

    This rigorous bound is below the bound $\sigma_\infty(n) \approx 6.95212 \log n$ that one expects to hold for almost all integers, which corresponds to a ones-ratio of $\frac{1}{2}$. The paper gives heuristic arguments suggesting that the method of this paper might prove $\sigma_\infty(n) \approx 6.95212 \log n$ holds for infinitely many n, but that it would likely require a search of $3x+1$ trees to depth at least 76. This would require a very large computation.

13. J. Arsac (1986), *Algorithmes pour vérifier la conjecture de Syracuse,* C. R. Acad. Sci. Paris **303**,Serie I, no. 4, (1986), 155–159. [Also: RAIRO, Inf. Théor. Appl. **21** (1987), 3–9.] (MR 87m:11128).

    This paper studies the computational complexity of algorithms to compute stopping times of $3x+1$ function on all integers below a given bound x.

14. C. Ashbacher (1992), *Further Investigations of the Wondrous Numbers,* J. Recreational Math. **24** (1992), 1–15.

    This paper numerically studies the "MU" functions $F_D(x)$ of Wiggin (1988) on $x \in \mathbb{Z}^+$ for $2 \le D \le 12$. It finds no exceptions for Wiggin's conjecture that all cycles of F_D on $\mathbb{Z}^+$ contain an integer smaller than D, for $x < 1.4 \times 10^7$. It tabulates integers in this range that have a large stopping time, and observes various patterns. These are easily explained by observing that, for $n \not\equiv 0 \pmod{D}$, $F_D^{(2)}(n) = \frac{n(D+1)-R}{D}$ if $n \equiv R \pmod{D}$, $-1 \le R \le D-2$, hence, for most n, $F_D^{(2)}(n) > n$, although F_D decreases iterates on the average.

15. A. O. L. Atkin (1966), *Comment on Problem* $63-13^*$, SIAM Review **8** (1966), 234–236.

    This comment gives more information of the problem of Klamkin (1963) concerning iteration of the original Collatz function, which is a permutation of the positive integers. Adding to the comment of Shanks (1965), he notes there is a method which in principle can determine all the cycles of a given period p of this map. This method determines upper and lower bounds on the integers that can appear in such a cycle. By computer calculation he shows that aside from the known cycles of periods 1, 2, 5 and 12 on the nonnegative integers, there are no other cycles of period less than 200.

    He gives an example casting some doubt on the heuristic of Shanks (1965) concerning the possible lengths of periods. He shows that for the related permutation $f(3n) = 4n+3, f(3n+1) = 2n, f(3n+2) = 4n+1$, which should obey a similar heuristic, that it has a cycle of period 94 (least term $n = 140$), and 94 is not a denominator of the continued fraction convergent to $\log_2 3$.

    He presents a heuristic argument asserting that the Collatz permutation should only have a finite number of cycles, since the iterates grow "on average" at an exponential rate.

16. M. R. Avedon (1997), *On primitive* 3*-smooth partitions of* n, Electronic J. Combinatorics **4** (1997), no.1 , 10pp. (MR 98a:11136).

    The author studies the number $r(n)$ of representations of n as sums of numbers of the form $2^a 3^b$ which are primitive (no summand divides another). Iterates of $3x+1$ function applied to n that get to 1 produces a representation of n of this kind. The author proves results about the maximal and average order of this function. See also Blecksmith, McCallum and Selfredge (1998) for more information.

17. R. B. Banerji (1996), *Some Properties of the* $3n+1$ *Function,* Cybernetics and Systems **27** (1996), 473–486.

    The paper derives elementary results on forward iterates of the $3x+1$ function viewed as binary integers, and on backward iterates of the map g taking odd integers to odd integers, given by

$$g(n) := \frac{3n+1}{2^k}\ , \quad \text{where} \quad 2^k || 3n+1\ .$$

Integers $n \equiv 0 (\text{mod } 3)$ have no preimages under g. If $n \not\equiv 0 (\text{mod } 3)$ define $g^{-1}(n)$ to be the unique integer t such that $g(t) = n$ and $t \not\equiv 5 (\text{mod } 8)$. Note that each odd n there are infinitely many $\tilde{t}$ with $g(\tilde{t}) = n$. If $d(n) = 4n+1$, these preimages are just $\{d^{(j)}g^{-1}(n) : j \geq 1\}$. The ternary expansion of $g^{-1}(n)$ is asserted to be computable from the ternary expansion of n by a finite automaton. The author conjectures that given any odd integer n, there is some finite k such that $(g^{-1})^{(k)}(n) \equiv 0 (\text{mod } 3)$. Here we are iterating the partially defined map

$$\begin{aligned} g^{-1}(6n+1) &= 8n+1\ , \\ g^{-1}(6n+5) &= 4n+3\ , \end{aligned}$$

and asking if some iterate is $0 (\text{mod } 3)$. The problem resembles Mahler's Z-number iteration [J. Australian Math. Soc. **8** (1968), 313–321].

18. E. Barone (1999), *A heuristic probabilistic argument for the Collatz sequence.* (Italian), Ital. J. Pure Appl. Math. **4** (1999), 151–153. (MR 2000d:11033).

    This paper presents a heuristic probabilistic argument which argues that iterates of the $3x+1$-function should decrease on average by a multiplicative factor $(\frac{3}{4})^{1/2}$ at each step. Similar arguments appear earlier in Lagarias (1985) and many other places, and trace back to the original work of Terras, Everett and Crandall.

19. E. Belaga (1998), *Reflecting on the* $3x+1$ *Mystery: Outline of a scenario,* U. Strasbourg preprint, 10 pages. http://www-irma.u-strasbg.fr/irma/publications/1998/98049.shtml

    This is an expository paper, discussing the possibility that the $3x+1$ conjecture is an undecidable problem. Various known results, pro and con, are presented.

20. E. Belaga and M. Mignotte (1999), *Embedding the* $3x+1$ *Conjecture in a* $3x+d$ *Context,* Experimental Math. **7**, No. 2 (1999), 145–151. (MR 200d:11034).

    The paper studies iteration on the positive integers of the $3x+d$ function

$$T(x) = \begin{cases} \dfrac{3x+d}{2} & \text{if } x \equiv 1 \pmod{2}\ , \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2}\ , \end{cases}$$

    where $d \geq -1$ and $d \equiv \pm 1 \pmod{6}$. It proves that there is an absolute constant c such that there are at most dk^c periodic orbits which contain at most k odd integers. Furthermore c is effectively computable. This follows using a transcendence result of Baker and Wüstholz [J. reine Angew. **442** (1993), 19–62.]

21. E. Belaga and M. Mignotte (2000), *Cyclic Structure of Dynamical Systems Associated with* $3x+d$ *Extensions of Collatz Problem,* U. Strasbourg preprint, 57 pp. http://www-irma.u-strasbg.fr/irma/publications/2000/00018.ps.gz

This paper is a theoretical and experimental study of the distribution of number and lengths of finite cycles to the $3x+d$ map, for $d \equiv \pm 1 \pmod 6$. It contains much data and a wealth of interesting results and conjectures.

22. S. Beltraminelli, D. Merlini and L. Rusconi (1994), *Orbite inverse nel problema del* $3n+1$, Note di matematica e fisica, Edizioni Cerfim Locarno **7** (1994), 325–357.

    [I have not seen this paper.]

23. L. Berg and G. Meinardus (1994), *Functional equations connected with the Collatz problem,* Results in Math. **25** (1994), 1–12. (MR 95d:11025).

    This paper reformulates the $3x+1$ problem in terms of generating functions for the $3x+1$ mapping. Let $F(z,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T^{(m)}(n) z^m y^n$, which converges for $|z| < 1$ and $|y| < \frac{2}{3}$. If $\omega = \exp\left(\frac{2\pi i}{3}\right)$, then

    $$F(z^3, y) = \frac{z^3}{(1-z^3)^2} + y\ F(z^6, y) + \frac{y}{3z} \sum_{j=0}^{2} \omega^j F(\omega^j z^2, y)\ .$$

    Consider the functional equation $h(z^3) = h(z^6) + \frac{1}{3z} \sum_{j=0}^{2} \omega^j h(\omega^j z^2)$. The $3x+1$ Conjecture is equivalent to the assertion that the only solutions of this equation holomorphic in $|z| < 1$ are $c_0 + \frac{c_1 z}{1-z}$ for complex constants c_0, c_1. If $f_m(z) = \sum_{n=0}^{\infty} T^{(m)}(n) z^n$ then $f_m(z) = p_m(z)(1-z^{2m})^{-2}$ for a polynomial $p_m(z) \in \mathbb{Z}[z]$, and various recurrence formulae for $p_m(z)$ are derived.

24. L. Berg and G. Meinardus (1995), *The 3n+1 Collatz Problem and Functional Equations,* Rostock Math. Kolloq. **48** (1995), 11-18. (MR 97e:11034).

    From Math. Reviews: "The authors obtain functional equations satisfied by the generating functions that they introduced in Berg and Meinardus (1994). They also report some results on the analytic versions of the Collatz problem."

25. D. J. Bernstein (1994), *A Non-Iterative 2-adic Statement of the* $3x+1$ *Conjecture,* Proc. Amer. Math. Soc., **121** (1994), 405–408. (MR 94h:11108).

    Let $\mathbb{Z}_2$ denote the 2-adic integers, and for $x \in \mathbb{Z}_2$ write $x = \sum_{i=0}^{\infty} 2^{d_i}$ with $0 \le d_0 < d_1 < d_2 < \cdots$. Set $\Phi(x) = -\sum_{j=0}^{\infty} \frac{1}{3^{j+1}} 2^j$. The map Φ is shown to be a homeomorphism of the 2-adic integers to itself, which is the inverse of the map Q_∞ defined in Lagarias (1985). The author proves that $\Phi \circ T \circ \Phi^{-1} = S$, where T is the $(3x+1)$-function on $\mathbb{Z}_2$, and S is the shift map

    $$S(x) = \begin{cases} \dfrac{x-1}{2} & \text{if } x \equiv 1 \pmod 2\ , \\ \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2\ . \end{cases}$$

    He shows that the $3x+1$ Conjecture is equivalent to the conjecture that $\mathbb{Z}^+ \subseteq \Phi(\frac{1}{3}\mathbb{Z})$. He rederives the known results that $\mathbb{Q} \cap \mathbb{Z}_2 \subseteq Q_\infty(\mathbb{Q} \cap \mathbb{Z}_2)$, and that Q_∞ is nowhere differentiable, cf. Müller (1991).

26. D. J. Bernstein and J. C. Lagarias (1996), *The* $3x+1$ *Conjugacy Map,* Canadian J. Math., **48** (1996), 1154-1169. (MR 98a:11027).

This paper studies the map $\Phi : \mathbb{Z}_2 \to \mathbb{Z}_2$ of Bernstein (1994) that conjugates the 2-adic shift map to the $3x+1$ function. This is the inverse of the map Q_∞ in Lagarias (1985); see also Akin (1993^+). The map $\bar{\Phi}_n \equiv \Phi \pmod{2^n}$ is a permutation of $\mathbb{Z}/2^n\mathbb{Z}$. This permutation is shown to have order 2^{n-4} for $n \geq 6$. Let $\hat{\Phi}_n$ denote the restriction of this permutation to $(\mathbb{Z}/2^n\mathbb{Z})^* = \{x : x \equiv 1 \pmod{2}\}$. The function Φ has two odd fixed points $x = -1$ and $x = 1/3$ and the 2-cycle $\{1, -1/3\}$, hence each $\hat{\Phi}_n$ inherits two 1-cycles and a 2-cycle coming from these points. Empirical evidence indicates that $\hat{\Phi}_n$ has about $2n$ fixed points for $n \leq 1000$. A heuristic argument based on this data suggests that -1 and $1/3$ are the only odd fixed points of Φ. The analogous conjugacy map $\Phi_{25,-3}$ for the '$25x-3$' problem is shown to have no nonzero fixed points.

27. J. Blazewitz and A. Pettorossi (1983), *Some properties of binary sequences useful for proving Collatz's conjecture,* J. Found. Control Engr. **8** (1983), 53–63. (MR 85e:11010, Zbl. 547.10000).

Studies the $3x+1$ Problem as a strong termination property of a term rewriting system. They view the problem as transforming binary strings into new binary strings and look in partcular at its action on the patterns 1^n, 0^n and $(10)^n$ occurring inside strings. The $3x+1$ map exhibits regular behavior relating these patterns.

28. Richard Blecksmith, Michael McCallum, and J. L. Selfridge (1998), *3-Smooth Representations of Integers,* American Math. Monthly **105** (1998), 529–543. (MR 2000a:11019).

A *3-smooth representation* of an integer n is a representation as a sum of distinct positive integers each of which has the form 2^a3^b, and no term divides any other term. This paper proves a conjecture of Erdos and Lewin that for each integer t all sufficiently large integers have a 3-smooth representation with all individual terms larger than t. They note a connection of 3-smooth representations to the $3x+1$-problem, which is that a number m iterates to 1 under the $3x+1$ function if and only if there are positive integers e and f such that $n = 2^e - 3^f m$ is a positive integer that has a 3-smooth representation with f terms in which there is one term exactly divisible by each power of three from 0 to $f-1$. The choice of e and f is not unique, if it exists.

29. C. Böhm and G. Sontacchi (1978), *On the existence of cycles of given length in integer sequences like* $x_{n+1} = x_n/2$ *if* x_n *even, and* $x_{n+1} = 3x_n + 1$ *otherwise,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **64** (1978), 260–264. (MR 83h:10030)

The authors are primarily concerned with cycles of a generalization of the $3x+1$ function. They consider the recursion in which $x_{n+1} = ax_n + b$, if a given recursive predicate $P(x_n)$ is true, and $x_{n+1} = cx_n + d$ if the predicate $P(x_n)$ false, where a, b, c, d and x_n are rational numbers. They observe that as a consequence of linearity alone there are at most 2^k possible cycles of period k, corresponding to all possible sequences of "true" and "false" of length n. Furthermore one can effectively determine the 2^n rationals that are the solutions to each of these equations and check if they give cycles. Thus in principle one can determine all cycles below any given finite bound. They observe that a rational number x in a cycle of the $3x+1$-function $T(\cdot)$ of period n necessarily has the form

$$x = \frac{\sum_{k=0}^{n-1} 3^{m-k-1} 2^{v_k}}{2^n - 3^m}$$

with $0 \le v_0 < v_1 < \ldots < v_m = n$. They deduce that every integer x in a cycle of length n necessarily has $|x| < 3^n$.

Further study of rational cycles of the $3x+1$ function appears in Lagarias (1990).

30. K. Borovkov and D. Pfeifer (2000), *Estimates for the Syracuse problem via a probabilistic model,* Theory of Probability and its Applications **45**, No. 2 (2000), 300–310. (MR 1 967 765).

    This paper studies a multiplicative random walk imitating the $3x+1$ iteration. Let X_0 be given and set $Y_j = X_0 X_1 \cdots X_j$ where each X_i for $i \ge 1$ are i.i.d. random variables assuming the value $\frac{1}{2}$ or $\frac{3}{2}$ with probability $\frac{1}{2}$ each. Let $\sigma_\infty(X_0, \omega)$ be a random variable equal to the smallest J such that $Y_j < 1$. Then $E[\sigma_\infty(X_0, \omega)] = (\frac{1}{2}\log\frac{4}{3})^{-1}\log X_0$ and the normalized variable

    $$\hat{\sigma}_\infty(X_0, \omega) = \frac{\sigma_\infty(X_0, \omega) - c_1 \log X_0}{c_2 (\log X_0)^{1/2}}$$

    with $c_1 = (\frac{1}{2}\log\frac{4}{3})^{-1} = 6.95212$ and $c_2 = c_1^{3/2}(\frac{1}{2}\log 3)$, has a distribution converging to a standard normal distribution as $X_0 \to \infty$ (Theorem 5). Various refinements of this result are given. Comparisons are made to empirical $3x+1$ data for values around $n \approx 10^6$, which show good agreement.

31. D. Boyd (1985), *Which rationals are ratios of Pisot sequences?,* Canad. Math. Bull. **28** (1985), 343–349. (MR 86j:11078).

    The Pisot sequence $E(a_0, a_1)$ is defined by $a_{n+2} = \left[\frac{a_{n+1}^2}{a_n} + \frac{1}{2}\right]$, where a_0, a_1 are integer starting values. If $0 < a_0 < a_1$ then $\frac{a_n}{a_n+1}$ converges to a limit θ as $n \to \infty$. The paper asks: which rationals $\frac{p}{q}$ can occur as a limit? If $\frac{p}{q} > \frac{q}{2}$ then $\frac{p}{q}$ must be an integer. If $\frac{p}{q} < \frac{q}{2}$ the question is related to a stopping time problem resembling the $3x+1$ problem.

32. B. Brent (2002), $3X+1$ *dynamics on rationals with fixed denominator,* `eprint: arXiv math.DS/0204170`.

    This paper reports on computer experiments looking for cycles with greatest common divior 1 for the $3x+k$ problem, for various $k \equiv \pm 1 \pmod 6$. It suggests that for $k = 7$, $k = 19$ and $k = 31$ there is only one such cycle on the positive integers. Various other statistics are reported on.

33. S. Brocco (1995), *A Note on Mignosi's Generalization of the* $3x+1$ *Problem,* J. Number Theory, **52** (1995), 173–178. (MR 96d:11025).

    F. Mignosi (1995) studied the function $T_\beta : \mathbb{N} \to \mathbb{N}$ defined by

    $$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \,, \end{cases}$$

where $\lceil x \rceil$ denotes the smallest integer $n \geq x$. He also formulated Conjecture C_β asserting that T_β has finitely many cycles and that every $n \in \mathbb{N}$ eventually enters a cycle under T_β. This paper shows that Conjecture C_β is false whenever β is a Pisot number or a Salem number. The result applies further to functions

$$T_{\beta,\alpha}(n) = \begin{cases} \lceil \beta n + \alpha \rceil & \text{if } n \equiv 1 \pmod 2 , \\ \dfrac{n}{2} & \text{if } n \equiv 0 \pmod 2 , \end{cases}$$

for certain ranges of values of β and α.

34. T. Brox (2000), *Collatz cycles with few descents,* Acta Arithmetica **92** (2000), 181–188. (MR 2001a:11032)

This paper considers the variant of the $3x + 1$ map, call it T_1, that divides out all powers of 2 at each step. A cycle is written $\{x_1, x_2, ...x_m\}$ where each x_i is an odd integer. A *descent* means $|T_1(x)| < |x|$. Let $|C|$ denote the number of (odd) elements in a $3x + 1$ cycle C, and let $d(C)$ denote the number of descents in C. The author proves that the the number of $3x + 1$ cycles C that have $d(C) < 2log|C|$ is finite. In particular the number of Collatz cycles with $d(C) < r$ is finite for any fixed r. The proof uses a bound of Baker and Feldman and is in principle effective. Thus for each r there exists an algorithm to determine all cycles with $d(C) < r$. This greatly strengthens the result of R. Steiner (1977) who detemined all cycles with one descent.

