Orthonormal Bases of Exponentials for the $n$-Cube

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Abstract

A compact domain $\Omega$ in $\mathbb{R}^n$ is a spectral set if there is some subset $\Lambda$ of $\mathbb{R}^n$ such that $\{\exp(2\pi i \langle \lambda, x \rangle) : \lambda \in \Lambda\}$ when restricted to $\Omega$ gives an orthogonal basis of $L^2(\Omega)$. The set $\Lambda$ is called a spectrum for $\Omega$. We give a criterion for $\Lambda$ being a spectrum of a given set $\Omega$ in terms of tiling Fourier space by translates of a suitable auxiliary set $D$. We apply this criterion to classify all spectra for the $n$-cube by showing that $\Lambda$ is a spectrum for the $n$-cube if and only if $\{\lambda + [0, 1]^n : \lambda \in \Lambda\}$ is a tiling of $\mathbb{R}^n$ by translates of unit cubes.

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1. Introduction

A compact set $\Omega$ in $\mathbb{R}^n$ of positive Lebesgue measure is a spectral set if there is some set of exponentials

$$B_\Lambda := \{ e^{2\pi i \langle \lambda , x \rangle} : \lambda \in \Lambda \}, \quad (1.1)$$

which when restricted to $\Omega$ gives an orthogonal basis for $L^2(\Omega)$, with respect to the inner product

$$\langle f , g \rangle_\Omega := \int_{\Omega} f(x)g(x)dx . \quad (1.2)$$

Any set $\Lambda$ that gives such an orthogonal basis is called a spectrum for $\Omega$. Only very special sets $\Omega$ in $\mathbb{R}^n$ are spectral sets. However when a spectrum exists, it can be viewed as a generalization of Fourier series, because for the $n$-cube $\Omega = [0, 1]^n$ the spectrum $\Lambda = \mathbb{Z}^n$ gives the standard Fourier basis of $L^2([0, 1]^n)$.

The main object of this paper is to relate the spectra of sets $\Omega$ to tilings in Fourier space. We develop such a relation and apply it to geometrically characterize all spectra for the $n$-cube $\Omega = [0, 1]^n$.

**Theorem 1.1.** The following conditions on a set $\Lambda$ in $\mathbb{R}^n$ are equivalent.

(i) The set $B_\Lambda = \{ e^{2\pi i \langle \lambda , x \rangle} : \lambda \in \Lambda \}$ when restricted to $[0, 1]^n$ is an orthonormal basis of $L^2([0, 1]^n)$.

(ii) The collection of sets $\{ \lambda + [0, 1]^n : \lambda \in \Lambda \}$ is a tiling of $\mathbb{R}^n$ by translates of unit cubes.

This result was conjectured by Jørgensen and Pedersen [6], who proved it in dimensions $n \leq 3$. We note that in high dimensions there are many "exotic" cube tilings. There are
aperiodic cube tilings in all dimensions $n \geq 3$, while in dimensions $n \geq 10$ there are cube
tilings in which no two cubes share a common $(n - 1)$-face (Lagarias and Shor [8]).

In the theorem above the $n$-cube $[0, 1]^n$ appears in both conditions (i) and (ii), but the
$n$-cube in (i) lies in the space domain $\mathbb{R}^n$ while the $n$-cube in (ii) lies in the Fourier domain
$(\mathbb{R}^n)^*$, so that they transform differently under linear change of variables. Thus Theorem 1.1
is equivalent to the following result.

**Theorem 1.2.** For any invertible linear transformation $A \in GL(n, \mathbb{R})$, the following conditions
are equivalent.

(i) $\Lambda \subset \mathbb{R}^n$ is a spectrum for $\Omega_A := A([0, 1]^n)$.

(ii) The collection of sets $\{\lambda + D_A : \lambda \in \Lambda\}$ is a tiling of $\mathbb{R}^n$, where $D_A = (A^T)^{-1}([0, 1]^n)$.

Our main result in §3 gives a necessary and sufficient condition for a general set $\Lambda$ to be
a spectrum of $\Omega$ in terms of a tiling of $\mathbb{R}^n$ by $\Lambda + D$ where $D$ is a specified auxiliary set in
Fourier space. The applicability of this result is restricted to cases where a suitable auxiliary
set $D$ exists. Theorem 1.2 is then proved in §4.

Spectral sets were originally studied by Fuglede [2], who related them to the problem of finding commuting self-adjoint extensions in $L^2(\Omega)$ of the set of differential operators
$-i\partial_{x_1}, \ldots, -i\partial_{x_n}$ defined on the common dense domain $C_c^{\infty}(\Omega)$. Our definition of spectrum
differs from his by a factor of $2\pi$. Fuglede showed that for sufficiently nice regions $\Omega$ each
spectrum $\Lambda$ of $\Omega$ (in our sense) has $2\pi \Lambda$ as a joint spectrum of a set of commuting self-adjoint
extensions of $-i\partial_{x_1}, \ldots, -i\partial_{x_n}$, and conversely; we state his result precisely in Appendix B.
He also showed that only very special sets $\Omega$ are spectral sets. Much recent work on spectral
sets is due to Jorgensen and Pedersen, see [4]–[6] and [12], [13].

Fuglede [2, p. 120] made the following conjecture.

**Spectral Set Conjecture.** A set $\Omega$ in $\mathbb{R}^n$ is a spectral set if and only if it tiles $\mathbb{R}^n$ by
translation.

This conjecture concerns tilings by $\Omega$ in the space domain; in contrast Theorem 1.2 above
describes spectra $\Lambda$ for the $n$-cube in terms of tilings in the Fourier domain by an auxiliary
set $D$. In general there does not seem to be any simple relation between sets of translations $T$
used to tile $\Omega$ in the space domain and the set of spectra $\Lambda$ for $\Omega$, see [5],[9],[13]. However our
main results in §3 indicate a relation between the Spectral Set Conjecture and tilings in the Fourier domain — this is discussed at the end of §3.

Theorem 1.2 also implies a result concerning sampling and interpolation of certain classes of entire functions. Given a compact set Ω of nonzero Lebesgue measure, let $B_2(Ω)$ denote the set of band-limited functions on Ω, which are those $L^2$ functions on $\mathbb{R}^n$ whose Fourier transform has compact support contained in Ω. Such a function is necessarily the restriction to $\mathbb{R}^n$ of an entire function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ of restricted growth, see Stein and Weiss [15, Theorem 4.9]. A set $Λ$ is a set of sampling for $B_2(Ω)$ if there is a positive constant $C$ such that each $f \in B_2(Ω)$ satisfies

$$\|f\|^2 \geq C \sum_{λ \in Λ} \|f(λ)\|^2. \quad (1.3)$$

A set $Λ$ is a set of interpolation for $B_2(Ω)$ if for each set of complex values $\{c_λ : λ \in Λ\}$ with $\sum |c_λ|^2 < \infty$ there is at least one function $f \in B_2(Ω)$ such that

$$f(λ) = c_λ, \text{ for each } \lambda \in Λ. \quad (1.4)$$

It is clear that a spectrum $Λ$ of a spectral set $Ω$ is both a set of sampling and a set of interpolation for $B_2(Ω)$. So Theorem 1.2 immediately yields:

**Theorem 1.3.** Given a linear transformation $Λ$ in $GL(n, \mathbb{R})$, set $Ω_Λ = Λ([0, 1]^n)$ and $D_Λ = (Λ^T)^{-1}([0, 1]^n)$. If $Λ + D_Λ$ is a tiling of $\mathbb{R}^n$, then $Λ$ is both a set of sampling and a set of interpolation for $B_2(Ω_Λ)$.

Note that the set $Λ$ has density exactly the Nyquist rate $|\text{det}(Λ)|$, as is required by results of Landau ([10], [11]) for sets of sampling and interpolation. In this connection see also Gröchenig and Razafinjatovo [3].

Theorem 1.2 also can be viewed as providing a collection of “nonharmonic Fourier series” expansions for $L^2$-functions on an affine image of the $n$-cube; see Young [16].