35. S. Burckel (1994), *Functional equations associated with congruential functions,* Theoretical Computer Science **123** (1994), 397–406. (MR 94m.11147).

The author proves undecidability results for periodically linear functions generalizing those of J. H. Conway [Proc. 1972 Number Theory Conference, Boulder, Colorado, 49–52]. A periodically linear function $f : \mathbb{Z} \to \mathbb{Z}$ is one which is a linear function on each congruence class $(\text{mod} L)$ for some finite L. The author shows it is undecidable whether a given function has $f^{(k)}(1) = 0$ for some $k \geq 1$, and also whether a given function has the property: for each $n \geq 1$, some $f^{(k)}(n) = 0$. He also shows that the $3x + 1$ conjecture is equivalent to a certain functional equation having only the trivial solution over the set of all power-series $R(z) = \sum_{n=}^{\infty} a_n z^n$ with all $a_i = 0$ or 1. The functional equation is

$$3z^3 R(z^3) - 3z^9 R(z^6) - R(z^2) - R(\omega z^2) - R(\omega^2 z^2) = 0$$

where $\omega = \exp(\frac{2\pi i}{3})$.

36. R. N. Buttsworth and K. R. Matthews (1990), *On some Markov matrices arising from the generalized Collatz mapping,* Acta Arithmetica **55** (1990), 43–57. (MR 92a:11016).

This paper studies maps $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i (\text{mod } d)$, where $r_i \equiv i m_i (\text{mod } d)$. In the case where g.c.d. $(m_0, \ldots, m_{d-1}, d) = 1$ it gives information about the structure of T-ergodic sets (mod m) as m varies. A set $S \subseteq \mathbb{Z}$ is *T-ergodic* (mod m) if it is a union of k congruence classes (mod m), $S = C_1 \cup \ldots \cup C_k$, such that $T(S) \subseteq S$ and there is an n such that $C_j \cap T^{(n)}(C_i) \neq \phi$ holds for all i and j. It characterizes them in many cases.

As an example, for

$$T(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2} \ , \\ \\ \dfrac{5x-3}{2} & \text{if } x \equiv 1 \pmod{3} \ , \end{cases}$$

the ergodic set (mod m) is unique and is $\{n : n \in \mathbb{Z} \text{ and } (n, m, 15) = 1\}$, i.e. it is one of $\mathbb{Z}, \mathbb{Z} - 3, \mathbb{Z} - 5\mathbb{Z}$ or $\mathbb{Z} - 3\mathbb{Z} - 5\mathbb{Z}$ as m varies. An example is given having infinitely many different ergodic sets (mod m) as m varies.

37. C. C. Cadogan (1984), *A note on the* $3x+1$ *problem,* Caribbean J. Math. **3** No. 2 (1984), 69–72. (MR 87a:11013).

    Using an observation of S. Znam, this paper shows that to prove the $3x + 1$ Conjecture it suffices to check it for all $n \equiv 1 \pmod{4}$. Contrast this with the obvious fact that to prove the $3x+1$ Conjecture it suffices to check it for all $n \equiv 3 \pmod{4}$. See Korec and Znam (1987) for stronger results in this direction.

38. C. C. Cadogan (1996), *Exploring the* $3x+1$ *problem I.*, Caribbean J. Math. Comput. Sci. **6** (1996), 10–18. (MR 2001k:11032)

    From Math. Reviews: "The author makes some observations concerning the $3x + 1$ problem." [I have not seen this paper.]

39. C. C. Cadogan (2003), *Trajectories in the* $3x+1$ *problem,* J. of Combinatorial Mathematics and Combinatorial Computing, **44** (2003), 177–187.

    This paper descrubes various pairs of trajectories that coalesce under the $3x + 1$ iteration. For example the trajectories of $3n + 1$ and $4n + 1$ coalesce, and the trajectory of $16k + 13$ coalesces with that of $3k + 4$. The main theorem 3.9 gives a certain infinite family of coalescences.

40. M. Chamberland (1996), *A Continuous Extension of the* $3x+1$ *Problem to the Real Line,* Dynamics of Continuous, Discrete and Impulsive Dynamical Systems, **2** (1996), 495–509. (MR 97f:39031).

    This paper studies the iterates on $\mathbb{R}$ of the function

$$\begin{aligned} f(x) &= \frac{\pi x}{2}\left(\cos\frac{\pi x}{2}\right)^2 + \frac{3x+1}{2}\left(\sin\frac{\pi x}{2}\right)^2 \\ &= x + \frac{1}{4} - \left(\frac{x}{2} + \frac{1}{4}\right)\cos \pi x \ . \end{aligned}$$

which interpolates the $3x + 1$ function $T(\cdot)$. A fact crucial to the analysis is that f has negative Schwartzian derivative $Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ on $\mathbb{R}^+$. On the interval $[0, \mu_1)$, where $\mu_1 = 0.27773\ldots$ all iterates of f contract to a fixed point 0. Here μ_n denotes the $n-th$ positive fixed point of f. The interval $[\mu_1, \mu_3]$ is invariant under f, where $\mu_3 = 2.44570\ldots$ and this interval includes the trivial cycle $A_1 = \{1, 2\}$. On this interval almost every point is attracted to one of two attracting cycles, which are A_1 and $A_2 = \{1.19253\ldots,\ 2.13865\ldots\}$. There is also an uncountable set of measure 0 on which the dynamics is "chaotic." On the interval $[\mu_3, \infty)$ the set of x that do not eventually iterate to a point in $[\mu_1, \mu_3]$ is conjectured to be of measure zero. The point μ_3 is proved to be a "homoclinic point," in the sense that for any $\epsilon > 0$ the iterates of $[\mu_3, \mu_3 + \epsilon)$ cover the whole interval (μ_1, ∞). It is shown that any nontrivial cycle of the $3x + 1$ function on the positive integers would be an attracting periodic orbit of f.

41. M. Chamberland (2003+), *An update on the* $3x + 1$ *problem (Catalan)*, Butletti de la Societat Catalana, to appear.

    This is a survey paper describing recent results on the $3x + 1$ problem, classified by area. An English version of the paper will eventually be posted on the author's webpage.

42. Busiso P. Chisala (1994), *Cycles in Collatz Sequences,* Publ. Math. Debrecen **45** (1994), 35–39. (MR 95h:11019).

    The author shows that for any m-cycle of the Collatz map on positive *rationals*, the least element is at least as large as $(2^{[m\theta]/m} - 3)^{-1}$, where $\theta = \log_2 3$. Using this result, he derives a lower bound for $3x + 1$ cycle lengths based on the continued fraction of $\theta = [1, a_1, a_2, a_3, \ldots]$, in which the n-th convergent is $\frac{p_n}{q_n}$, and the intermediate convergent denominator q_n^i is $iq_{n+1} + q_n$ for $0 \le i < a_{n+1}$. If the $3x + 1$ conjecture is true for $1 \le n \le N$, and $N \ge (2^{C(i,k)} - 3)^{-1}$, where $C(i,k) = \frac{\lceil q_k^i \theta \rceil}{q_k^i}$, then there are no nontrivial cycles of the $3x+1$ function on $\mathbb{Z}^+$ containing less than q_k^{i+1} *odd* terms. Using the known bound $N = 2^{40} \doteq 1.2 \times 10^{12}$, the author shows that there are at least $q_{15} = 10\ 787\ 915$ odd terms in any cycle of the $3x + 1$ function on $\mathbb{Z}^+$.
    [Compare these results with those of Eliahou (1993).]

43. Dean Clark (1995), *Second-Order Difference Equations Related to the Collatz* $3n+1$ *Conjecture,* J. Difference Equations & Appl., **1** (1995), 73–85. (MR 96e:11031).

    The paper studies the integer-valued recurrence $\frac{x_{n+1}+x_n}{2}$ if $x_{n+1} + x_n$ is even, and $x_n = \frac{b|x_{n+1}-x_n|+1}{2}$ if $x_{n+1} + x_n$ is odd, for $b \ge 1$ an odd integer. For $b = 1, 3, 5$ all recurrence sequences stabilize at some fixed point depending on x_1 and x_2, provided that $x_1 = x_2 \equiv \frac{b+1}{2} \pmod{b}$. For $b \ge 7$ there exist unbounded trajectories, and periodic trajectories of period ≥ 2. In the "convergent" cases $b = 3$ or 5 the iterates exhibit an interesting phenomenon, which the author calls *digital convergence,* where the low order digits in base b of x_n successively stabilize before the high order bits stabilize.

44. D. Clark and J. T. Lewis (1995), *A Collatz-Type Difference Equation,* Proc. Twenty-sixth Internationsal Conference on Combinatorics, Graph Theory and Computing (Boca Raton 1995), Congr. Numer. **111** (1995), 129-135. (MR 98b:11008).

    This paper studies the difference equation

$$x_n = \begin{cases} \dfrac{x_{n-1} + x_{n-2}}{2} & \text{if } x_{n-1} + x_{n-2} \text{ is even }, \\ \\ x_{n-1} - x_{n-2} & \text{if } x_{n-1} + x_{n-2} \text{ is odd }. \end{cases}$$

Without loss of generality $(x_0, x_1) = 1$. For such initial conditions the recurrence always converges to one of the 1-cycles 1 or -1 or to the 6-cycle $\{3, 2, -1, -3, -2, -1\}$.

45. T. Cloney, E. C. Goles and G. Y. Vichniac (1987), The $3x + 1$ Problem: a Quasi-Cellular Automaton, Complex Systems **1**(1987), 349–360. (MR 88d:68080).

    The paper presents computer graphics pictures of binary expansions of $\{T^{(i)}(m) : i = 1, 2, \ldots\}$ for "random" large m, using black and white pixels to represent 1 resp. 0 (mod 2). It discusses patterns seen in these pictures. There are no theorems.

46. L. Collatz (1986), *On the Origin of the* $(3n + 1)$*-Problem,* J. of Qufu Normal University, Natural Science Edition **12** (1986) No. 3, 9–11 (Chinese, transcribed by Zhi-Ping Ren). [German translation: Über den Ursprung des $(3n + 1)$-Problems, by Zhang-Zheng Yu 1991, U. Hamburg]

    Lothar Collatz describes his interest since 1928 in iteration problems represented using associated graphs and hypergraphs. He describes the structure of such graphs for several different problems. He states that he invented the $3x + 1$ problem and publicized it in many talks. He says: "Because I couldn't solve it I never published anything. In 1952 when I came to Hamburg I told it to my colleague Prof. Dr. Helmut Hasse. He was very interested in it. He circulated the problem in seminars and in other countries."

47. J. H. Conway (1972), *Unpredicatable Iterations*, In: Proc. 1972 Number Theory Conference, University of Colorado, Boulder, CO. 1972, pp. 49–52. (MR 52 #13717).

    This paper states the $3x+1$ problem, and shows that a more general function iteration problem similar in form to the $3x + 1$ problem is computationally undecidable.

    The paper considers functions $f : \mathbb{Z} \to \mathbb{Z}$ for which there exists a finite modulus N and rational numbers $\{a_j : 0 \le j \le N - 1\}$ such that

    $$g(n) = a_j n \quad \text{if} \;\; n \equiv j \pmod{N}.$$

    In order that the map take integers to integers it is necessary that the denominator of a_j divide $gcd(j, N)$. The computationally undecidable question becomes: Given an $f(\cdot)$ in this class and an input value $n = 2^k$ decide whether or not some iterate of n is a power of 2. More precisely, he shows that for any recursive function $f(n)$ there exists a choice of $g(\cdot)$ such that for each n there holds $2^{f(n)} = g^k(2^n)$ for some $k \ge 1$ and this is the smallest value of k for which the iterate is a power of 2. It follows that there is no decision procedure to recognize if the iteration of such a function, starting from input $n = 2^j$, will ever encounter another power of 2, particularly whether it will encounter the value 1.

    The proof uses an encoding of computations in the exponents e_p of a multiplicative factorization $n = 2^{e_2} \cdot 3^{e_3} \cdot ..$, in which only a fixed finite number of exponents $(e_2, e_3, ..., e_{p_r})$ control the computation, corresponding to the primes dividing the numerators and denominators of all a_j. The computation is based on a machine model with a finite number registers storing integers of arbitrary size, (the exponents $(e_2, e_3, ..., e_{p_r})$) and there is a finite state controller. These are called *Minsky machines* in the literature, and are described in Chapter 11 of M. Minsky, *Computation: Finite and Infinite Machines*, Prentice-Hall: Englewood Cliffs, NJ 1967 (especially Sec. 11.1).

Conway (1987) later formalized this computational model as FRACTRAN, and also constructed a universal function $f(\cdot)$.

48. J. H. Conway (1987), *FRACTRAN- A Simple Universal Computing Language for Arithmetic,* in: *Open Problems in Communication and Computation* (T. M. Cover and B. Gopinath, Eds.), Springer-Verlag: New York 1987, pp. 3–27. (MR 89c:94003).

FRACTRAN is a method of universal computation based on Conway's earlier analysis in "Unpredictable Iterations." Successive computations are done by multiplying the current value of the computation, an integer, by one of a finite list of fractions, according to a definite rule which guarantees that the resulting value is still an integer. When the computation starts with input 2^n it is said to *halt* at the first value encountered which is a power of 2, say $2^{f(n)}$, and $f(n)$ is undefined if the program never halts. A FRACTRAN program is regarded as computing the function $\{f(n) \ : \ n \in \mathbb{Z}_{>0}\}$. FRACTRAN programs can compute any partial recursive function. The paper gives a number of examples of FRACTRAN programs, e.g. for computing the decimal digits of π, and for computing the successive primes. The prime producing algorithm was described earlier in Guy (1983b).

Section 11 of the paper discusses generalizations of the 3x+1 problem encoded as FRACTRAN programs, including a fixed such function for which the halting problem is undecidable.

49. R. E. Crandall (1978), *On the "$3x + 1$" problem*, Math. Comp. **32** (1978), 1281–1292. (MR 58 #494).

This paper studies iteration of the "3x+1" map and more generally the "$qx + r$" map

$$T_{q,r}(x) = \begin{cases} \frac{qx+r}{2} & \text{if } x \equiv 1 \pmod{2} . \\ \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} . \end{cases}$$

in which $q > 1$ and $r \geq 1$ are both odd integers. He actually considers iteration of the map $C_{q,r}(\cdot)$ acting on the domain of positive odd integers, given by

$$C_{q,r}(x) = \frac{qx + r}{2^{e_2(qx+r)}},$$

where $e_2(x)$ denotes the highest power of 2 dividing x.

Most results of the paper concern the map $C_{3,1}(\cdot)$ corresponding to the $3x$+1 map. He first presents a heuristic probabilistic argument why iterates of $C_{3,1}(\cdot)$ should decrease at an exponential rate, based on this he formulates a conjecture that the number of steps $H(x)$ starting from x needed to reach 1 under iteration of $C_{3,1}(\cdot)$ should be approximately $H(x) \approx \frac{\log x}{\log \frac{16}{9}}$ for most integers. He proves that the number of odd integers n taking exactly h steps to reach 1 is at least $\frac{1][h!}{(} \log_2 x)^h$. He deduces that the function $\pi_1(x)$ which counts the number of odd integers below x that eventually reach 1 under iteration of $C_{3,1}(\cdot)$ has $\pi_1(x) > x^c$ for a positive constant c. (He does not compute its value, but his proof seems to give $c = 0.05$.) He shows there are no cycles of length less than 17985 aside from the trivial cycle, using approximations to $\log_2 3$.

Concerning the "$qx + r$" problem, he formulates the conjecture that, aside from $(q, r) = (3, 1)$, every map $C_{q,r}(\cdot)$ has at least one orbit that never visits 1. He proves that this conjecture is true whenever $r \geq 3$, and in the remaining case $r = 1$ he proves it for $q = 5$, $q = 181$ and $q = 1093$. For the first two cases he exhibits a periodic orbit not containing 1, while for $q = 1093$ he uses the fact that there are no numbers of height 2 above 1, based on the congruence $2^{q-1} \equiv 1 \pmod{q^2}$. (This last argument would apply as well to $q = 3511$.) He argues the conjecture is true in the remaining cases because a heuristic probabilistic argument suggests that for each $q \geq 5$ the "$qx + 1$" problem should have a divergent trajectory.

50. J. L. Davison (1977), *Some Comments on an Iteration Problem*, Proc. 6-th Manitoba Conf. On Numerical Mathematics, and Computing (Univ. of Manitoba-Winnipeg 1976), Congressus Numerantium XVIII, Utilitas Math.: Winnipeg, Manitoba 1977, pp. 55–59. (MR 58 #31773).

    The author considers iteration of the map $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2, n > 1 \\ \frac{3n+1}{2} & \text{if } n \equiv 0 \pmod 2 \\ 1 & \text{if } n = 1, \end{cases}$$

    This is essentially the $3x + 1$ function, except for $n = 1$. He calls a sequence of iterates of this map a *circuit* if it starts with an odd number n, produces a sequence of odd numbers followed by a sequence of even numbers, ending at an odd number n^*. A circuit is a *cycle* if $n = n^*$.

    Based on computer evidence, he conjectures that the number of circuits required during the iteration of a number n to 1 is at most $K \log n$, for some absolute constant K. He presents a probabilistic heuristic argument in support of this conjecture.

    He asks whether a circult can ever be a cycle. He shows that this question can be formulated as the exponential Diophantine equation: There exists a circuit that is a cycle if and only if there exist positive integers (k, l, h) satisfying

$$(2^{k+l} - 3^k)h = 2^l - 1.$$

    The trivial cycle $\{1, 2\}$ of the $3x + 1$ map corresponds to the solution $(k, l, h) = (1, 1, 1)$, Davison states he has been unable to find any other solutions. He notes that the $5x + 1$ problem has a circuit that is a cycle.

    Steiner (1978) subsequently showed that $(1, 1, 1)$ is the only positive solution to the exponential Diophantine equation above.

51. P. Devienne, P. Lebègue, J.-C. Routier (1993), *Halting Problem of One Binary Horn Clause is Undecidable, Proceedings of STACS 1993,* Lecture Notes in Computer Science No. **665**, Springer–Verlag 1993, pp. 48–57. (MR 95e:03114).

    The halting problem for derivations using a single binary Horn clause for reductions is shown to be undecidable, by encoding Conway's problem of iterating periodically linear functions having no constant terms [Proc. 1972 Number Theory Conf., Boulder, Univ. of

Colorado, pp. 49–52]. In contrast, the problem of whether or not ground can be reached using reductions by a single binary Horn clause is decidable. [M. Schmidt-Schauss, Theor. Comp. Sci. **59** (1988), 287–296.]

52. J. M. Dolan, A. F. Gilman and S. Manickam (1987), *A generalization of Everett's result on the Collatz* $3x+1$ *problem,* Adv. Appl. Math. **8** (1987), 405–409. (MR 89a:11018).

    This paper shows that for any $k \geq 1$, the set of $m \in \mathbb{Z}+$ having k distinct iterates $T^{(i)}(m) < m$ has density one.

53. J. P. Dumont and C. A. Reiter (2001), *Visualizing Generalized 3x+1 Function Dynamics,* Computers and Graphics **25** (2001), 883–898.