We end this introduction with three remarks. First, in comparison with other spectral sets, the $n$-cube $[0, 1]^n$ has an enormous variety of spectra $Λ$. It seems likely that a “generic” spectral set has a unique spectrum, up to translations. Second, the tiling result in §3 applies to more general sets $Ω$ than linearly transformed $n$-cubes $Ω_Λ = Λ([0, 1]^n)$; a one-dimensional example is $Ω = [0, 1] \cup [2, 3]$. Third, there are open questions remaining in explicitly describing

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2It can be shown that “generic” fundamental domain $Ω$ of a full rank lattice $L$ in $\mathbb{R}^n$ has a unique spectrum $Λ = L^*$, the dual lattice.
the commuting self-adjoint extensions of $-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}$ in $L^2([0,1]^n)$ that correspond to cube tilings; see Appendix B.

Appendix A to the paper addresses the question of whether an orthogonal cube packing in $\mathbb{R}^n$ can be extended to a cube tiling; Appendix B describes the connection of spectral sets and commuting partial differential operators.

**Notation.** For $x \in \mathbb{R}^n$, let $\|x\|$ denote the Euclidean length of $x$. We let

$$B(x; T) := \{ y : \|y - x\| \leq T \}$$

denote the ball of radius $T$ centered at $x$. The Lebesgue measure of a set $\Omega$ in $\mathbb{R}^n$ is denoted $m(\Omega)$. The Fourier transform $\hat{f}(u)$ is normalized by

$$\hat{f}(u) := \int_{\mathbb{R}^n} e^{-2\pi i(u,x)} f(x) \, dx .$$

Throughout the paper we let

$$e_\lambda(x) := e^{2\pi i(\lambda,x)}, \quad \text{for} \quad x \in \mathbb{R}^n . \quad (1.5)$$

Note that other authors ([2] [6]) define $e_\lambda(x)$ without the factor $2\pi$.

2. **Orthogonal Sets of Exponentials and Packings**

We consider packings and tilings in $\mathbb{R}^n$ by compact sets $\Omega$ of the following kind.

**Definition 2.1.** A compact set $\Omega$ in $\mathbb{R}^n$ is a regular region if it has positive Lebesgue measure $m(\Omega) > 0$, is the closure of its interior $\Omega^o$, and has a boundary $\partial\Omega = \Omega \setminus \Omega^o$ of measure zero.

**Definition 2.2.** If $\Omega$ is a regular region, then a discrete set $\Lambda$ is a packing set for $\Omega$ if the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ have disjoint interiors. It is a tiling set if in addition the union of the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ covers $\mathbb{R}^n$. In these cases we say $\Lambda + \Omega$ is a packing or tiling of $\mathbb{R}^n$ by $\Omega$, respectively.

To a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $\mathbb{R}^n$ we associate the exponential function

$$e_\lambda(x) := e^{2\pi i(\lambda,x)}, \quad \text{for} \quad x \in \mathbb{R}^n . \quad (2.1)$$

Given a discrete set $\Lambda$ in $\mathbb{R}^n$, we set

$$B_\Lambda := \{e_\lambda(x) : \lambda \in \Lambda\} . \quad (2.2)$$
Now suppose that \( \mathcal{B}_\Lambda \) restricted to a regular region \( \Omega \) gives an orthogonal set of exponentials in \( L^2(\Omega) \). We derive conditions that the points of \( \Lambda \) must satisfy. Let
\[
\chi_\Omega(x) = \begin{cases} 
1 & \text{for } x \in \Omega \\
0 & \text{for } x \not\in \Omega 
\end{cases}
\] (2.3)
be the characteristic function of \( \Omega \), and consider its Fourier transform
\[
\hat{\chi}_\Omega(u) = \int_{\mathbb{R}^n} e^{-2\pi i \langle u, x \rangle} \chi_\Omega(x) dx, \quad u \in \mathbb{R}^n.
\] (2.4)
Since \( \Omega \) is compact the function \( \hat{\chi}_\Omega(u) \) is an entire function of \( u \in \mathbb{C}^n \). We denote the set of real zeros of \( \hat{\chi}_\Omega(u) \) by
\[
Z(\Omega) := \{ u \in \mathbb{R}^n : \hat{\chi}_\Omega(u) = 0 \}.
\] (2.5)

**Lemma 2.1.** If \( \Omega \) is a regular region in \( \mathbb{R}^n \) then a set \( \Lambda \) gives an orthogonal set of exponentials \( \mathcal{B}_\Lambda \) in \( L^2(\Omega) \) if and only if
\[
\Lambda - \Lambda \subseteq Z(\Omega) \cup \{ 0 \}.
\] (2.6)

**Proof.** For distinct \( \lambda, \lambda' \in \Lambda \) we have
\[
\hat{\chi}_\Omega(\lambda - \lambda') = \int_{\mathbb{R}^n} e^{-2\pi i \langle \lambda - \lambda', x \rangle} \chi_\Omega(x) dx
\]
\[
= \int_{\Omega} e^{-2\pi i \langle \lambda - \lambda', x \rangle} e^{2\pi i \langle \lambda', x \rangle} dx
\]
\[
= \langle e_{\lambda}, e_{\lambda'} \rangle_\Omega.
\] (2.7)

If (2.6) holds, then \( \langle e_{\lambda}, e_{\lambda'} \rangle_\Omega = 0 \), and conversely. \( \blacksquare \)

This lemma implies that the points of \( \Lambda \) cannot be too close together. Since \( \hat{\chi}_\Omega(0) = m(\Omega) > 0 \), the continuity of \( \hat{\chi}_\Omega(u) \) implies that there is some ball \( B(0; R) \) around \( 0 \) that includes no point of \( Z(\Omega) \), hence \( |\lambda - \lambda'| \geq R \) for all \( \lambda, \lambda' \in \Lambda, \lambda \neq \lambda' \).

**Definition 2.3.** Let \( \Omega \) be a regular region in \( \mathbb{R}^n \). A regular region \( D \) is said to be an orthogonal packing region for \( \Omega \) if
\[
(D^o - D^o) \cap Z(\Omega) = \emptyset.
\] (2.8)

**Lemma 2.2.** Let \( \Omega \) be a regular region in \( \mathbb{R}^n \) and let \( D \) be an orthogonal packing region for \( \Omega \). If a set \( \Lambda \) gives an orthogonal set of exponentials \( \mathcal{B}_\Lambda \) in \( L^2(\Omega) \) then \( \Lambda \) is a packing set for \( D \).
Proof. If \( \lambda \neq \lambda' \in \Lambda \) then Lemma 2.1 gives \( \lambda - \lambda' \in Z(\Omega) \). By definition of an orthogonal packing region we have \( D^\circ \cap (D^\circ + u) = \emptyset \) for all \( u \in Z(\Omega) \) hence

\[
D^\circ \cap (D^\circ + \lambda - \lambda') = \emptyset ,
\]
as required. ■

As indicated above, each regular region \( \Omega \) has an orthogonal packing region \( D \) given by a ball \( B(0; T) \) for small enough \( T \). The larger we can take \( D \), the stronger the restrictions imposed on \( \Lambda \).

**Lemma 2.3.** If \( \Omega \) is a spectral set, and \( D \) is an orthogonal packing region for \( \Omega \), then

\[
m(D)m(\Omega) \leq 1 .
\]

**Proof.** Let \( \Lambda \) be a spectrum for \( \Omega \). Then \( \Lambda \) is a set of sampling for \( B_2(\Omega) \), so the density results of Landau [10] (see also Gröchenig and Razafinjatovo [3]) give

\[
d(\Lambda) = \liminf_{n \to \infty} \frac{1}{(2T)^n} \#(\Lambda \cap [-T,T]^n) \geq m(\Omega) .
\]

Now \( \Lambda + D \) is a packing of \( \mathbb{R}^n \), hence if \( R = \text{diam}(D) \), we have

\[
\frac{m(D)}{(2T)^n} \#(\Lambda \cap [-T,T]^n) = \frac{1}{(2T)^n} m \left( \bigcup_{\lambda} (\lambda + D) : \lambda \in \Lambda \cap [-T,T]^n \right) \\
\leq \frac{m([-T + R, T + R]^n)}{(2T)^n} = \left( 1 + \frac{R}{2T} \right)^n .
\]