    This paper describes numerical and graphical experiments iterating generalizations of the $3x+1$ function. It plots basins of attraction and false color pictures of escape times for various generalizations of the $3x+1$ function to the real line, as in Chamberland (1996), and to the complex plane, as in Letherman, Schleicher and Wood (1999). It introduces a new generalization to the complex plane , the *winding 3x+1 function*,

$$W(z) := \frac{1}{2}\left(3^{mod_2(z)}z + mod_2(z)\right),$$

    in which

$$mod_2(z) := \frac{1}{2}(1 - e^{\pi i z}) = (\sin\frac{\pi z}{2})^2 - \frac{i}{2}\sin\pi z,$$

    Plots of complex basins of attraction for Chamberland's function appear to have a structure resembling the Mandlebrot set, while the basins of attraction of the winding $3x+1$ function seems to have a rather different structure. The programs were written in the computer language J.

54. J. P. Dumont and C. A. Reiter (2003), *Real dynamics of a* 3*-power extension of the* $3x+1$ *function,* Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis **10** (2003), 875–893.

    This paper studies the real dynamics of the function $T(x) := \frac{1}{2}(3^{mod2(x)}x + mod2(x))$, in which $mod(2)(x) := (\sin\frac{1}{2}\pi x)^2$. This function agrees with the $3x+1$-function on the integers. The authors show that this function has negative Schwartzian derivative on the region $x > 0$. They study its periodic orbits and critical points, and show that any cycle of positive integers is attractive. They define an extension of the notion of total stopping time and conjecture that around each odd integer there is an open set of reals having the same total stopping time.

55. P. Eisele and K. P. Hadeler (1990), *Game of Cards, Dynamical Systems, and a Characterization of the Floor and Ceiling Functions,* Amer. Math. Monthly **97** (1990), 466–477. (MR 91h:58086).

    Studies iteration of the mappings $f(x) = a + \lceil\frac{x}{b}\rceil$ on $\mathbb{Z}$ where a, b are positive integers. These are periodical linear functions (mod b). For $b \geq 2$, every trajectory becomes eventually constant or reaches a cycle of order 2.

56. S. Eliahou (1993), *The $3x+1$ problem: New Lower Bounds on Nontrivial Cycle Lengths,* Discrete Math., **118**(1993), 45–56. (MR 94h:11017).

    The author shows that any nontrivial cycle on $\mathbb{Z}+$ of the $3x+1$ function $T(x)$ has period $p = 301994A + 17087915B + 85137581C$ with A, B, C nonnegative integers where $B \geq 1$, and at least one of A or C is zero. Hence the minimal possible period length is at least 17087915. The method uses the continued fraction expansion of $\log_2 3$, and the truth of the $3x+1$ Conjecture for all $n < 2^{40}$.

57. P. D. T. A. Elliott (1985), *Arithmetic Functions and Integer Products,* Springer-Verlag, New York 1985. (MR 86j:11095)

    An *additive function* is a function with domain $\mathbb{Z}^+$, which satisfies $f(ab) = f(a) + f(b)$ if $(a,b) = 1$. In Chapters 1–3 Elliott studies additive functions having the property that $|f(an+b) - f(An+B)| \leq c_0$ for all $n \geq n_0$, for fixed positive integers a, b, A, B with $\det \begin{bmatrix} a & b \\ A & B \end{bmatrix} \neq 0$, and deduces that $|f(n)| \leq c_1(\log n)^3$. For the special case $\begin{bmatrix} a & b \\ A & B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ an earlier argument of Wirsching yields a bound $|f(n)| \leq c_2(\log n)$. On page 19 Elliott indicates that the analogue of Wirsching's argument for $\begin{bmatrix} a & b \\ A & B \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ leads to the $3x+1$ function, and implies that $|f(n)| \leq c_3\sigma_\infty(n)$. A strong form of the $3x+1$ Conjecture claims that $\sigma_\infty(n) \leq c_4 \log n$, see Lagarias and Weiss (1992). Elliott proves elsewhere by other arguments that in fact $|f(n)| \leq c_4 \log n$ holds. [P. D. T. A. Elliott, J. Number Theory **16** (1983), 285–310]

58. C. J. Everett (1977), *Iteration of the number theoretic function* $f(2n) = n, f(2n+1) = 3n+2$, Advances in Math. **25** (1977), 42–45. (MR 56#15552).

    This is one of the first research papers specifically on the $3x+1$ function. Note that $f(\cdot)$ is the $3x+1$-function $T(\cdot)$. The author shows that the set of positive integers n having some iterate $T^{(k)}(n) < n$ has natural density one. The result was obtained independently and contemporaneously by Terras (1976).

59. C. Farruggia, M. Lawrence and B. Waterhouse (1996), The elimination of a family of periodic parity vectors in the $3x+1$ problem, Pi Mu Epsilon J. **10** (1996), 275–280.

    This paper shows that the parity vector 10^k is not the parity vector of any integral periodic orbit of the $3x+1$ mapping whenever $k \geq 2$. (For $k = 2$ the orbit with parity vector 10 is $\{1, 2\}$.

60. M. R. Feix, A. Muriel, D. Merlini, and R. Tartini (1995), The $(3x+1)/2$ Problem: A Statistical Approach, in: *Stochastic Processes, Physics and Geometry II,* Locarno 1991. (Eds: S. Albeverio, U. Cattaneo, D. Merlini) World Scientific, 1995, pp. 289–300.

    Heuristic study of random walk models imitating "average" behavior of the $3x+1$ function.

61. M. R. Feix, M. Muriel and J. L. Rouet (1994), *Statistical Properties of an Iterated Arithmetic Mapping,* J. Stat. Phys. **76** (1994), 725–741. (MR 96b:11021).

    This paper interprets iteration of the $3x+1$ map as exhibiting a "forgetting" mechanism concerning the iterates (mod 2^k), i.e. after k iterations starting from elements it draws from a fixed residue class (mod 2^k), the iterate $T^k(n)$ is uniformly distributed (mod 2^k). It proves that certain associated $2^k \times 2^k$ matrices M_k has $(M_k)^k = J_{2^k}$ where J_{2^k} is the doubly-stochastic $2^k \times 2^k$ matrix having all entries equal to 2^{-k}.

62. P. Filipponi (1991), *On the $3n+1$ Problem: Something Old, Something New,* Rendiconti di Mathematica, Serie VII, Roma **11** (1991), 85–103. (MR 92i:11031).

    Derives by elementary methods various facts about coalescences of trajectories and divergent trajectories. For example, the smallest counterexample n_0 to the $3x+1$ Conjecture, if one exists, must have $n_0 \equiv 7, 15, 27, 31, 39, 43, 63, 75, 79, 91 (\mathrm{mod}\ 96)$. [The final Theorem 16 has a gap in its proof, because formula (5.11) is not justified.]

63. Z. Franco (1990), *Diophantine Approximation and the $qx+1$ Problem,* Ph.D. Thesis, Univ. of Calif. at Berkeley 1990. (H. Helson, Advisor).

    This thesis considers iteration of the $qx+1$ function defined by $C_q(x) = \frac{qx+1}{2^{\mathrm{ord}_2(qx+1)}}$, where $2^{\mathrm{ord}_2(y)} || y$, and both q and x are odd integers. The first part of the thesis studies a conjecture of Crandall (1978), and the results appear in Franco and Pomerance (1995). The second part of the thesis gives a method to determine for a fixed q whether there are any orbits of period 2, i.e. solutions of $C_q^{(2)}(x) = x$, and it shows that for $|q| < 10^{11}$, only $q = \pm 1, \pm 3, 5, -11, -91$, and 181 have such orbits. The method uses an inequality of F. Beukers, [Acta Arith. **38** (1981) 389–410].

64. Z. Franco and C. Pomerance (1995), *On a Conjecture of Crandall Concerning the $qx+1$ Problem,* Math. Comp. **64** (1995), 1333–1336. (MR9 5j:11019).

    This paper considers iterates of the $qx+1$ function $C_q(x) = \frac{qx+1}{2^{\mathrm{ord}_2(qx+1)}}$, where $2^{\mathrm{ord}_2(y)} || y$ and both q and x are odd integers. Crandall (1978) conjectured that for each odd $q \geq 5$ there is some $n > 1$ such that the orbit $\{C_q^{(k)}(n) : k \geq 0\}$ does not contain 1, and proved it for $q = 5, 181$ and 1093. This paper shows that $\{q :$ Crandall's conjecture is true for $q\}$ has asymptotic density 1, by showing the stronger result that the set $\{q : C^{(2)}(m) \neq 1$ for all $m \in \mathbb{Z}\}$ has asymptotic density one.

65. D. Gale (1991), *Mathematical Entertainments: More Mysteries,* Mathematical Intelligencer **13**, No. 3, (1991), 54–55.

    This paper discusses the possible undecidability of the $3x+1$ Conjecture, and also whether the orbit containing 8 of the original Collatz function

$$U(n) = \begin{cases} \dfrac{3}{2}n & \text{if } n \equiv 0 \pmod 2 \\ \dfrac{3}{4}n + \dfrac{1}{4} & \text{if } n \equiv 1 \pmod 4 \\ \dfrac{3}{4}n - \dfrac{1}{4} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

is infinite.

66. G.-G. Gao (1993), *On consecutive numbers of the same height in the Collatz problem,* Discrete Math., **112** (1993), 261–267. (MR 94i:11018).

    This paper proves that if there exists one k-tuple of consecutive integers all having the same height and same total stopping time, then there exists infinitely many such k-tuples. (He attributes this result to P. Penning.) There is a 35654-tuple starting from $2^{500} + 1$. He conjectures that the set $\{n : C^{(k)}(n) = C^{(k)}(n+1) \text{ for some } k \leq \log_2 n\}$ has natural density one, and proves that it has a natural density which is at least 0.389.

67. M. V. Garcia and F. A. Tal (1999), *A note on the generalized* $3n+1$ *problem,* Acta Arith. **90**, No. 3 (1999), 245–250. (MR 2000i:11019).

    This paper studies the generalized $3x + 1$ function, defined for $m > d \geq 2$ by

$$H(x) = \begin{cases} \dfrac{x}{d} & \text{if } x \equiv 0 \pmod d\ , \\ \dfrac{mx - \pi(ma)}{2} & \text{if } x \equiv a \pmod d,\ a \neq 0, \end{cases}$$

    where $\pi(x)$ denotes projection (mod d) onto a fixed complete set of residues (mod d). The *Banach density* of a set $B \subset \mathbb{Z}^+$ is

$$\rho_b(B) = \limsup_{n \to \infty} (max_{a \in \mathbb{Z}^+} \frac{\sharp(B \cap \ \{a, a+1, ..., a+n-1\})}{n}).$$

    The Banach density is always defined and is at least as large as the natural density of the set B , if it exists. Call two integers m_1 and m_2 *equivalent* if there is some positive integer k such that $H^{(k)}(m_1) = H^{(k)}(m_2)$. The authors assume that $m < d^{d/d-1}$, a hypothesis which implies that almost all integers have some iterate which is smaller, and which includes the $3x+1$ function as a special case. They prove that if $\mathcal{P}$ is any complete set of representatives of equivalence classes of $\mathbb{Z}^+$ then the Banach density of $\mathcal{P}$ is zero. As a corollary they conclude that the Banach density of the orbit of any integer n under such a map H is zero. In particular, the Banach density of any divergent trajectory for such a map is zero.

68. Martin Gardner (1972), *Mathematical Games*, Scientific American **226** (1972), 114–118.

    This artice is one of the first places the $3x + 1$ problem is stated in print. Gardner attributes it to a technical report issued by M. Beeler, R. Gosper, and R. Schroeppel, HAKMEM, Memo 239, Artificial Intelligence Laboratory, M.I.T., 1972, p. 64.

69. L. E. Garner (1981), *On the Collatz* $3n+1$ *algorithm*, Proc. Amer. Math. Soc. **82** (1981), 19–22. (MR 82j:10090).

    The *coefficient stopping time* $\kappa(n)$ introduced by Terras (1976) is the least iterate k such that $T^{(k)}(n) = \alpha(n)n + \beta(n)$, with $\alpha(n) < 1$. Here $\alpha(n) = \frac{3^{a(n)}}{2^n}$ where $a(n)$ is the number of iterates $T^{(j)}(n) \equiv 1 (\bmod\ 2)$ with $0 \le j < k$. One has $\kappa(n) \le \sigma(n)$, where $\sigma(n)$ is the stopping time of n, and the Coefficient Stopping Time Conjecture of Terras (1976) asserts that $\kappa(n) = \sigma(n)$ for all $n \ge 2$. This paper proves that $\kappa(n) < 105,000$ implies that $\kappa(n) \le \sigma(n)$. The proof methods used are those of Terras (1976), who proved the conjecture holds for $\kappa(n) < 2593$. They invove the use of the continued fraction expansion of $\log_2 3$ and the truth of the $3x+1$ Conjecture for $n < 2.0 \times 10^9$.

70. L. E. Garner (1985), *On heights in the Collatz* $3n+1$ *problem,* Discrete Math. **55** (1985), 57–64. (MR 86j:11005).

    This paper shows that infinitely many pairs of consecutive integers have equal (finite) heights and equal total stopping times. To do this he studies how trajectories of consecutive integers can coalesce. Given two consecutive integers m, $m+1$ having $T^{(i)}(m) \neq T^{(i)}(m+1)$ for $i < k$ and $T^{(k)}(m) = T^{(k)}(m+1)$, associate to them the pair $(\mathbf{v}, \mathbf{v}')$ of $0-1$ vectors of length k encoding the parity of $T^{(i)}(m)$ (resp. $T^{(i)}(m+1)$) for $0 \le i \le k-1$. Call the set $\mathcal{A}$ of pairs $(\mathbf{v}, \mathbf{v}')$ obtained this way *admissible pairs.* Garner exhibits collections $\mathcal{B}$ and $\mathcal{S}$ of pairs of equal-length $0-1$ vectors $(\mathbf{b}, \mathbf{b}')$ and $(\mathbf{s}, \mathbf{s}')$ called *blocks* and *strings*, respectively, which have the properties: If $(\mathbf{v}, \mathbf{v}') \in \mathcal{A}$ and $(\mathbf{b}, \mathbf{b}') \in \mathcal{B}$ then the concatenated pair $(\mathbf{bv}, \mathbf{b}'\mathbf{v}') \in \mathcal{A}$, and if $(\mathbf{s}, \mathbf{s}') \in \mathcal{S}$ then $(\mathbf{sv}', \mathbf{s}'\mathbf{v}) \in \mathcal{A}$. Since $(001, 100) \in \mathcal{A}$, $(10, 01) \in \mathcal{B}$ and $(000011, 101000) \in \mathcal{S}$, the set $\mathcal{A}$ is infinite. He conjectures that: (1) a majority of all positive integers have the same height as an adjacent integer (2) arbitrarily long runs of integers of the same height occur.

71. H. Glaser and H.-G. Wiegand (1989), *Das-ULAM Problem-Computergestütze Entdeckungen,* DdM **2** (1989), 114–134.

    [I have not seen this paper.]

72. David Gluck and Brian D. Taylor (2002), *A new statistic for the* $3x+1$ *problem,* Proc. Amer. Math. Soc. **130** (2002), 1293–1301. (MR 2002k:11031).

    This paper considers iterations of the Collatz function $C(x)$. If $\mathbf{a} = (a_1, a_2, ..., a_n)$ is a finite Collatz trajectory starting from a_1, with $a_n = 1$ being the first time 1 is reached, they assign the statistic

    $$C(\mathbf{a}) = \frac{a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_na_1}{a_1^2 + a_2^2 + ... + a_n^2}.$$

    They prove that $\frac{9}{13} < C(\mathbf{a}) < \frac{5}{7}$. They find sequences of starting values that approach the upper and lower bounds, given that the starting values terminate.

73. G. Gonnet (1991), *Computations on the* $3n+1$ *Conjecture,* MAPLE Newsletter

    This paper describes how to write computer code to efficiently compute $3x+1$ function iterates for very large x using MAPLE. It displays a computer plot of the total stopping function for $n < 4000$, revealing an interesting structure of well-spaced clusters of points.

74. J. Goodwin (2002+), *Results on the Collatz conjecture,* preprint.

    This paper partitions the inverse iterates of 1 under the $3x + 1$ map into various subsets, and studies their internal recursive structure.

75. R. K. Guy (1981), *Unsolved Problems in Number Theory*, Springer-Verlag, New York 1981.

    The $3x + 1$ Problem is discussed as problem E16.

76. R. K. Guy (1983a) *Don't try to solve these problems!*, Amer. Math. Monthly **90** (1983), 35–41.

    The $3x + 1$ problem is stated as Problem 2. Problem 3 concerns cycles in the original Collatz problem, for which see Klamkin (1963). Problem 4 asks the question : Let S be the smallest set of positive integers containing 1 which is closed under $x \mapsto 2x$, $x \mapsto 3x + 2$, $x \mapsto 6x + 3$. Does this set have a positive lower density?

    The article gives some brief history of work on the $3x+$ problem. It mentions at second hand a statement of P. Erdős regarding the $3x + 1$ problem: "Mathematics is not yet ripe enough for such questions."

77. R. K. Guy (1983b), *Conway's prime producing machine*, Math. Magazine **56** (1983), 26–33. (MR 84j:10008).

    This paper gives a function $g(\cdot)$ of the type in Conway (1972) having the follwing property. If p_j denotes the j-th prime, given in increasing order, then starting from the value $n = 2^{p_j}$ and iterating under $g(\cdot)$, the first power of 2 that is encountered in the iteration is $2^{p_{j+1}}$.

    He shows that the associated register machine uses only four registers. See also the paper Conway (1987) on FRACTRAN.

78. R. K. Guy (1986), *John Isbell's Game of Beanstalk and John Conway's Game of Beans Don't Talk,* Math. Magazine **59** (1986), 259–269. (MR 88c:90163).

    John Isbell's game of Beanstalk has two players alternately make moves using the rule

$$n_{i+1} = \begin{cases} \dfrac{n_1}{2} & \text{if } n_i \equiv 0 \pmod 2 \ , \\ 3n_i \pm 1 & \text{if } n_i \equiv 1 \pmod 2 \ , \end{cases}$$

    where they have a choice if n_i is odd. The winner is the player who moves to 1. In Conway's game the second rule becomes $\dfrac{3n \pm 1}{2^*}$, where 2^* is the highest power of 2 that divides the numerator. It is unknown whether or not there are positions from which neither player can force a win. If there are then the $3x + 1$ problem must have a nontrivial cycle or a divergent trajectory.

79. L. Halbeisen and N. Hungerbühler (1997), *Optimal bounds for the Length of rational Collatz cycles,* Acta Arithmetica **78** (1997), 227-239. (MR 98g:11025).

The paper presents "optimal" upper bounds for the size of a minimal element in a rational cycle of length k for the $3x+1$ function. These estimates improve on Eliahou (1993) but currently do not lead to a better linear bound for nontrivial cycle length than the value 17,087,915 obtained by Eliahou. They show that if the $3x+1$ Conjecture is verified for $1 \le n \le 212,366,032,807,211$, which is about 2.1×10^{14}, then the lower bound on cycle length jumps to 102,225,496.