Letting \( T \to \infty \) and taking the lim inf yields

\[
m(D)d(\Lambda) \leq 1,
\]
which with (2.10) yields (2.9). ■

In §3 we give a self-contained proof of Lemma 2.3. The inequality (2.9) of Lemma 2.3 does not hold for general sets \( \Omega \). In fact the set \( \Omega = [0, 1] \cup [2, 2 + \theta] \) for suitable irrational \( \theta \) has a Fourier transform \( \hat{\chi}_\Omega(\xi) \) which has no real zeros, so \( Z(\Omega) = \emptyset \), and any regular region \( D \) is an orthogonal packing region for \( \Omega \).

In view of Lemma 2.3 we introduce the following terminology.

**Definition 2.4.** An orthogonal packing region \( D \) for a regular region \( \Omega \) is tight if

\[
m(D) = \frac{1}{m(\Omega)} .
\]

**Lemma 2.4.** Let \( D \) be a tight orthogonal packing region for a regular region \( \Omega \). Then for any \( A \in GL(n, \mathbb{R}) \) the set \( (A^T)^{-1}(D) \) is a tight orthogonal packing region for \( A(\Omega) \).
Proof. Since \(\hat{\chi}_{A(\Omega)}(u) = |\det(A)|\hat{\chi}_\Omega(A^T u)\), \(Z(A(\Omega)) = (A^T)^{-1}Z(\Omega)\). Hence \((A^T)^{-1}(D)\) is an orthogonal packing region for \(A(\Omega)\). It is tight because

\[
m((A^T)^{-1}D) = \frac{1}{|\det(A^T)|} m(D) = \frac{1}{|\det(A^T)|} \frac{1}{m(\Omega)} = \frac{1}{m(A(\Omega))}.
\]

There are many spectral sets which have tight orthogonal packing regions. For our main result in §4 we show that if \(\Omega_A = A([0,1]^n)\) is an affine image of the \(n\)-cube, with \(A \in GL(n, \mathbb{R})\), then

\[
D := (A^T)^{-1}([0,1]^n)
\]

(2.14)

is a tight orthogonal packing region for \(\Omega_A\). Another example in \(\mathbb{R}^1\) is the region

\[
\Omega = [0,1] \cup [2,3].
\]

(2.15)

In this case we can take

\[
D = \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right].
\]

(2.16)

Indeed \(\chi_\Omega(x)\) is the convolution of \(\chi_{[0,1]}(x)\) with the sum of two delta functions \(\delta_0 + \delta_2\). Thus

\[
\hat{\chi}_\Omega(x) = (1 + e^{-4\pi ix})\hat{\chi}_{[0,1]}(x).
\]

(2.17)

From this it is easy to check that the zero set is given by

\[
Z(\Omega) = (\mathbb{Z} \setminus \{0\}) \cup \left(\frac{1}{4} + \mathbb{Z}\right) \cup \left(-\frac{1}{4} + \mathbb{Z}\right),
\]

(2.18)

that \(D\) is an orthogonal packing region for \(\Omega\), and, since \(m(D) = \frac{1}{7} = \frac{1}{m(\Omega)}\), that \(D\) is tight. A spectrum for \(\Omega\) is \(\Lambda = \mathbb{Z} \cup \left(\mathbb{Z} + \frac{1}{7}\right)\).

Lemma 2.3 together with the spectral set conjecture lead us to propose:

**Conjecture 2.1.** If \(\Omega\) tiles \(\mathbb{R}^n\) by translations, and \(D\) is an orthogonal packing region for \(\Omega\), then

\[
m(\Omega)m(D) \leq 1.
\]

(2.19)

3. Spectra and Tilings

A main result of this paper is the following criterion which relates spectra to tilings in the Fourier domain.
**Theorem 3.1.** Let $\Omega$ be a regular region in $\mathbb{R}^n$, and let $\Lambda$ be such that the set of exponentials $B_\Lambda$ is orthogonal for $L^2(\Omega)$. Suppose that $D$ is a regular region with

$$m(D)m(\Omega) = 1$$

(3.1)

such that $\Lambda + D$ is a packing of $\mathbb{R}^n$. Then $\Lambda$ is a spectrum for $\Omega$ if and only if $\Lambda + D$ is a tiling of $\mathbb{R}^n$.