[The $3x+1$ conjecture now verified up to 1.9×10^{17} by E. Roosendaal, see the comment on Oliveira e Silva (1999).]

80. G. Hasenjager (1990), *Hasse's Syracuse-Problem und die Rolle der Basen,* in: *Mathesis rationis. Festschrift für Heinrich Schepers,* Nodus Publications: Müllernster 1990 (ISBN 3-89323-229-X), 329–336.

This paper, written for a philosopher's anniversary, comments on the complexity of the $3x+1$ problem compared to its simple appearance. It suggests looking for patterns in the iterates written in base 3 or base 27, rather than in base 2. Contains heuristic speculations and no theorems.

81. H. Hasse (1975), *Unsolved Problems in Elementary Number Theory,* Lectures at University of Maine (Orono), Spring 1975. Mimeographed notes.

Hasse discusses the $3x+1$ problem on pp. 23–33. He states that Thompson has proved the Finite Cycles Conjecture, but this seems not to be the case, as no subsequent publication has appeared.

Hasse's circulation of the problem motivated some of the first publications on it. He proposed a class of generalized $3x+1$ maps studied in Möller (1978) and Heppner (1978).

82. B. Hayes (1984), *Computer recreations: The ups and downs of hailstone numbers*, Scientific American **250**, No. 1 (1984), 10–16.

The author introduces the $3x+1$ problem to a general audience under yet another name — hailstone numbers.

83. E. Heppner (1978), *Eine Bemerkung zum Hasse-Syracuse Algorithmus*, Archiv. Math. **31** (1978), 317–320. (MR 80d:10007)

This paper studies iteration of generalized $3+1$ maps that belong to a class formulated by H. Hasse. This class consists of maps depending on parameters of the form

$$T(n) = T_{m,d,R}(n) := \begin{cases} \dfrac{mn + r_j}{d} & \text{if } n \equiv j \pmod{d},\ 1 \le j \le d-1 \\[2ex] \dfrac{x}{d} & \text{if } n \equiv 0 \pmod{d}\ . \end{cases}$$

in which the parameters (d, m) satisfy $d \ge 2$, $gcd(m,d) = 1$, and the set $R = \{r_j : 1 \le j \le d-1\}$ has each $r_j \equiv -mj \pmod{d}$. The qualitative behavior of iterates of these maps are shown to depend on the relative sizes of m and d.

Heppner proves that if $m < d^{d/d-1}$ then almost all iterates get smaller, in the following quantitative sense: There exist positive real numbers δ_1, δ_2 such that for any $x > d$, and $N = \lfloor \frac{\log x}{\log d} \rfloor$, there holds

$$\#\{n \le x : \ T^{(N)}(n) \ge nx^{-\delta_1}\} = O(x^{1-\delta_2}).$$

He also proves that if $m > d^{d/d-1}$ then almost all iterates get larger, in the following quantitative sense: There exist positive real numbers δ_3, δ_4 such that for any $x > d$, and $N = \lfloor \frac{\log x}{\log d} \rfloor$, there holds

$$\#\{n \le x : \ T^{(N)}(n) \le nx^{\delta_3}\} = O(x^{1-\delta_4}).$$

In these results the constants δ_j depend on d and m only while the implied constant in the O-symbols depends on m, d and R. This results improve on those of Möller (1978).

84. F. Jarvis (1989), *13, 31 and the $3x+1$ problem,* Eureka **49** (1989), 22–25.

This paper studies the function $g(n) = h(n+1) - h(n)$, where $h(n)$ is the height of n, and observes empirically that $g(n)$ appears unusually often to be representable as $13x + 31y$ with small values of x and y. It offers a heuristic explanation of this observation in terms of Diophantine approximations to $\log_6 2$. Several open problems are proposed, mostly concerning $h(n)$.

85. J. A. Joseph (1998), *A chaotic extension of the Collatz function to $\mathbb{Z}_2[i]$,* Fibonacci Quarterly **36** (1998), 309–317. (MR 99f:11026).

This paper studies the function 1 $\mathcal{F} : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$ given by

$$\mathcal{F}(\alpha) = \begin{cases} \dfrac{\alpha}{2} & \text{if } \alpha \in [0] \ , \\ \dfrac{3\alpha + 1}{2} & \text{if } \alpha \in [1] \ , \\ \dfrac{3\alpha + i}{2} & \text{if } \alpha \in [i] \ , \\ \dfrac{3\alpha + 1 + i}{2} & \text{if } \alpha \in [1+i] \ , \end{cases}$$

where $[\alpha]$ denotes the equivalence class of α in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]$. The author proves that the map $\tilde{T}$ is chaotic in the sense of Devaney [*A First Course in Dynamical Sysems: Theory and Experiment*, Addison-Wesley 1992]. He shows that $\tilde{T}$ is not conjugate to $T \times T$ via a $\mathbb{Z}_2$-module isomorphism, but is topologically conjugate to $T \times T$. This is shown using an analogue $\tilde{Q}_\infty$ of the $3x+1$ conjugacy map Q_∞ studied in Lagarias (1985), Theorem L and in Bernstein and Lagarias (1996).

86. Immo O. Kerner (2000), *Die Collatz-Ulam-Kombination CUK*, Rostocker Informatik-Berichte No. 24 (2000), preprint.

This paper attibutes the Collatz problem to Collatz in 1937. Let $c(n)$ be a function counting the number of iterations of the Collatz function $C(n)$ to get to 1, with $c(1) = 0$ and $c(3) = 7$, and so on. It reviews some basic results on the Collatz function, and introduces an auxiliary two-variable function $B(n,t)$ to describe the iteration, and plots some graphs of its behavior.

87. M. S. Klamkin (1963), *Problem* $63-13^*$, SIAM Review **5** (1963), 275–276.

He states the problem: "Consider the infinite permutation

$$P \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & ... \\ 1 & 3 & 2 & 5 & 7 & 4 & ,,, \end{pmatrix}$$

taking $n \mapsto f(n)$ where $f : \mathbb{N}^+ \to \mathbb{N}^+$ is given by $f(3n) = 2n, f(3n-1) = 4n-1, f(3n-2) = 4n-3$. We now write P as a product of cycles

$$P \equiv (1)\ (2,3)\ (4,5,7,9,6)\ (8,11,15,...$$

It is conjectured that the cycle $(8, 11, 15, ...)$ is infinite. Other problems concerning P are:

(a) Does the permutation P consist of finitely many cycles?

(b) Are there any more finite cycles than those indicated? "

This function was the original function proposed by L. Collatz in his private notes in 1932. See Shanks (1965) and Atkin (1966) for comments on this problem. This problem remains unsolved concerning the orbit of $n = 8$ and part (a). Concerning (b) one more cycle was found, of period 12 with smallest element $n = 144$. Atkin (1966) presents a heuristic argument suggesting there are finitely many cycles.

Remark. About the same time M. Klamkin and A. L. Tritter proposed a another problem concerning the orbit structure of a different infinite permutation of the integers (*Problem 5109*, Amer. Math. Monthly **70** (1963), 572–573). For this integer permutation all orbits are cycles. A solution to this problem was given by G. Bergman, Amer. Math. Monthly **71** (1964), 569–570.

88. A. V. Kontorovich and Ya. G. Sinai (2002), *Structure Theorem for (d,g,h)-maps,* Bull. Braz. Math. Soc. (N.S.) **33** (2002), 213–224. (MR 2003k:11034).

This paper studies (d,g,h)-maps, in which $g > d \geq 2$, with g relatively prime to d, and $h(n)$ is a periodic integer-valued function with period d, with $h(n) \equiv -n (\bmod\ d)$ and $0 < |h(n)| < g$. The (d,g,h)-map is defined on $\mathbb{Z} \backslash d\mathbb{Z}$ by

$$T(x) := \frac{gx + h(gx)}{d^k}, \quad \text{with} \quad d^k || gx + h(gx).$$

A path of m iterates can be specified by the values $(k_1, k_2, ..., k_m)$ and a residue class ϵ (mod dg), and set $k = k_1 + k_2 + ... + k_m$. The structure theorem states that exactly $(d-1)^m$ triples (q, r, δ) with $0 \leq q < d^k$, $0 < r < g^m$ and $\delta \in E = \{j : 1 \leq j < dg, d \nmid j, g \nmid j\}$ produce a given path, and for such a triple (q, r, δ) and all $x \in \mathbb{Z}$,

$$T^{(m)}(gd(d^k x + q) + \epsilon) = gd(g^m x + r) + \delta.$$

They deduce that, as $m \to \infty$, a properly logarithmically scaled version of iterates converges to a Brownian motion with drift $\log g - \frac{d}{d-1} \log d$. More precisely, fix m, and choose points $0 = t_0 < t_1 < ... < t_r = 1$ and set $m_i = \lfloor t_i m \rfloor$ and $y_i = \log T^{(m_i)}(x)$. Then the values $y_i - y_{i-1}$ converge to a Brownian path. These results imply that when the drift is negative, almost all trajectories have a finite stopping time with $|T^{(m)}(x)| < |x|$.

89. I. Korec (1992), *The $3x+1$ Problem, Generalized Pascal Triangles, and Cellular Automata,* Math. Slovaca, **42** (1992), 547–563. (MR 94g:11019).

    This paper shows that the iterates of the Collatz function $C(x)$ can actually be encoded in a simple way by a one-dimensional nearest-neighbor cellular automaton with 7 states. The automaton encodes the iterates of the function $C(x)$ written in base 6. The encoding is possible because the map $x \to 3x+1$ in base 6 does not have carries propagate. (Compare the $C(x)$-iterates of $x = 26$ and $x = 27$ in base 6.) The $3x+1$ Conjecture is reformulated in terms of special structural properties of the languages output by such cellular automata.

90. I. Korec (1994), *A Density Estimate for the $3x+1$ Problem,* Math. Slovaca **44** (1994), 85–89. (MR 95h:11022).

    Let $S_\beta = \{n : \text{some } T^{(k)}(n) < n^\beta\}$. This paper shows that for any $\beta > \frac{\log 3}{\log 4} \doteq .7925$ the set S_β has density one. The proof follows the method of E. Heppner [Arch. Math. **31** (1978) 317-320].

91. I. Korec and S. Znam (1987), *A Note on the $3x+1$ Problem,* Amer. Math. Monthly **94** (1987), 771–772. (MR 90g:11023).

    This paper shows that to prove the $3x+1$ Conjecture it suffices to verify it for the set of all numbers $m \equiv a \pmod{p^n}$, for any fixed $n \geq 1$, provided that 2 is a primitive root $\pmod{p}$ and $(a,p) = 1$. This set has density p^{-n}.

92. I. Krasikov (1989), *How many numbers satisfy the $3x+1$ Conjecture?,* Internatl. J. Math. & Math. Sci. **12** (1989), 791–796. (MR 91c:11013).

    This paper shows that the number of integers $\leq x$ for which the $3x+1$ function has an iterate that is 1 is at least $x^{3/7}$. More generally, if $\theta_a(x) = \{n \leq x : \text{ some } T^{(k)}(n) = a\}$, then he shows that, for $a \not\equiv 0 \pmod 3$, $\theta_a(x)$ contains at least $x^{3/7}$ elements, for large enough x. For each fixed $k \geq 2$ this paper derives a system of difference inequalities based on information (mod 3^k). The bound $x^{3/7}$ was obtained using $k = 2$, and by using larger values of k better exponents can be obtained. This was done in Applegate and Lagarias (1995b) and Krasikov and Lagarias (2003).

93. I. Krasikov and J. C. Lagarias (2003), *Bounds for the $3x+1$ problem using difference inequalities*, Acta Arithmetica **109** (2003), no. 3, 237–258. (arXiv eprint: `math.NT/0205002 v1 30 Apr 2002`).

    This paper deals with the problem of obtaining lower bounds for $\pi_a(x)$, the counting function for the number of integers $n \leq x$ that have some $3x+1$ iterate $T^{(k)}(n) = a$. It improves the nonlinear programming method given in Applegate and Lagarias (1995b) for extracting lower bounds from the inequalities of Krasikov (1989). It derives a nonlinear program family directly from the Krasikov inequalities (mod 3^k) whose associated lower bounds are expected to be the best possible derivable by this approach. The nonlinear program for $k = 11$ gives the improved lower bound: If $a \not\equiv 0 \pmod 3$, then $\pi_a(x) > x^{.841}$ for all sufficiently large x. The interest of the new nonlinear program family is the (not yet realized) hope of proving $\pi_a(x) > x^{1-\epsilon}$ by this approach, taking a sufficiently large k.

94. J. R. Kuttler (1994), *On the* $3x+1$ *Problem,* Adv. Appl. Math. **15** (1994), 183–185. (Zbl. 803.11018.)

    The author states the oft-discovered fact that $T^{(k)}(2^k n - 1) = 3^k n - 1$ and derives his main result that if r runs over all odd integers $1 \le r \le 2^k - 1$ then $T^{(k)}(2^k n + r) = 3^p n + s$, in which $p \in \{1, 2, \ldots, k\}$ and $1 \le s \le 3^p$, and each value of p occurs exactly $\binom{k-1}{p-1}$ times. Thus the density of integers with $T^{(k)}(n) > n$ is exactly $\frac{\alpha}{2^k}$, where $\alpha = \sum_{p > \theta k} \binom{k-1}{p-1}$, and $\theta = \log_2 3$. This counts the number of inflating vectors in Theorem C in Lagarias (1985).

95. J. C. Lagarias (1985), *The* $3x+1$ *problem and its generalizations,* Amer. Math. Monthly **92** (1985), 3–23. (MR 86i:11043).

    This paper is a survey with extensive bibliography of known results on $3x+1$ problem and related problems up to 1984. It also contains improvements of previous results and some new results, including in particular Theorems D, E, F, L and M.

    Theorem O has several misprints. The method of Conway (1972) gives, for any partial recursive function f, a periodic piecewise linear function g, with the property: iterating g with the starting value 2^n will never be a power of 2 if $f(n)$ is undefined, and will eventually reach a power of 2 if $f(n)$ is defined, and the first such power of 2 will be $2^{f(n)}$. Thus, in parts (ii) and (iii) of Theorem O, occurrences of n must be replaced by 2^n. On page 15, the thirteenth partial quotient of $\log_2(3)$ should be $q_{13} = 190537$.

    An updated version of this paper is accessible on the World-Wide Web:
    http://www.cecm.sfu.ca/organics/papers/lagarias/index.html, or http://www.mathsoft.com (unsolved problems menu)

96. J. C. Lagarias (1990), *The set of rational cycles for the* $3x+1$ *problem,* Acta Arithmetica **56** (1990), 33–53. (MR 91i:11024).

    This paper studies the sets of those integer cycles of

$$T_k(x) = \begin{cases} \dfrac{3x+k}{2} & \text{if } x \equiv 1 \pmod{2}, \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2}\ , \end{cases}$$

    for positive $k \equiv \pm 1 \pmod{6}$ which have $(x, k) = 1$. These correspond to rational cycles $\frac{x}{k}$ of the $3x+1$ function T. It conjectures that every T_k has such an integer cycle. It shows that infinitely many k have at least $k^{1-\epsilon}$ distinct such cycles of period at most $\log\ k$, and infinitely many k have no such cycles having period length less than $k^{1/3}$. Estimates are given for the counting function $C(k, y)$ counting the number of such cycles of T_k of period $\le y$, for all $k \le x$ with $y = \beta \log x$. In particular $C(k, 1.01k) \le 5k(\log\ k)^5$.

97. J. C. Lagarias (1999), *How random are 3x + 1 function iterates?,* in: *The Mathemagician and Pied Puzzler: A Collection in Tribute to Martin Gardner*, A. K. Peters, Ltd.: Natick, Mass. 1999, pp. 253–266.

    This paper briefly summarizes results on extreme trajectories of 3x+1 iterates, including those in Applegate and Lagarias (1995a), (1995b),(1995c) and Lagarias and

Weiss (1992). It also presents some large integers n found by V. Vyssotsky whose trajectories take $c\ log\ n$ steps to iterate to 1 for various $c > 35$. For example $n =$ 37 66497 18609 59140 59576 52867 40059 has $\sigma_\infty(n) = 2565$ and $\gamma(n) = 35.2789$. It also mentions some unsolved problems.

The examples above were the largest values of γ for $3x + 1$ iterates known at the time, but are now superseded by examples of Roosendaal (2003+) achieving $\gamma = 36.716$.

98. J. C. Lagarias, H. A. Porta and K. B. Stolarsky (1993), *Asymmetric Tent Map Expansions I. Eventually Periodic Points,* J. London Math. Soc., **47** (1993), 542–556.
(MR 94h:58139).

This paper studies the set of eventually periodic points $\mathrm{Per}(T_\alpha)$ of the asymmetric tent map

$$T_\alpha(x) = \begin{cases} \alpha x & \text{if } 0 < x \le \dfrac{1}{\alpha} \\[2ex] \dfrac{\alpha}{\alpha - 1}(1 - x) & \text{if } \dfrac{1}{\alpha} \le x < 1 \ , \end{cases}$$

where $\alpha > 1$ is real. It shows that $\mathrm{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0, 1]$ for those α such that both α and $\frac{\alpha}{\alpha-1}$ are Pisot numbers. It finds 11 such numbers, of degree up to four, and proves that the set of all such numbers is finite.

It conjectures that the property $\mathrm{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0, 1]$ holds for certain other α, including the real root of $x^5 - x^3 - 1 = 0$. The problem of proving that $\mathrm{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0, 1]$ in these cases appears analogous to the problem of proving that the $3x + 1$ function has no divergent trajectories.

[C. J. Smyth, *There are only eleven special Pisot numbers*, (Bull. London Math. Soc. **31** (1989) 1–5) proved that the set of 11 numbers found above is the complete set.]

99. J. C. Lagarias, H. A. Porta and K. B. Stolarsky (1994), *Asymmetric Tent Map Expansions II. Purely Periodic Points,* Illinois J. Math. **38** (1994), 574–588.
(MR 96b:58093, Zbl 809:11042).

This paper continues Lagarias, Porta and Stolarsky (1993). It studies the set $\mathrm{Fix}(T_\alpha)$ of purely periodic points of the asymmetric tent map $T_\alpha(\cdot)$ and the set $\mathrm{Per}_0(T_\alpha)$ with terminating T_α-expansion, in those cases wwhen α and $\frac{\alpha}{1-\alpha}$ are simultaneously Pisot numbers. It shows that $\mathrm{Fix}(T_\alpha) \subseteq \{\gamma \in \mathbb{Q}(\alpha)$ and $\sigma(\alpha) \in A_\alpha^\sigma$ for all embeddings $\sigma : \mathbb{Q}(\alpha) \to \mathbb{C}$ with $\sigma(\alpha) \ne \alpha\}$, in which each set A_α^σ is a compact set in $\mathbb{C}$ that is the attractor of a certain hyperbolic iterated function system. It shows that equality holds in this inclusion in some cases, and not in others. Some related results for $\mathrm{Per}_0(T_\alpha)$ are established.

100. J. C. Lagarias and N. J. A. Sloane (2003+), *Approximate squaring,* preprint posted on arXiv: `math.NT/0309389`.

This paper studies iteration of the "approximate squaring" map $f(x) = x\lceil x \rceil$, and asks the question whether for a rational starting value $x_0 = r > 1$ some iterate is an integer. It conjectures that the answer is always "yes", and proves it for rationals r with denominator 2. It shows that this holds for most rationals having a fixed denominator

$d \geq 3$ with an exceptional set of integers below x of size at most $O(x^{\alpha_d})$ for certain constants $0 < \alpha_d < 1$. It then considers a variant of this problem on the p-adic numbers, where an exceptional set exists and is shown to have Hausdorff dimension equal to α_p.

The paper also studies the iteration of "approximate multiplication" maps $f_r(x) = r\lceil x \rceil$, where r is a fixed rational number. It conjectures that for $r > 1$ all but a finite number of integer starting values have some subsequent iterate that is an integer, and proves this for rationals r with denominator 2. It shows for rationals r with denominator d that the size of the exceptional set of integers below x that have no integer in their forward orbit under f_r has cardinality at most $O(x^{\beta_d})$ with $\beta_d = \frac{\log(d-1)}{\log d}$. It suggests that this conjecture is likely to be hard in the general case, by noting an analogy with iteration of the map appearing in Mahler's Z-number problem, see Mahler (1968).

101. J. C. Lagarias and A. Weiss (1992), *The* $3x+1$ *Problem: Two Stochastic Models,* Annals of Applied Probability **2** (1992), 229–261. (MR 92k:60159).

This paper studies two stochastic models that mimic the "pseudorandom" behavior of the $3x+1$ function. For both models it proves that there is a constant $c_0 = 41.677...$ such that with probability one for any $\epsilon > 0$ only finitely many m take $(c_0 + \epsilon)\log m$ iterations to take the value 1, while infinitely many m take at least $(c_0 - \epsilon)\log m$ iterations to do so. This prediction is shown to be consistent with empirical data for the $3x+1$ function.

It also studies the maximum excursion

$$t(n) := \max_{k \geq 1} T^{(k)}(n)$$

and conjectures that $t(n) < n^{2+o(1)}$ as $n \to \infty$. An analog of this conjecture is proved for one stochastic model. The conjecture $t(n) \leq n^{2+o(1)}$ is consistent with empirical data of Leavens and Vermeulen (1992) for $n < 10^{12}$.

102. G. T. Leavens and M. Vermeulen (1992), $3x+1$ *Search Programs,* Computers & Mathematics, with Applications **24**, No. 11,(1992), 79–99. (MR 93k:68047).

This paper describes methods for computing $3x+1$ function iterates, and gives results of extensive computations done on a distributed network of workstations, taking an estimated 10 CPU-years in total. The $3x+1$ Conjecture is verified for all $n < 5.6 \times 10^{13}$. Presents statistics on various types of extremal trajectories of $3x+1$ iterates in this range. Gives a detailed discussion of techniques used for program optimization.

[This bound for verifying the $3x+1$ conjecure is now superseded by that of Oliveira e Silva (1999).]

103. G. M. Leigh (1986), A Markov process underlying the generalized Syracuse algorithm, Acta Arithmetica **46** (1986), 125–143. (MR 87i:11099)

This paper considers mappings $T(x) = \frac{m_i x - r_i}{d}$ if $x \equiv i \pmod{d}$, where all $r_i \equiv im_i \pmod{d}$. It presents conjectural formulae for limiting densities that divergent trajectories (should they exist) spend in a residue class $j \pmod{m}$.

104. S. Letherman, D. Schleicher and R. Wood (1999), *On the* $3X+1$ *problem and holomorphic dynamics,* Experimental Math. **8**, No. 3 (1999), 241–251. (MR 2000g:37049.)

This paper studies the class of entire functions

$$f_h(z) := \frac{z}{2} + (z + \frac{1}{2})(\frac{1 + \cos \pi z}{2}) + \frac{1}{\pi}(\frac{1}{2} - \cos \pi z) \sin \pi z + h(z)(\sin \pi z)^2,$$

in which $h(z)$ is an arbitrary entire function. Each function in this class reproduces the $3x + 1$-function on this integers, and the set of integers is contained in the set of critical points of the function. The simplest such function takes $h(z)$ to be identically zero, and is denoted $f_0(z)$. The authors study the iteration properties of this map in the complex plane $\mathbb{C}$. They show that $\mathbb{Z}$ is contained in the Fatou set of f_h. There is a classification of the connected components of the Fatou set of an entire function into six categories: (1) (periodic) immediate basins of (super-) attracting periodic points, (2) (periodic) immediate basins of attraction of rationally indifferent periodic points, (3) (periodic) Siegel disks, (4) Baker domains, (5) preperiodic components of any of the above, and (6) wandering domains. The following results all apply to the function f_0, and some of them are proved for more general f_h. The Fatou component of any integer must be of type (1), (5) or (7). The existence of a divergent trajectory is shown equivalent to f_0 containing a wandering domain containing some integer. No two integers are in the same Fatou component, except possibly -1 and -2. The real axis contains points of the Julia set between any two integers, except possibly -1 and -2. It is known that holomorphic dynamics of an entire function f is controlled by its critical points. The critical points of f_0 on the real axis consist of the integers together with $-\frac{1}{2}$. The authors would like to choose h to reduce the number of other critical points, to simplify the dynamics. However they show that any map f_h must contain at least one more critical point in addition to the critical points on the real axis (and presumably infinitely many.) The authors compare and contrast their results with the real dynamics studied in Chamberland (1996).

105. D. Levy (2003+), *Injectivity and Surjectivity of Collatz Functions*, Discrete Math., to appear.

This paper gives necessary and sufficient conditions on members of a class of generalized Collatz maps of the form $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i \pmod{d}$ to be injective maps, resp. surjective maps, on the integers. These give as a corollary a criterion of Venturini (1997) for such a map to be a permutation of the integers.

The author frames some of his results in terms of concepts involving integer matrices. He introduces a notion of *gcd matrix* if its elements can be written $M_{ij} = gcd(m_i, m_j)$ and a *difference matrix* if its elements can be written $M_{ij} = m_i - m_j$. Then he considers a relation that M is a *total non-divisor* of N if $M_{ij} \nmid N_{ij}$ for all i, j. Then the author's condition for injectivity of a generalized Collatz map above is that the $d \times d$ gcd matrix $M_{ij} = gcd(m_i, m_j)$ is a total non-divisor of the $d \times d$ difference matrix $N_{ij} = q_i - q_j$, with $q_j = \frac{r_j - j m_j}{d}$.

A very interesting result of the author is an explicit example of an injective function $T(\cdot)$ in the class above which has a (provably) divergent trajectory, and which has iterates both increasing and decreasing in size. This particular map T is not surjective.

106. K. Mahler (1968), *An unsolved problem on the powers of* 3/2, J. Australian Math. Soc. **8** (1968), 313–321. (MR 37 #2694).

A Z-number is a real number ξ such that $0 \le (\frac{3}{2})^k \xi \le \frac{1}{2}$ holds for all $k \ge 1$, where x denotes the fractional part of x. Do Z-numbers exist? The Z-number problem was originally proposed by Prof. Saburo Uchiyama (Tsukuba Univ.) according to S. Ando (personal communication), and was motivated by a connection with the function $g(k)$ in Waring's problem, for which see G. H. Hardy and E. M. Wright , An Introduction to the Theory of Numbers (4-th edition), Oxford Univ. Press 1960, Theorem 393 ff.

Mahler shows that existence of Z-numbers relates to a question concerning the iteration of the function

$$g(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2 . \\ \\ \dfrac{3x}{2} & \text{if } x \equiv 0 \pmod 2 . \end{cases}$$

Mahler showed that a Z-number exists in the interval $[n, n+1)$ if and only if no iterate $g^{(k)}(n) \equiv 3 \pmod 4$. He uses this relation to prove that the number of Z-numbers below x is at most $O(x^{0.7})$. He conjecures that no Z-numbers exist, a problem which is still unsolved.

[L. Flatto (*Z-numbers and β transformations, pp. 181–201 in: Symbolic Dynamics and Applications (New Haven, CT 1991)*, Contemp. Math. Vol. 135, AMS: Providence RI 1992) subsequently improved Mahler's upper bound on Z-numbers below x to $O(x^{\theta})$, with $\theta = \log_2 \frac{3}{2} \approx 0.59$.]

107. J. Marcinkowski (1999), *Achilles, Turtle, and Undecidable Boundedness Problems for Small DATALOG programs*, SIAM J. Comput. **29** (1999), 231–257. (MR 2002d:68035).

DATALOG is the language of logic programs without function symbols. A DATALOG program consists of a finite set of Horn clauses in the language of first order logic without equality and without functions.

The author introduces *Achilles-Turtle machines*, which model the iteration of *Conway functions*, as introduced in Conway (1972). These are functions $g : \mathbb{N} \to \mathbb{N}$ having the form

$$g(n) = \frac{a_j}{q_j} n \quad \text{if } n \equiv j (\text{mod } p),\ 0 \le j \le p-1,$$

where for each j, $a_j \ge 0, q_j \ge 1$ are integers with $q_j | gcd(j,p)$ and with $\frac{a_j}{q_j} \le p$. He cites Devienne, Lebégue and Routier (1993) for the idea of relating Conway functions to Horn clauses. In Section 2.3 he explicitly describes an Achilles-Turtle machine associated to computing the Collatz function.

The paper proves that several questions concerning the uniform boundedness of computations are undecidable. These include uniform boundedness for ternary linear programs; unfiorm boundedness for single recursive rule ternary programs; and uniform boundedness of single rule programs.

[These results give no information regarding the $3x+1$ function itself.]

108. D. Marcu (1991), *The powers of two and three,* Discrete Math. **89** (1991), 211-212. (MR 92h:11026).

Erdös raised the question: "Does there exist an integer $m \neq 0, 2, 8$ such that 2^m

is a sum of distinct integral powers of 3?" This paper shows that if $N(T)$ denotes the number of such $m \leq T$ then $N(T) \leq 2.495T^{\theta}$ where $\theta = \frac{\log 2}{\log 3}$.

109. M. Margenstern (2000), *Frontier between decidability and undecidability: a survey*, Universal Machines and Computations (Metz 1998), Theor. Comput. Sci. **231** (2000), 217–251. (MR 2001g:03079).

    This paper surveys results concerning the gap between decidability and undecidability, as measured by the number of states or symbols used in a Turing machine, and by other complexity measures. It surveys such results for computing specific functions. These include results on the number of states and symbols required to compute the $3x+1$ function, as given in Margenstern and Matiyasevich (1999).

110. M. Margenstern and Y. Matiyasevich (1999), *A binomial representation of the* $3X+1$ *problem,* Acta Arithmetica **91**, No. 4 (1999), 367–378. (MR 2001g:11015).

    This paper encodes the $3x+1$ problem as a logical problem using one universal quantifier and existential quantifiers, with an arithmetical formula using polynomials and binomial coefficients. The authors observe that the use of such expressions in a language with binomial coefficients often leads to shorter formulations than are possible in a language just allowing polynomial equations and quantifiers. They give three equivalent restatements of the $3x+1$ Conjecture in terms of quantified binomial coefficient equations.

111. K. R. Matthews (1992), *Some Borel measures associated with the generalized Collatz mapping,* Colloq. Math. **53** (1992), 191–202. (MR 93i:11090).

    This paper studies maps of the form

    $$T(x) = \frac{m_i x - r_i}{d} \quad \text{if} \quad x \equiv i \pmod{d} \ .$$

    These extend first to maps on the d-adic integers $\mathbb{Z}_d$, and then further to maps on the polyadic integers $\hat{\mathbb{Z}}$. Here $\hat{\mathbb{Z}}$ is the projective limit of the set of homomorphisms $\phi_{n,m} : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ where $m|n$. The open sets $B(j,m) := \{x \in \hat{\mathbb{Z}} : x \equiv j (\text{mod } m)\}$ put a topology on $\hat{\mathbb{Z}}$, which has a Haar measure $\sigma(B(j,m)) = \frac{1}{m}$. The paper proves conjectures of Buttsworth and Matthews (1990) on the structure of all ergodic open sets (mod m). In particular the ergodic sets link together to give finitely many projective systems, each giving T-invariant measure on $\hat{\mathbb{Z}}$. The paper gives examples where there are infinitely many ergodic sets.

112. K. R. Matthews (2001+), *The generalized* $3x+1$ *mapping,* preprint, 23pp., downloadable as pdf file from: `http://www.maths.uq.edu.au/~krm/interests.html`

    This paper discusses the behavior of $3x+1$-like mappings and surveys many example functions as considered in Matthews and Watts (1984, 1985), Leigh (1986), Leigh and Mattnews (1987) and Matthews (1992), see also Venturiri (1992). A particularly tantalizing example is

$$U(x) = \begin{cases} 7x+3 & \text{if } x \equiv 0 \pmod 3 \\ \dfrac{7x+2}{3} & \text{if } x \equiv 1 \pmod 3 \\ \dfrac{x-2}{3} & \text{if } x \equiv 2 \pmod 3 \ . \end{cases}$$

Almost all trajectories contain an element $n \equiv 0$ (mod 3) and once a trajectory enters the set $\{n : n \equiv 0 \pmod 3\}$ it stays there. Matthews offers \$100 to show that if a trajectory has all iterates $U^{(k)}(x) \equiv \pm 1 \pmod 3$ then it must eventually enter one of the cycles $\{1, -1\}$ or $\{-2, -4, -2\}$. The paper also considers some maps on the rings of integers of an algebraic number field, for example $U : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$ given by

$$U(\alpha) = \begin{cases} \dfrac{(1-\sqrt{2})\alpha}{\sqrt{2}} & \text{if } \alpha \equiv 0 \pmod{(\sqrt{2})} \ , \\ \dfrac{3\alpha+1}{\sqrt{2}} & \text{if } \alpha \equiv 1 \pmod{(\sqrt{2})} \ . \end{cases}$$

The author conjectures that if $U^{(k)}(\alpha) = x_k + y_k\sqrt{2}$ is a divergent trajectory, then $x_k/y_k \to -\sqrt{2}$ as $k \to \infty$.

113. K. R. Matthews and G. M. Leigh (1987), *A generalization of the Syracuse algorithm to $F_q[x]$,* J. Number Theory **25** (1987), 274–278. (MR 88f:11116).

This paper defines mappings analogous to the $3x+1$ function on polynomials over finite fields, e.g. $T(f) = \frac{f}{x}$ if $f \equiv 0 \pmod x$ and $\frac{(x+1)^3 f+1}{x}$ if $f \equiv 1 \pmod x$, over GF(2). It proves that divergent trajectories exist for certain such maps. These divergent trajectories have a regular behavior.

114. K. R. Matthews and A. M. Watts (1984), *A generalization of Hasse's generalization of the Syracuse algorithm,* Acta. Arithmetica **43** (1984), 167–175. (MR 85i:11068).

This paper studies functions $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i \pmod d$, where all m_i are positive integers and $r_i \equiv i m_i \pmod d$. It is shown that if $\{T^{(k)}(m) : k \geq 0\}$ is unbounded and uniformly distributed (mod d) then $m_1 m_2 \cdots m_d > d^d$ and $\lim_{k\to\infty} |T^{(k)}(m)|^{1/k} = \frac{1}{d}(m_1 \cdots m_d)^{1/d}$. The function T is extended to a mapping on the d-adic integers and is shown to be strongly mixing, hence ergodic, on $\mathbb{Z}_d$. The trajectories $\{T^{(k)}(\omega : k \geq 0\}$ for almost all $\omega \in \mathbb{Z}_d$ are equidistributed (mod d^k) for all $k \geq 1$.

115. K. Matthews and A. M. Watts (1985), *A Markov approach to the generalized Syracuse algorithm,* Acta Arithmetica **45** (1985), 29–42. (MR 87c:11071).

This paper studies the same functions T considered in Matthews and Watts (1984). Given a modulus m one associates to T a row-stochastic matrix $Q = [q_{jk}]$ in which j, k index residue classes (mod m) and q_{jk} equals $\frac{1}{md}$ times the number of residue classes (mod md) which are $\equiv k \pmod m$ and whose image under T is $\equiv j \pmod m$. It gives sufficient conditions for the entries of Q^l to be the analogous probabilities associated to the iterated mapping $T^{(l)}$. Matthews and Watts conjecture that if $\mathcal{S}$ is an ergodic set of residues (mod m) and $(\alpha_i;\ i \in \mathcal{S})$ is the corresponding stationary vector on $\mathcal{S}$, and if $A = \Pi_{i\in\mathcal{S}} \left(\frac{m_i}{d}\right) \alpha_i < 1$ then all trajectories of T starting in $\mathcal{S}$ will eventually be periodic, while if $A > 1$ almost all trajectories starting in $\mathcal{S}$ will diverge. Some numerical examples are given.

116. Karl Heinz Metzger (1999), *Zyklenbestimmung beim* $(bm + 1)$*-Algorithmus*, PM (Praxis der Mathematik in Der Schule) **41** (1999), 25–27.

[I have not seen this paper.]

117. Karl Heinz Metzger (2000), *Untersuchungen zum* $(3n+1)$*-Algorithmus. Teil II: Die Konstruktion des Zahlenbaums,* PM (Praxis der Mathematik in Der Schule) **42** (2000), 27–32.

According to Metzger (2003) this paper discusses trees of inverse iterates ending at a given number.

[I have not seen this paper.]

118. Karl Heinz Metzger (2003), *Untersuchungen zum* $(3n + 1)$*-Algorithmus, Teil III: Gesetzmässigkeiten der Ablauffolgen,* PM (Praxis der Mathematik in Der Schule) **45** (2003), No. 1, 25–32.

This paper gives a description for iterates of the Collatz function $C(n)$. It views the iterates (mod 6).

119. P. Michel (1993), *Busy Beaver Competition and Collatz-like Problems,* Archive Math. Logic **32** (1993), 351–367. (MR 94f:03048).

The Busy Beaver problem is to find that Turing machine which, among all k-state Turing machines, when given the empty tape as input, eventually halts and produces the largest number $\Sigma(k)$ of ones on the output tape. The function $\Sigma(k)$ is well-known to be non-recursive. This paper shows that the current Busy Beaver record-holder for 5-state Turing machine computes a Collatz-like function. This machine M_5 of Marxen and Buntrock [Bulletin EATCS No. **40** (1990) 247–251] has $\Sigma(M_5) = 47,176,870$. Michel shows that M_5 halts on all inputs if and only if iterating the function $g(3m) = 5m + 6$, $g(3m + 1) = 5m + 9$, $g(3m + 2) =\uparrow$ eventually halts at $\uparrow$ for all inputs. Similar results are proved for several other 5-state Turing machines. The result of Mahler [J. Aust. Math. Soc. **8** (1968) 313-321] on Z-numbers is restated in this framework: If a Z-number exists then there is an input m_0 such that the iteration $g(2m) = 3m$, $g(4m + 1) = 6m + 2$, $g(4m + 3) =\uparrow$ never halts with $\uparrow$ when started from m_0.

120. F. Mignosi (1995), *On a Generalization of the* $3x + 1$ *Problem,* J Number Theory, **55** (1995), 28–45. (MR 96m:11016).

For real $\beta > 1$, define the function $T_\beta : \mathbb{N} \to \mathbb{N}$ by

$$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & \text{if } n \equiv 1 \pmod 2 \\ \dfrac{n}{2} & \text{if } n \equiv 0 \pmod 2\ . \end{cases}$$

Then $\beta = \frac{3}{2}$ gives the $(3x+1)$-function. Conjecture C_β asserts that T_β has finitely many cycles and every $n \in \mathbb{N}$ eventually enters a cycle under iteration of T_β. The author shows that, for any fixed $0 < \epsilon < 1$, and for β either transcendental or rational with an even denominator, if $1 < \beta < 2$, then the set $S(\epsilon, \beta) = \{n : \text{some } T_\beta^{(k)}(n) < \epsilon n\}$ has natural density one, while if $\beta > 2$ then $S^*(\epsilon, \beta) = \{n : \text{some } T_\beta^{(k)}(n) > \epsilon^{-1} n\}$ has natural density one. For certain algebraic β different behavior may occur, and Conjecture C_β can sometimes be settled. In particular Conjecture C_β is true for $\beta = \sqrt{2}$ and false for $\beta = \frac{1+\sqrt{5}}{2}$.

121. T. Mimuro (2001), *On certain simple cycles of the Collatz conjecture,* SUT Journal of Mathematics, **37**, No. 2 (2001), 79–89. (MR 2002j:11018).

The paper shows there are only finitely many positive integer cycles of the $3x+1$ function whose symbol sequence has the form $1^i\ (10^j)^k$, where i, j, k vary over nonnegative integers. (The symbol seqence is read left to right.) This result includes the trivial cycle starting from $n = 1$, whose symbol sequence is (10), where $(i, j, k) = (0, 1, 1)$. Let the periodic orbit has period $p = i + k(j+1)$ terms, of which $d = i + k$ are odd. The author shows by elementary arguments that there are no integer orbits of the above type with $\frac{3}{4} > \frac{3^d}{2^p}$, and the trivial cycle is the unique solution with $\frac{3}{4} = \frac{3^d}{2^p}$. Using bounds from transcendence theory (linear forms in logarithms) he shows that there are finitely many values of (i, j, k) giving an integer orbit with $1 > \frac{3^d}{2^p} > \frac{3}{4}$, with an effective bound on their size. Any orbit on the positive integers necessarily has $1 > \frac{3^d}{2^p}$, so the result follows.

For other papers using transcendence theory to classify some types of periodic orbits, see Steiner (1978), Belaga and Mignotte (1999), and Brox (2000).

Remark. There are two known integer orbits on the negative integers of the author's form. They are $n = -1$ with symbol seqence (1), where $(i, j, k) = (1, *, 0)$, and $\frac{3^d}{2^p} = \frac{3}{2}$, and $n = -5$ with symbol sequence (110) where $(i, j, k) = (1, 1, 1)$, and $\frac{3^d}{2^p} = \frac{9}{8}$. The author's finiteness result might conceivably be extended to the range $\frac{3}{2} \ge \frac{3^d}{2^p} \ge 1$ and so cover them.

122. M. Misiurewicz and A. Rodriguez (2004), Real $3X + 1$, Proc. Amer. Math. Soc., to appear.

The authors consider the semigroup generated by the two maps $T_1(x) = \frac{x}{2}$ and $T_2(x) = \frac{3x+1}{2}$. They show this semigroup is a free semigroup on two generators. The forward orbit of a positive input x_0 under this semigroup is

$$O^+(x_0) := \{T_{i_1} \circ \ldots \circ T_{in}(x_0) : n \ge 1,\ \text{each } i_k \in \{0, 1\}\}.$$

They show that each orbit $O^+(x(_0)$ is dense on $(0, \infty)$. Furthermore they show that starting from x_0 one can get an iterate $T^{(n_0)}(x_0)$ within a given error ϵ of a given value y while remaining in the bounded region

$$\min(x, y - \epsilon \le T_{i_1} \circ \cdots T_{i_j}(x_0) \le \max(11x + 4, 4y - x).\quad 1 \le j \le n_0.$$

They show that orbits having a periodic point are dense in $(0, \infty)$. Finally they show that the group of homeomorphisms of the line generated by T_1, T_2 consists of all maps

$x \mapsto 2^k 3^l x + \frac{m}{2^i 3^j}$, in which k, l, m are integers and i, j are nonnegative. Thus it is not a free group. integers.

123. H. Möller (1978), Uber Hasses Verallgemeinerung des Syracuse-Algorithmus (Kakutanis Problem), Acta Arith. **34** (1978), No. 3, 219–226. (MR 57 #16246).

This paper studies for parameters d, m with $d \geq 2, m \geq 1$ and $(m, d) = 1$ the class of maps $H : \mathbb{Z}^+ \backslash \mathbb{Z}^+ \to \mathbb{Z}^+ \backslash d\mathbb{Z}^+$ having the the form

$$H(x) = \frac{1}{d^{a(x)}}(mx - r_j) \quad \text{when} \quad rx \equiv j \pmod{m},\ 1 \leq j \leq d-1,$$

in which $R(d) := \{r_j :\ 1 \leq j \leq d-1\}$ is any set of integers satisfying $r_j \equiv mj (\bmod\ d)$ for $1 \leq j \leq d-1$, and $d^{a(x)}$ is the maximum power of d dividing $mx - r_j$. The $3x+1$ problem corresponds to $d = 2, m = 3$ and $R(d) = \{-1\}$. The paper shows that if

$$m \leq d^{d/(d-1)}$$

then the set of positive integers n which have some iterate $H^{(k)}(n) < n$, has full natural density $1 - \frac{1}{d}$ in the set $\mathbb{Z}^+ \backslash d\mathbb{Z}^+$. He conjectures that when $m \leq d^{d/(d-1)}$ the exceptional set of positive integers which don't satisfy the conditon is finite.

The author's result generalizes that of Terras(1976) and Everett (1977). In a note added in proof the author asserts that the proofs of Terras (1976) are faulty. Terras's proofs seem essentially correct to me, and in response Terras (1979) provided further details.

124. Kenneth G. Monks (2002), $3X + 1$ Minus the +, Discrete Math. Theor. Comput. Sci. **5** (2002), 47–53. (MR 2203f:11030).

This paper formulates a FRACTRAN program (see Conway(1987)) of the form $R_i(n) \equiv r_i n$ if $n \equiv i (\bmod\ d)$, such that the $3x+1$ Conjecture is true if and only if the R-orbit of 2^m contains 2, for all positive integers m. He determines information on the behavior under iteration of the function $R(n)$ for all positive integers n, not just powers of 2. He deduces information on the possible structure of an integer $3x+1$ cycle (for the function $T(\cdot)$), namely that the sum of its even elements must equal the sum of its odd elements added to the number of its odd elements. He notes that this fact can be deduced directly without using the FRACTRAN encoding.

125. Kenneth G. Monks and Jonathan Yazinski (2003+), *The Autoconjugacy of the* $3x+1$ *function,* Discrete Mathematics, to appear.

This paper studies the iteration of the $3x+1$ map $T(x)$ on the 2-adic integers $\mathbb{Z}_2$. It shows that the set of $\text{Aut}(T) = \{U \in \text{Aut}(\mathbb{Z}_2) : UTU^{-1} = T\}$ consists of the identity map and a map $\Omega = \Phi \circ V \circ \Phi^{-1}$ where $V(x) = -1 - x$ is the map reversing the bits in a 2-adic integer and Φ is the $3x+1$ Conjugacy map studied in Bernstein and Lagarias (1996). It formulates the Autoconjugacy conjecture that $\Omega(\mathbb{Q}_{odd}) \subseteq \mathbb{Q}_{odd}$, and proves this is equivalent to no rational number with odd denominator having a divergent T-orbit. It defines a notion of self-conjugate cycle under the $3x+1$ map, which is a periodic orbit C such that $\Omega(C) = C$. It proves that $\{1, 2\}$ is the only self-conjugate cycle of integers. It shows that all self-conjugate cycles consist of positive rational numbers.

126. Helmut A. Müller (1991), *Das '$3n+1$' Problem,* Mitteilungen der Math. Ges. Hamburg **12** (1991), 231–251. (MR 93c:11053).

This paper presents basic results on $3x+1$ problem, with some overlap of Lagarias (1985), and it presents complete proofs of the results it states. it contains new observations on the $3x+1$ function T viewed as acting on the 2-adic integers $\mathbb{Z}_2$. For $\alpha \in \mathbb{Z}_2$ the 2-adic valuation $|\alpha|_2 = 2^{-j}$ if $2^j || \alpha$. Müller observes that $T(\alpha) = \sum_{n=0}^{\infty} (-1)^{n-1}(n+1)2^{n-2}\binom{\alpha}{n}$ where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. The function $T(\alpha)$ is locally constant but not analytic. Define the function $Q_\infty : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $Q_\infty(\alpha) = \sum_{i=0}^{\infty} a_i 2^i$ where $a_i = 0$ if $|T^{(i)}(\alpha)|_2 < 1$ and $a_i = 1$ if $|T^{(i)}(\alpha)|_2 = 1$. Lagarias (1985) showed that this function is a measure-preserving homeomorphism of $\mathbb{Z}_2$ to itself. Müller proves that Q_∞ is nowhere differentiable.

127. Helmut A. Müller (1994), *Über eine Klasse 2-adischer Funktionen im Zusammenhang mit dem "$3x+1$"-Problem,* Abh. Math. Sem. Univ. Hamburg **64** (1994), 293–302. (MR 95e:11032).

Let $\alpha = \sum_{j=0}^{\infty} a_{0,j} 2^j$ be a 2-adic integer, and let $T^{(k)}(\alpha) = \sum_{j=0}^{\infty} a_{k,j} 2^j$ denote its k-th iterate. Müller studies the functions $Q_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ defined by $Q_j(\alpha) = \sum_{k=0}^{\infty} a_{kj} 2^k$. He proves that each function Q_j is continuous and nowhere differentiable. He proves that the function $f : \mathbb{Z}_2 \to \mathbb{Q}_2$ given by $f = \sum_{j=1}^{N} A_j Q_j(\alpha)$ with constants $A_j \in \mathbb{Q}_2$ is differentiable at a point α with $T^{(k)}(\alpha) = 0$ for some $k \geq 0$ if and only if $2A_0 + A_1 = 0$ and $A_2 = A_3 = \ldots = A_N = 0$.

128. T. Oliveira e Silva (1999), *Maximum Excursion and Stopping Time Record-Holders for the $3x+1$ Problem: Computational Results,* Math. Comp. **68** No. 1 (1999), 371-384, (MR 2000g:11015).

This paper reports on computations that verify the $3x+1$ conjecture for $n < 3 \cdot 2^{53} = 2.702 \times 10^{16}$. It also reports the values of n that are champions for the quantity $\frac{t(n)}{n}$, where

$$t(n) := \sup_{k \geq 1} T^{(k)}(n) \ .$$

In this range all $t(n) \leq 8n^2$, which is consistent with the conjecture $t(n) \leq n^{2+o(1)}$ of Lagarias and Weiss (1992).

[Remark: As of April 2000 his computation verified the 3x+1 conjecture up to $100 \cdot 2^{50} > 1.125 \times 10^{17}$. See his webpage at: `http://www.ieeta.pt/t̃os/`; email: `tos@ieeta.pt`. These computations are now superseded by the distributed search algorithm computations of Eric Roosendaal (2003+), which has the current record value for verifying the $3x+1$ conjecture.]

129. C. Pickover (1989), *Hailstone $3n+1$ Number graphs,* J. Recreational Math. **21** (1989), 120–123.

This paper gives two-dimensional graphical plots of $3x+1$ function iterates revealing "several patterns and a diffuse background of chaotically-positioned dots."

130. M. Pierantoni and V. Curcic (1996), *A transformation of iterated Collatz mappings,* Z. Angew Math. Mech. *76*, Suppl. 2 (1996), 641–642. (Zbl. 900.65373).

[I have not seen this paper.]

131. Susana Puddu (1986), *The Syracuse problem (Spanish),* 5th Latin American Colloq. on Algebra – Santiago 1985, Notas Soc. Math. Chile **5** (1986), 199–200. (MR88c:11010).

This note considers iterates of the Collatz function $C(x)$. It shows every positive m has some iterate $C^k(m) \equiv 1 \pmod 4$. If $m \equiv 3 \pmod 4$ the smallest such k must have $C^k(m) \equiv 5 \pmod{12}$.

132. Wei Xing Qiu (1997) , *Observations on the* $3x+1$ *problem* (Chinese), J. Shanghai Univ. Nat. Sci. bf 3 (1997), No. 4, 462–464.

[I have not seen this paper.]

133. Lee Ratzan (1973), *Some work on an Unsolved Palindromic Algorithm*, Pi Mu Epsilon Journal **5** (Fall 1973), 463–466.

The $3x+1$ problem and a generalization were proposed as an Undergraduate Research Project by the editor (David Kay) in Pi Mu Epsilon Journal **4** (1972), p. 338. The author gives computer code to test the conjecture, and used it to verify the $3x+1$ Conjecture up to 31,910. He notices some patterns in two consecutive numbers having the same total stopping time and makes conjectures when they occur.

134. D. Rawsthorne (1985), *Imitation of an iteration,* Math. Magazine **58** (1985), 172–176. (MR 86i:40001).

This paper proposes a multiplicative random walk that models imitating the "average" behavior of the $3x+1$ function and similar functions. It compares the mean and standard deviation that this model predicts with empirical $3x+1$ function data, and with data for several similar mappings, and finds good agreement between them.

135. Eric Roosendaal (2003+), *On the* $3x+1$ *problem*, web document, available at: `http://personal.computrain.nl/eric/wondrous/`

The author maintains an ongoing distributed search program for verifying the $3x+1$ Conjecture to new records and for searching for extremal values for various quantities associated to the $3x+1$ function. These include quantities termed the glide, delay, residue, completeness, and gamma. Many people are contributing time on their computers to this project.

As of Oct. 4, 2003 the $3x+1$ Conjecture is verified up to $235 \times 2^{50} \approx 2.64 \times 10^{17}$. The largest value $\gamma(n)$ found so far is 36.716918 at $n = 7,219,136,416,377,236,271,195 \approx 7.2 \times 10^{21}$.

[The current record for verification of the $3x+1$ conjecture published in archival literature is that of Oliveira e Silva (1999).]

136. J. L. Rouet and M. R. Feix (2002), *A generalization of the Collatz problem. Building cycles and a stochastic approach,* J. Stat. Phys. **107**, No. 5/6 (2002), 1283–1298. (MR 2003i:11035).

The paper studies the class of functions $U(x) = (l_i x + m_i)/n$ if $x \equiv i \pmod{n}$, with $il_i + m_i \equiv 0 \pmod{n}$. These functions include the $3x+1$ function as a special case. They show that there is a bijection between the symbolic dynamics of the first k iterations and the last k digits of the input x written in base n if and only if n is relatively prime to the product of the l_i. They show that for fixed n and any given $\{m_i : 1 \le j \le k\}$ and can find a set of coefficients $\{l_i : 0 \le i \le n-1\}$ and $\{m_i : 0 \le i \le n-1\}$ with n relatively prime to the product of the l_i which give these values as a k-cycle, $U(m_i) = m_{i+1}$ and $U(m_k) = m_0$. They give numerical experiments indicating that for maps of this kind on k digit inputs (written in base n) "stochasticity" persists beyond the first k iterations. For the $3x+1$ problem itself (with base $n = 2$), it is believed that "stochasticity" persists for about c_0 k iterations, with $c_0 = \frac{2}{\log_2(4/3)} = 4.8187$, as described in Borovkov and Pfeifer (2000), who also present supporting numerical data.

137. O. Rozier (1990), *Demonstraton de l'absence de cycles d'une certain forme pour le probléme de Syracuse,* Singularité **1** no. 3 (1990), 9–12.

This paper proves that there are no cycles except the trivial cycle whose iterates (mod 2) repeat a pattern of the form $1^m 0^{m'}$. Such cycles are called *circuits* by R. P. Steiner, who obtained this result in 1978. Rozier's proof uses effective transcendence bounds similar to Steiner's, but is simpler since he can quote the recent bound

$$\left|\frac{\log 3}{\log 2} - \frac{p}{q}\right| > q^{-15},$$

when $(p, q) = 1$, which appears in M. Waldschmidt, Equationes diophantiennes et Nombres Transcendents, Revue Palais de la Découverte, **17**, No. 144 (1987) 10–24.

138. O. Rozier (1991), *Probleme de Syracuse: "Majorations" Elementaires des Cycles,* Singularité **2**, no. 5 (1991), 8–11.

This paper proves that if the integers in a cycle of the $3x+1$ function are grouped into blocks of integers all having the same parity, for any k there are only a finite number of cycles having $\le k$ blocks. The proof uses the Waldschmidt result $\left|\frac{\log 3}{\log 2} - \frac{p}{q}\right| > q^{-15}$.

139. Jurgen W. Sander (1990), *On the* $(3N+1)$*-conjecture,* Acta Arithmetica **55** (1990), 241–248. (MR 91m:11052).

The paper shows that the number of integers $\le x$ for which the $3x+1$ Conjecture is true is at least $x^{3/10}$, by extending the approach of Crandall (1978). (Krasikov (1989) obtains a better lower bound $x^{3/7}$ by another method.) This paper also shows that if the $3x+1$ Conjecture is true for all $n \equiv \frac{1}{3}(2^{2k} - 1) \pmod{2^{2k-1}}$, for any fixed k, then it is true in general.

140. B. G. Seifert (1988), *On the arithmetic of cycles for the Collatz-Hasse ('Syracuse') conjectures,* Discrete Math. **68** (1988), 293–298. (MR 89a:11031).

This paper gives criteria for cycles of $3x+1$ function to exist, and bounds the smallest number in the cycle in terms of the length of the cycle. Shows that if the $3x+1$ Conjecture is true, then the only positive integral solution of $2^l - 3^r = 1$ is $l = 2, r = 1$.

[All integer solutions of $2^l - 3^r = 1$ have been found unconditionally. This can be done by a method of Størmer, Nyt. Tidsskr. Math. B **19** (1908) 1–7. It was done as a special case of various more general results by S. Pillai, Bull. Calcutta Math. Soc. **37** (1945), 15–20 (MR# 7, 145i); W. LeVeque, Amer. J. Math. **74** (1952), 325–331 (MR 13, 822f); R. Hampel, Ann. Polon. Math. **3** (1956), 1–4 (MR 18, 561c); etc.]

141. Jeffrey O. Shallit and David W. Wilson (1991), *The "$3x + 1$" Problem and Finite Automata,* Bulletin of the EATCS (European Association for Theoretical Computer Science), No. 46, 1991, pp. 182–185.

A set S of positive integers is said to be *2-automatic* if the binary representations of the integers in S form a regular language $L_S \subseteq \{0,1\}^*$. Let S_i denote the set of integers n which have some $3x+1$ function iterate $T^{(j)}(n) = 1$, and whose $3x+1$ function iterates include exactly i odd integers ≥ 3. The sets S_i are proved to be 2-automatic for all $i \geq 1$.

142. D. Shanks (1965), *Comments on Problem* $63 - 13^*$, SIAM Review **7** (1965), 284–286.

This note gives comments on Problem $63 - 13^*$, proposed by Klamkin (1963). This problem concerns the iteration of Collatz's original function, which is a permutation of the integers. He states that these problems date back at least to 1950, when L. Collatz mentioned them in personal conversations at the International Math. Congress held at Harvard University. He gives the results of a computer search of the orbit of 8 for Collatz's original function, observing that it reaches numbers larger than 10^{10}. He observes there are known cycles of length 1, 2, 5 and 12, the last having smallest element $n = 144$. He observes that this seems related to the fact that the continued fraction expansion of $\log_2 3$ has initial convergents having denominators 1, 2, 5, 12, 41, ... However he does not know of any cycle of period 41. He notes that it is not known whether the only cycle lengths that can occur must be denominators of such partial quotients. Later Atkin (1966) proved there exists no cycle of period 41.

143. D. Shanks (1975), *Problem* #5, Western Number Theory Conference 1975, Problem List, (R. K. Guy, Ed.).

The problem concerns iteration of the Collatz function $C(n)$. Let $l(n)$ count the number of distinct integers appears in the sequence of iterates $C^{(k)}(n)$ of $n \geq 1$, assuming it eventually enters a cycle. Thus $l(1) = 3, l(2) = 3, l(3) = 8$, for example. Set $S(N) = \sum_{n=1}^{N} l(n)$. The problem asks whether it is true that

$$S(N) = AN \log N + BN + o(N) \quad \text{as} \quad N \to \infty,$$

where

$$A = \frac{3}{2} \log \frac{4}{3} \approx 5.21409 \text{ and } B = A(1 - \log 2) \approx 1.59996.$$

[This problem formalizes the result of a heuristic probabilistic calculation which assumes $3x + 1$ Conjecture to be true. In order for $S(N)$ to remain finite, there must be no divergent trajectories.]

144. Ya. G. Sinai (2003), *Statistical* $(3X + 1)$*-Problem*, Comm. Pure Appl. Math. **56** No. 7 (2003), 1016–1028. (`arXiv eprint: math.DS/0201102 v2 13 Jan 2003`, 15 pages).

This paper analyzes iterations of the variant of the $3x + 1$ map that removes all powers of 2 at each step, so takes odd integers to odd integers. It gives a structure theorem for the form of the iterates having a given symbolic dynamics. The discussion in section 5 can be roughly stated as asserting: There is an absolute constant $c > 0$ such that "most" $3x + 1$ trees (mod 3^m) contain at most $e^{c\sqrt{m \log m}} 2^{2m}/3^m$ nodes whose path to the root node has length at most $2m$ and which has exactly m odd iterates. Here one puts a probability density on such trees which for a given tree counts the number of such nodes divided by the total number of such nodes summed over all trees, and "most" means that the set of such trees having the property contains $1 - O(1/m)$ of the total probability, as $m \to \infty$. (The total number of such nodes is 2^{2m}, and the total number of trees is $2 \cdot 3^{m-1}$.) Furthermore at least $\frac{1}{m}$ of the probability is distributed among trees having at most $M^c 2^{2m}/3^m$ such nodes. From this latter result follows an entropy inequality (Theorem 5.1) which is the author's main result: The entropy H_m of this probability distribution satisfies $H_m \geq m \log 3 - O(\log\ m)$. For comparison the uniform distribution on $[1, 3^m]$ has the maximal possible entropy $H = m \log 3$. He conjectures that the entropy satisfies $H_m \geq m \log 3 - O(1)$.

145. Ya. G. Sinai (2003+), *Uniform distribution in the* $(3x + 1)$ *problem*, preprint.

This paper shows that although the trees of inverse $3x + 1$ iterates vary greatly in size, under sufficiently strong averaging over long enough intervals the total number of leaves are approximately uniformly distributed. (See the paper for the precise formulation of the result.)

146. M. K. Sinisalo (2003+), *On the minimal cycle lengths of the Collatz sequences*, preprint.

This paper shows that the minimal length of a nontrivial cycle of the $(3x + 1)$-function on the positive integers is at least 630,138,897. It uses a method similar to that of Eliahou (1993), and takes advantage of the verification of the $3x + 1$ conjecture below the bound 2.70×10^{16} of Oliveira e Silva (1999). It also considers bounds for cycles of the $(3x - 1)$-function.

147. R. P. Steiner (1978), *A Theorem on the Syracuse Problem*, Proc. 7-th Manitoba Conference on Numerical Mathematics and Computing (Univ. Manitoba-Winnipeg 1977), Congressus Numerantium XX, Utilitas Math.: Winnipeg, Manitoba 1978, pp. 553–559. (MR 80g:10003).

This paper studies periodic orbits of the $3x + 1$ map, and a problem raised by Davidson (1976). A sequence of iterates $\{n_1, n_2, ..., n_p, n_{p+1}\}$ with $T(n_j) = n_{j+1}$ is called by Davidson (1977) a *circuit* if it consists of a sequence of odd integers $\{n_1, n_2, ..., n_j\}$ followed by a sequence of even integers $\{n_{j+1}, n_{j+2}, ..., n_p\}$, with $n_{p+1} = T(n_p)$ an odd integer. A circuit is a *cycle* if $n_{p+1} = n_1$.

This paper shows that the only circuit on the positive integers that is a cycle is $\{1, 2\}$. It uses the observation of Davison (1977) that these corresponds to positive solutions (k, l, h) to the exponential Diophantine equation

$$(2^{k+l} - 3^k)h = 2^l - 1.$$

The paper shows that the only solution to this equation in positive integers is $(k, l, h) = (1, 1, 1)$. The proof uses results from transcendence theory, Baker's method of linear forms in logarithms (see A. Baker, *Transcendental Number Theory*, Cambridge Univ. Press 1975, p. 45.)

[One could prove similarly that there are exactly two circuits that are cycles on the non-positive integers, namely $\{-1\}$, and $\{-5,-7,-10\}$. These correspond to the solutions to the exponential Diophantine equation $(k,l,h) = (2,1,-1)$ and $(0,2,1)$, respectively.]

148. R. P. Steiner (1981a), *On the "$Qx+1$" Problem, Q odd*, Fibonacci Quarterly **19** (1981), 285–288. (MR 84m:10007a)

This paper studies the $Qx+1$-map

$$h(n) = \begin{cases} \frac{Qn+1}{2} & \text{if } n \equiv 1 \pmod 2, n > 1 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 1 & \text{if } n = 1, \end{cases}$$

when Q is odd and $Q > 3$. It proves that the only circuit which is a cycle when $Q = 5$ is $\{13, 208\}$, that there is no circuit which is a cycle for $Q = 7$. Baker's method is again used, as in Steiner (1978).

149. R. P. Steiner (1981b), *On the "$Qx+1$" Problem, Q odd II*, Fibonacci Quarterly **19** (1981), 293–296. (MR 84m:10007b).

This paper continues to study the $Qx+1$-map

$$h(n) = \begin{cases} \frac{Qn+1}{2} & \text{if } n \equiv 1 \pmod 2, n > 1 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 1 & \text{if } n = 1, \end{cases}$$

when Q is odd and $Q > 3$. It makes general remarks on the case $Q > 7$, and presents data on from the computation of $\log_2 \frac{5}{2}$ and $\log_2 \frac{7}{2}$ used in the proofs in part I.

150. K. S. Stolarsky (1998), *A prelude to the $3x+1$ problem,* J. Difference Equations and Applications **4** (1998), 451–461. (MR 99k:11037).

This paper studies a purportedly "simpler" analogue of the $3x+1$ function. Let $\phi = \frac{1+\sqrt{5}}{2}$. The function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is given by

$$\begin{cases} f(\lfloor n\phi \rfloor) = \lfloor n\phi^2 \rfloor + 2 \ , & \text{all } \ n \geq 1 \ , \\ f(\lfloor n\phi^2 \rfloor) = n \ . & \text{all } \ n \geq 1 \ . \end{cases}$$

This function is well-defined because the sets $A = \{\lfloor n\phi \rfloor : n \geq 1\}$ and $B = \{\lfloor n\phi^2 \rfloor : n \geq 1\}$ form a partition of the positive integers. (This is a special case of Beatty's theorem.) The function f is analogous to the $3x+1$ function in that it is increasing on the domain A and decreasing on the domain B. The paper shows that almost all trajectories diverge to $+\infty$, the only exceptions being a certain set $\{3, 7, 18, 47, \ldots\}$ of density zero which converges to the two-cycle $\{3,7\}$. Stolarsky determines the complete symbolic dynamics of f. The possible symbol sequences of orbits are $B^{\ell}(AB)^{k}A^{\infty}$ for some $\ell, k \geq 0$, for divergent trajectories, and $B^{\ell}(AB)^{\infty}$ for some $\ell \geq 0$, for trajectories that reach the two-cycle $\{3,7\}$.

151. Gy. Targonski (1991), *Open questions about KW-orbits and iterative roots,* Aequationes Math. **41** (1991), 277–278.

The author suggests the possibility of applying a result of H. Engl (An analytic representation for self-maps of a countably infinite set and its cycles, Aequationes Math. **25** (1982), 90-96, MR85d.04001) to bound the number of cycles of the $3x+1$ problem. Engl's result expresses the number of cycles as the geometric multiplicity of 1 as an eigenvalue of a map on the sequence space l^1.

152. R. Terras (1976), *A stopping time problem on the positive integers,* Acta Arithmetica **30** (1976), 241–252. (MR 58 #27879).

This is the first significant research paper to appear that deals directly with the $3x+1$ function. The $3x+1$ function was however the motivation for the paper Conway (1972). The main result of this paper was obtained independently and contemporaneously by Everett (1977).

A positive integer n is said to have *stopping time* k if the k-th iterate $T^{(k)}(n) < n$, and $T^{(j)}(n) \geq n$ for $1 \leq j < k$. The author shows that the set of integers having stopping time k forms a set of congruence classes (mod 2^k), minus a finite number of elements. He shows that the set of integers having a finite stopping time has natural density one. Some further details of this proof were supplied later in Terras (1979).

This paper introduces the notion of the *coefficient stopping time* $\kappa(n)$ of an integer $n > 1$. Write $T^{(k)}(n) = \alpha(n)n + \beta(n)$ with $\alpha(n) = \frac{3^{a(n)}}{2^n}$, where $a(n)$ is the number of iterates $T^{(j)}(n) \equiv 1 (\bmod\ 2)$ with $0 \leq j < k$. Then $\kappa(n)$ is defined to be the least $k \geq 1$ such that $T^{(k)}(n) = \alpha(n)n + \beta(n)$ has $\alpha(n) < 1$, and $\kappa(n) = \infty$ if no such value exists. It is clear that $\kappa(n) \leq \sigma(n)$, where $\sigma(n)$ is the stopping time of n. Terras formulates the *Coefficient Stopping Time Conjecture*, which asserts that $\kappa(n) = \sigma(n)$ for all $n \geq 2$. He proves this conjecture for all values $\kappa(n) \leq 2593$. This can be done for $\kappa(n)$ below a fixed bound by upper bounding $\beta(n)$ and showing bounding $\kappa(n) < 1 - \delta$ for suitable δ and determining the maximal value $\frac{3^l}{2^j} < 1$ possible with j below the given bound. The convergents of the continued fraction expansion of $\log_2 3$ play a role in determining the values of j that must be checked.

153. R. Terras (1979), *On the existence of a density*, Acta Arithmetica **35** (1979), 101–102. (MR 80h:10066).

This paper supplies additional details concerning the proof in Terras (1976) that the set of integers having an infinite stopping time has asymptotic density zero. The proof in Terras (1976) had been criticized by Möller (1978).

154. B. Thwaites (1985), *My conjecture,* Bull. Inst. Math. Appl. **21** (1985), 35–41. (MR86j:11022).

The author states that he invented the $3x+1$ problem in 1952. He derives basic results about iterates and makes conjectures on the "average" behavior of trajectories.

[It appears that L. Collatz was the first to propose the $(3x+1)$-problem, see Collatz (1986).]

155. B. Thwaites (1996), *Two Conjectures, or how to win £ 1000,* Math. Gazette **80** (1996), 35–36.

    One of the two conjectures is the $3x+1$-problem, for which the author offers the stated reward.

156. C. W. Trigg, C. W. Dodge and L. F. Meyers (1976), *Comments on Problem 133*, Eureka (now Crux Mathematicorum) **2**, No. 7 (August-Sept.) (1976), 144–150.

    Problem 133 is the $3x+1$ problem. It was proposed by K. S. Williams (Concordia Univ.), who said that he was shown it by one of his students. C. W. Trigg gives some earlier history of the problem. He remarks that in 1970 H. S. M. Coxeter offered a prize of \$50 for proving the $3x+1$ Conjecture and \$100 for finding a counterexample, in his talk: "Cyclic Sequences and Frieze Patterns" (The Fourth Felix Behrend Memorial Lecture in Mathematics), The University of Melbourne, 1970. There was a discussion of the problem in several issues of *Popular Computing* No. 1 (April 1973) 1–2; No. 4 (July 1973) 6–7; No. 13 (April 1974) 12–13; No. 25 (April 1975), 4–5.

157. T. Urvoy (2000), *Regularity of congruential graphs*, in: *Mathematical Foundations of Computer Science 2000 (Bratislava)*, Lecture Notes in Computer Science Vol. 1893, Springer: Berlin, 2000, pp. 680–689. (MR 2002d:68083).

    Math Reviews summary: "The aim of this article is to make a link between the congruential systems investigated by Conway and the theory of infinite graphs. We first compare the graphs of congruential systems with a well-known family of infinite graphs: the regular graphs of finite degree considered by Muller and Shupp, and by Courcelle. We first consider congruential systems as word rewriting systems to extract some subfamilies of congruential systems, the q-p congruential systems, representing the regular graphs of finite degree. We then prove the nonregularity of the Collatz graph."

158. G. Venturini (1982), *Sul Comportamento delle Iterazioni di Alcune Funzioni Numeriche,* Rend. Sci. Math. Institute Lombardo **A 116** (1982), 115–130. (MR 87i:11015; Zbl. 583.10009).

    The author studies functions $g(n) = a_r n + b_r$ for $n \equiv r \pmod{p}$ where a_r $(0 \le r \le p)$ are positive rationals with denominator p. He mainly treats the case that the a_r take two distinct values. If $\tau = (a_0 a_1 \cdots a_{p-1})^{1/p}$ has $\tau < 1$ then for almost all n there is some k with $g^{(k)}(n) < n$, while if $\tau > 1$ then the iterates tend to increase. [The Zentralblatt reviewer says that proofs are incomplete but contain an interesting idea. Rigorous versions of these results have since been established, see Lagarias (1985), Sect. 3.2.]

159. G. Venturini (1989), *On the $3x+1$ Problem*, Adv. Appl. Math **10** (1989), 344–347. (MR 90i:11020).

This paper shows that for any fixed ρ with $0 < \rho < 1$ the set of $m \in \mathbb{Z}^+$ which either have some $T^{(i)}(m) = 1$ or some $T^{(i)}(m) < \rho m$ has density one.

160. G. Venturini (1992), *Iterates of Number Theoretic Functions with Periodic Rational Coefficients (Generalization of the* $3x+1$ *problem),* Studies in Applied Math. **86** (1992), 185–218. (MR 93b:11102).

This paper studies iteration of maps $g : \mathbb{Z} \to \mathbb{Z}$ of the form $g(m) = \frac{1}{d}(a_r m + b_r)$ if $m \equiv r \pmod{d}$ for $0 \leq r \leq d-1$, where all $a_r,\ b_r \in \mathbb{Z}$, and $d \geq 2$ is an arbitrary integer. These maps generalize the $(3x+1)$-function, and include a wider class of such functions than Venturini (1989). The author is concerned with classifying g-ergodic sets S of such g which are finite unions of congruence classes. For example, the mapping $g(3m) = 2m$, $g(3m+1) = 4m+3$ and $g(3m+2) = 4m+1$ is a permutation and has $S = \{m : m \equiv 0 \text{ or } 5 (\text{mod } 10)\}$ as a g-ergodic set. He then associates a (generally finite) Markov chain to such a g-ergodic set, whose stationary distribution is used to derive a conjecture for the distribution of iterates $\{g^{(k)}(n_0 : k \geq 1\}$ in these residue classes for a randomly chosen initial value n_0 in S. For the example above the stationary distribution is $p_0 = \frac{1}{3}$ and $p_5 = \frac{2}{3}$. One can obtain also conjectured growth rates $a(g|_S)$ for iterates of a randomly chosen initial value in a g-ergodic set S. For the example above one obtains $a(g|s) = (\frac{4}{3})^{2/3}(\frac{2}{3}) \cong 1.0583$. The author's methods are similar in spirit to Matthews and Watts [Acta Arith. **43** (1984) 167–175 and **45** (1985), 29–42]. The author classifies the maps g into classes $G_v(d)$, for $v = 0, 1, 2, \ldots$, and also a class $G_\infty(d)$, according to how the various a_r of iterates $g(v)$ have common factors with d, and obtains results for more general functions than Matthews and Watts studied. He cannot handle all maps in the exceptional class $G_\infty(d)$, which sometimes lead to infinite Markov chains. The class $G_\infty(d)$ presumably contains functions constructed by J. H. Conway [Proc. 1972 Number Theory Conf., U. of Colorado, Boulder 1972, 39-42] which have all $b_r = 0$ and which can encode computationally undecidable problems. The author's Corollary to Theorem 6 has an incomplete proof: It remains an open problem whether a $\mathbb{Z}$-permutation having an ergodic set S with $a(g|_S) > 1$ contains any orbit that is infinite. Sections 6 and 7 of the paper contain many interesting examples of Markov chains associated to such functions g; these examples are worth looking at for motivation before reading the rest of the paper.

161. G. Venturini (1997), *On a Generalization of the* $3x+1$ *problem,* Adv. Appl. Math. **19** (1997), 295–305. (MR 98j:11013).

This paper considers mappings $T(x) = \frac{t_r x - u_r}{p}$ when $x \equiv r \pmod{p}$, having $t_r \in \mathbb{Z}^+$, and $u_r \equiv r t_r \ (\text{mod } p)$. It shows that if gcd $(t_0 t_1 \ldots t_{p-1}, p) = 1$ and $t_0 t_1 \ldots t_{p-1} < p^p$ then for any fixed ρ with $0 < \rho < 1$ almost all $m \in \mathbb{Z}$ have an iterate k with $|T^{(k)}(m)| < \rho |m|$. The paper also considers the question: when are such mappings T permutations of $\mathbb{Z}$? It proves they are if and only if $\sum_{r=0}^{p-1} \frac{1}{t_r} = 1$ and $T(r) \not\equiv T(s) \pmod{(t_r, t_s)}$ for $0 \leq r < s \leq p-1$. The geometric-harmonic mean inequality implies that $t_0 \ldots t_{p-1} > p^p$ for such permutations, except in the trivial case that all $t_i = p$.

162. C. Viola (1983), *A problem of arithmetic* (Italian), Archimede **35** (1983), 37–39. (MR 85j:11024).

From Math Reviews: The author presents, with interesting commentaries, a survey of papers on a problem of Collatz (also known as the problem of Kakutani and/or

Ulam), the generalized formulation of which is given in the quoted paper of C. Bohm and G. Sontacchi (1978).

163. S. Wagon (1985), The Collatz problem, Math. Intelligencer **7**, No. 1, (1985), 72-76. (MR 86d:11103, Zbl. 566.10008).

Studies a random walk imitation of the "average" behavior of the $3x + 1$ function, computes its expected value and compares it to data on $3x + 1$ iterates.

164. B. C. Wiggin (1988), *Wondrous Numbers – Conjecture about the* $3n + 1$ *family,* J. Recreational Math. **20**, No. 2 (1988), 52–56.

This paper calls Collatz function iterates "Wondrous Numbers" and attributes this name to D. Hofstadter, *Gödel, Escher, Bach.* He proposes studying iterates of the class of "MU" functions

$$F_D(x) = \begin{cases} \dfrac{x}{D} & \text{if } x \equiv 0 \pmod{D} , \\[2ex] (D+1)x - j & \text{if } x \equiv j \pmod{D}, \quad 1 \le j \le D-2 , \\[2ex] (D+1)x + 1 & \text{if } x \equiv -1 \pmod{D} , \end{cases}$$

F_D is the Collatz function for $D = 2$. Wiggin's analogue of the $3x + 1$ conjecture for a given $D \ge 2$ is that all iterates of $F_D(n)$ for $n \ge 1$ reach some number smaller than D. Somewhat surprisingly, no D is known for which this is false. It could be shown false for a given D by exhibiting a cycle with all members $> D$; no such cycles exist for $2 \le D \le 12$, and $x < 3 \times 10^4$.

165. G. J. Wirsching (1993), *An Improved Estimate Concerning* $3N + 1$ *Predecessor Sets,* Acta Arithmetica, **63** (1993), 205–210. (MR 94e:11018).

This paper shows that, for all $a \not\equiv 0 \pmod{3}$, the set $\theta_a(x) := \{n \le x : \text{some } T^{(k)}(x) = a\}$ has cardinality at least $x^{\cdot 48}$, for sufficiently large x. This is achieved by exploiting the inequalities of Krasikov (1989).

166. G. J. Wirsching (1994), *A Markov chain underlying the backward Syracuse algorithm,* Rev. Roumaine Math. Pures Appl. **39** (1994), no. 9, 915–926. (MR 96d:11027).

The author constructs from the inverse iterates of the $3x + 1$ function ('backward Syracuse algorithm') a Markov chain defined on the state space $[0, 1] \times \mathbb{Z}_3^\times$, in which $\mathbb{Z}_3^\times$ is the set of invertible 3-adic integers. He lets $g_n(k, a)$ count the number of "small sequence" preimages of an element $a \in \mathbb{Z}_3^\times$ at depth $n + k$, which has n odd iterates among its preimages, and with symbol sequence $0^{\alpha_0} 1 0^{\alpha_1} \cdots 1 0^{\alpha_{n-1}} 1$, with $\sum_{i=0}^{n} \alpha_i = k$ satisfying the "small sequence" condition $0 \le \alpha_j < 2 \cdot 3^{j-1}$ for $0 \le j \le n - 1$. These quantities satisfy a functional equation

$$g_n(k, a) = \frac{1}{2 \cdot 3^{n-1}} \sum_{j=0}^{2 \cdot 3^{n-1}} g_{n-1}\left(k - j \ , \ \frac{2^{j+1} a - 1}{3}\right) .$$

He considers the “renormalized” quantities

$$\hat{g}\left(\frac{k}{n}, a\right) := \frac{1}{\Gamma_n} g_n(k, a) \ ,$$

with

$$\Gamma_n = 2^{1-n} 3^{-\frac{1}{2}(n-1)(n-2)} (3^n - n) \ ,$$

He obtains, in a weak limiting sense, a Markov chain whose probability density of being at (x, a) is a limiting average of $\hat{g}(x, a)$ in a neighborhood of (x, a) as the neighborhood shrinks to (x, α). One step of the backward Syracuse algorithm induces (in some limiting average sense) a limiting Markovian transition measure, which has a density taking the form of a product measure $\frac{3}{2}\chi_{[\frac{x}{3}, \frac{x+2}{3}]} \otimes \phi$, in which $\chi_{[\frac{x}{3},\ \frac{x+2}{3}]}$ is characteristic function of the interval $\left[\frac{x}{3},\ \frac{x+2}{3}\right]$ and ϕ is a nonnegative integrable function on $\mathbb{Z}_3^{\times}$ (which may take the value $+\infty$).

The results of this paper are included in Chapter IV of Wirsching (1996b).

167. G. J. Wirsching (1996), *On the Combinatorial Structure of* $3N+1$ *Predecessor Sets,* Discrete Math. **148** (1996), 265–286. (MR 97b:11029).

This paper studies the set $P(a)$ of inverse images of an integer a under the $3x+1$ function. Encode iterates $T^{(k)}(n) = a$ by a set of nonnegative integers $(\alpha_0, \alpha_1, \ldots, \alpha_\mu)$ such that the 0-1 vector $\mathbf{v}$ encoding $\{T^{(j)}(n) \pmod 2\!:\ 0 \le j \le k-1\}$ has $\mathbf{v} = 0^{\alpha_0} 1\ 0^{\alpha_1} \cdots 1\ 0^{\alpha_\mu}$. Wirsching studies iterates corresponding to “small sequences,” which are ones with $0 \le \alpha_i < 2 \cdot 3^{i-1}$. He lets $G_\mu(a)$ denote the set of “small sequence” preimages n of a having a fixed number μ of iterates $T^{(j)}(n) \equiv 1 \pmod 2$, and shows that $|G_\mu(a)| = 2^{\mu-1} 3^{1/2(\mu-1\ldots\mu-2)}$. He lets $g_a(k, \mu)$ denote the number of such sequences with $k = \alpha_0 + \alpha_1 + \cdots + \alpha_\mu$, and introduces related combinatorial quantities $\psi(k, \mu)$ satisfying $\sum_{0 \le l \le k} \psi(l, \mu - 1) \ge g_a(k, \mu)$. The quantities $\psi(k, \mu)$ can be asymptotically estimated, and have a (normalized) limiting distribution

$$\psi(x) = \lim_{\mu \to \infty} \frac{3^\mu - \mu}{2^\mu 3^{1/2\mu(\mu-1)}} \psi([3^\mu - \mu)x], \mu).$$

$\psi(x)$ is supported on $[0, 1]$ and is a C^∞-function. He suggests a heuristic argument to estimate the number of “small sequence” preimages of a smaller than $2^n a$, in terms of a double integral involving the function $\psi(x)$.

The results of this paper are included in Chapter IV of Wirsching (1998).

168. G. J. Wirsching (1997), $3n+1$ *Predecessor Densities and Uniform Distribution in* $\mathbb{Z}_3^*$, *Proc. Conference on Elementary and Analytic Number Theory* (in honor of E. Hlawka), Vienna, July 18-20, 1996, (G. Nowak and H. Schoissengeier, Eds.), 1996, pp. 230-240. (Zbl. 883.11010).

This paper formulates a kind of equidistribution hypothesis on 3-adic integers under backwards iteration by the $3x+1$ mapping, which, if true, would imply that the set of integers less than x which iterate under T to a fixed integer $a \not\equiv 0 \pmod 3$ has size at least $x^{1-\epsilon}$ as $x \to \infty$, for any fixed $\epsilon > 0$.

169. G. J. Wirsching (1998a), *The Dynamical System Generated by the* $3n+1$ *Function*, Lecture Notes in Math. No. 1681, Springer-Verlag: Berlin 1998. (MR 99g:11027).

This volume is a revised version of the author's Habilitationsscrift (Katholische Universität Eichstätt 1996). It studies the problem of showing that for each positive integer $a \not\equiv 0 \pmod 3$ a positive proportion of integers less than x iterate to a, as $x \to \infty$. It develops an interesting 3-adic approach to this problem.

Chapter I contains a history of work on the $3x+1$ problem, and a summary of known results.

Chapter II studies the graph of iterates of T on the positive integers, ("Collatz graph") and, in particular studies the graph $\Pi^a(\Gamma_T)$ connecting all the inverse iterates $P(a)$ of a given positive integer a. This graph is a tree for any noncyclic value a. Wirsching uses a special encoding of the symbolic dynamics of paths in such trees, which enumerates symbol sequences by keeping track of the successive blocks of 0's. He then characterizes which graphs $\Pi^a(\Gamma_T)$ contain a given symbol sequence reaching the root node a. He derives "counting functions" for such sequences, and uses them to obtain a formula giving a lower bound for the function

$$P_T^n(a) := \{m : 2^n a \leq m < 2^{n+1}a \text{ and some iterate } \quad T^{(j)}(m) = a\} \ ,$$

which states that

$$|P_T^n(a)| := \{m : 2^n(a) := \sum_{\ell=0}^{\infty} e_\ell(n + \lfloor \log_2 \left(\frac{3}{2}\right) \ell \rfloor \ , \quad a) \ , \tag{1}$$

where $e_\ell(k,a)$ is a "counting function." (Theorem II.4.9). The author's hope is to use (1) to prove that

$$|P_T^n(a)| > c(a)2^n \ , \quad n = 1, 2, \ldots \ . \tag{2}$$

for some constant $c(a) > 0$.

Chapter III studies the counting functions appearing in the lower bound above, in the hope of proving (2). Wirsching observes that the "counting functions" $e_\ell(k,a)$, which are ostensibly defined for positive integer variables k, ℓ, a, actually are well-defined when the variable a is a 3-adic integer. He makes use of the fact that one can define a 'Collatz graph' $\Pi^a(\Gamma_T)$ in which a is a 3-adic integer, by taking a suitable limiting process. Now the right side of (1) makes sense for all 3-adic integers, and he proves that actually $s_n(a) = +\infty$ on a dense set of 3-adic integers a. (This is impossible for any integer a because $|P_T^n(a)| \leq 2^n a$.) He then proves that $s_n(a)$ is a nonnegative integrable function of a 3-adic variable, and he proves that its expected value $\bar{s}_n$ is explicitly expressible using binomial coefficients. Standard methods of asymptotic analysis are used to estimate $\bar{s}_n$, and to show that

$$\liminf_{n\to\infty} \frac{\bar{s}_n}{2^n} > 0 \ , \tag{3}$$

which is Theorem III.5.2. In view of (1), this says that (2) ought to hold in some "average" sense. He then proposes that (2) holds due to the:

Heuristic Principle. *As $n \to \infty$, $s_n(a)$ becomes relatively close to $\bar{s}_n$, in the weak sense that there is an absolute constant $c_1 > 0$ such that*

$$s_n(a) > c_1 \bar{s}_n \quad \textit{for all } n > n_0(a) \ . \tag{4}$$

One expects this to be true for all positive integers a. A very interesting idea here is that it might conceivably be true for all 3-adic integers a. If so, there is some chance to rigorously prove it.

Chapter IV studies this heuristic principle further by expressing the counting function $e_\ell(k,a)$ in terms of simpler counting functions $g_\ell(k,a)$ via a recursion

$$e_\ell(k,a) = \sum_{j=0}^{k} p_\ell(k-j) g_\ell(j,a) \ ,$$

in which $p_\ell(m)$ counts partitions of m into parts of a special form. Wirsching proves that properly scaled versions of the functions g_ℓ "converge" (in a rather weak sense) to a limit function which is independent of a, namely

$$g_\ell(k,a) \approx 2^{\ell-1} 3^{\frac{1}{2}(\ell^2-5\ell+2)} \psi\left(\frac{k}{3^\ell}\right) \ , \tag{5}$$

for "most" values of k and a. (The notion of convergence involves integration against test functions.) The limit function $\psi : [0,1] \to \mathbb{R}$ satisfies the functional-differential equation

$$\psi'(t) = \frac{9}{2}(\psi(3t) - \psi(3t-2)) \ . \tag{6}$$

He observes that ψ has the property of being C^∞ and yet is piecewise polynomial, with infinitely many pieces, off a set of measure zero. There is a fairly well developed theory for related functional-differential equations, cf. V. A. . Rvachev, Russian Math. Surveys **45** No. 1 (1990), 87–120. Finally, Wirsching observes that if the sense of convergence in (5) can be strengthened, then the heuristic principle can be deduced, and the desired bound (2) would follow.

170. G. J. Wirsching (1998b), *Balls in Constrained Urns and Cantor-Like Sets,* Z. für Analysis u. Andwendungen **17** (1998), 979-996. (MR 2000b:05007).

This paper studies solutions to an integral equation that arises in the author's analysis of the $3x+1$ problem (Wirsching (1998a)). Berg and Krüppel (J. Anal. Appl. **17** (1998), 159–181) showed that the integral equation

$$\phi(x) = \frac{q}{q-1} \int_{qx-q+1}^{qx} \phi(y) dy$$

subject to ϕ being supported in the interval [0, 1] and having $\int_0^1 \phi(y)dy = 1$ has a unique solution whenever $q > 1$. The function $\phi(y)$ is a C^∞-function. In this paper Wirsching shows that a certain iterative procedure converges to this solution when $q > \frac{3}{2}$. He also shows that for $q > 2$ the function $\phi(y)$ is piecewise polynomial off a Cantor-like set of measure zero. The case of the $3x+1$ problem corresponds to the choice $q = 3$.

171. G. J. Wirsching (2000), *Über das* $3n+1$ *Problem,* Elem. Math. **55** (2000), 142–155. (MR 2002h:11022).

This is a survey paper, which discusses the origin of the $3n+1$ problem and results on the dynamics of the $3x+1$ function.

172. G. J. Wirsching (2001), *A functional differential equation and* $3n+1$ *dynamics,* in: *Topics in Functional Differential and Functional Difference Equations (Lisbon 1999),* (T. Faria,

E. Frietas, Eds.), Fields Institute Communications No. 29, Amer. Math. Soc. 2001, pp. 369–378. (MR 2002b:11035).

This paper explains how a functional differential equation arises in trying to understand $3n+1$ dynamics. as given in Wirsching (1998a). It analyzes some properties of its solutions.

173. G. J. Wirsching (2003) *On the problem of positive predecessor density in* $3N+1$ *dynamics,* Disc. Cont. Dynam. Syst. **9** (2003), no. 3, 771–787.

This paper discusses an approach to prove positive predecessor density, which formulates three conjectures which, if proved, would establish the result. This approach presents in more detail aspects of the approach taken in the author's Springer Lecture Notes volume, Wirsching (1998a).

174. Jia Bang Wu (1992), *The trend of changes of the proportion of consecutive numbers of the same height in the Collatz problem* (Chinese), J. Huazhong (Central China) Univ. Sci. Technol. **20**, No. 5, (1992), 171–174. (MR 94b:11024, Zbl. 766.11013).

From Math Reviews: This paper studies the density of consecutive numbers having the same height. Set $d(2^N) := \frac{1}{2^N-1}\{n < 2^N : h(n) = h(n+1)\}$. It computes $d(2^N)$ for $N \leq 24$ and asserts that $d(2^N)$ is increasing with N for $N \leq 24$. The longest k-tuple of consecutive numbers of the same height for $n \leq 2^{30}$ is a 176-tuple with initial value $n = 722\ 067\ 240$.

175. Jia Bang Wu (1993), *On the consecutive positive integers of the same height in the Collatz problem* (Chinese), Math. Appl., suppl. **6** (1993), 150–153. (MR 1 277 568)

[I have not seen this paper.]

176. Jia Bang Wu (1995), *The monotonicity of pairs of coalescence numbers in the Collatz problem.* (Chinese), J. Huazong Univ. Sci. Tech. **23** (1995), suppl. II, 170–172. (MR 1 403 509)

[I have not seen this paper.]

177. M. Yamada (1980), *A convergence proof about an integral sequence*, Fibonacci Quarterly **18** (1980), 231–242. (MR 82d:10026)

This paper claims a proof of the $3x+1$ Conjecture. However the proof is in error, with specific mistakes pointed out in the Math. Review. In particular, Lemma 7 (iii) and Lemma 8 are false.

178. Zhao Hua Yang (1998), *An equivalent set for the* $3x+1$ *conjecture*(Chinese), J. South China Normal Univ. Natur. Sci. Ed. 1998, no. 2, 66–68. (MR 2001f:11040).

From Math. Reviews: The $3x+1$ Conjecture is true if, for any fixed $k \geq 1$, it is true for all integers of the special form $3 + \frac{10}{3}(4^k - 1) \pmod{2^{2k+2}}$. Related results were shown by Korec and Znam (1987).

179. R. Zarnowski (2001), *Generalized inverses and the total stopping time of Collatz sequences,* Linear and Multilinear Algebra **49** (2001), 115–130. (MR 2003b:15011).

The $3x + 1$ iteration is formulated in terms of a denumberable Markov chain with transition matrix P. The $3x + 1$ Conjecture is reformulated in terms of the limiting behavior of P^k. The group inverse $A^\sharp$ to an $n \times n$ matrix A is defined by the properties $AA^\sharp = A^\sharp A$, $AA^\sharp A = A$ and $A^\sharp AA^\sharp = A^\sharp$, and is unique when it exists. Now set $A = I - P$, an infinite matrix. Assuming there are no nontrivial cycles, the group inverse $A^\sharp$ exists, and satisfies $\lim_{k\to\infty} P^k = I - AA^\sharp$. An explicit formula is given for $A^\sharp$.

180. Chuan Zhong Zhou (1995), *Some discussion on the* $3x + 1$ *problem.* (Chinese) J. South China Normal Univ. Natur. Sci. Ed. 1995, No. 3, 103–105. (MR 97h:11021).

From Math. Reviews: Given a positive integer, define $C(n) = 3n + 1$ if n is odd, $C(n) = n/2$ if n is even, and let $H = \{n \in \mathbb{Z}_{>0} \ : \ C^{(k)}(n) = 1, \text{ some } k \geq 1\}$. The $3x + 1$ Conjecture asserts that $H = \mathbb{Z}_{>0}$. This paper extends a result of B. Y. Hong [Hubei Shifan Xueyan Xuebao (Siran Kexue) 1986, no. 1, 1–5.] by showing that $\min\{n : \ n \notin H\}$ has $n \equiv 7, 15, 27, 31, 39, 63, 79$ or $91 \pmod{96}$. It also proves that if $2^{2m+1} - 1 \in H$ then $2^{2m+2} - 1 \in H$. Finally it suggests that the numerical computation of $C(n)$ could be simplified if it were performed in base 3.

181. Chuan Zhong Zhou (1997), *Some recurrence relations connected with the* $3x + 1$ *problem* (Chinese), J. South China Normal Univ. Natur. Sci. Ed. 1997, no. 4, 7–8.

[I have not seen this paper.]

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