**Proof.** $\Rightarrow$. Suppose first that $\Lambda$ is a spectrum for $\Omega$. Pick a “bump function” $\gamma(x) \in C_c^\infty(\Omega)$, and set

$$\gamma_t(x) = e^{-2\pi i \gamma(x)}, \quad \text{for } t \in \mathbb{R}^n.$$ 

By hypothesis $B_\Lambda = \{e_\lambda(x) : \lambda \in \Lambda\}$ is orthogonal and complete for $L^2(\Omega)$. Thus, on $\Omega$, we have

$$\gamma_t(x) \sim \sum_{\lambda \in \Lambda} \left( \frac{e^{-2\pi i \langle \lambda, x \rangle}}{\|e_\lambda\|_2} \right) \gamma_t(x) = e^{2\pi i \langle \lambda, x \rangle},$$

(3.2)

with coefficients

$$\frac{\langle e^{2\pi i \langle \lambda, x \rangle}, \gamma_t(x) \rangle}{\|e_\lambda\|_2} = \frac{1}{m(\Omega)} \int_{\Omega} e^{-2\pi i \langle \lambda, x \rangle} \gamma_t(x) dx = \frac{1}{m(\Omega)} \int_{\mathbb{R}^n} e^{-2\pi i \langle \lambda + t, x \rangle} \gamma_t(x) dx = \frac{1}{m(\Omega)} \hat{\gamma}(\lambda + t),$$

(3.3)

where $m(\Omega)$ is the Lebesgue measure of $\Omega$. Since $\gamma$ is a smooth function,

$$\|\hat{\gamma}(u)\| \leq C\|u\|^{-n-2}, \quad \text{for } u \in \mathbb{R}^n \text{ with } \|u\| \geq 1.$$ 

(3.4)

This fact, plus the “well-spaced” property of $\Lambda$ shows that the right side of (3.2) converges absolutely and uniformly on $\mathbb{R}^n$, for fixed $t$, to a continuous function. Since $\gamma_t(x)$ is continuous, we have

$$\gamma_t(x) = \frac{1}{m(\Omega)} \sum_{\lambda \in \Lambda} \hat{\gamma}(\lambda + t) e^{2\pi i \langle \lambda, x \rangle},$$

(3.5)

This yields, for all $t \in \mathbb{R}^n$, that

$$\gamma(x) = e^{2\pi i \langle \lambda, x \rangle} \gamma_t(x) = \frac{1}{m(\Omega)} \sum_{\lambda \in \Lambda} \hat{\gamma}(\lambda + t) e^{2\pi i \langle \lambda + t, x \rangle},$$

(3.6)

The series on the right side of (3.6) converges absolutely and uniformly for all $x \in \mathbb{R}^n$ and for $t$ in any fixed compact subset of $\mathbb{R}^n$, but is only guaranteed to agree with $\gamma(x)$ for $x \in \Omega$. 

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We now integrate both sides of (3.6) in $t$ over all $t \in D$ to obtain:

$$m(D)\hat{\gamma}(x) = \frac{1}{m(\Omega)} \int_D \sum_{\lambda \in \Lambda} \hat{\gamma}(\lambda + t)e^{2\pi i (\lambda + t,x)} dt$$

$$= \frac{1}{m(\Omega)} \sum_{\lambda \in \Lambda} \int_D \hat{\gamma}(\lambda + t)e^{2\pi i (\lambda + t,x)} dt$$

$$= \frac{1}{m(\Omega)} \int_{\Lambda + D} \hat{\gamma}(u)e^{2\pi i (u,x)} du, \quad \text{for all } x \in \Omega. \quad (3.7)$$

In the last step we used the fact that the translates $\lambda + D$ overlap on sets of measure zero, because $\Lambda + D$ is a packing of $\mathbb{R}^n$. Since $m(D) = \frac{1}{m(\Omega)}$, (3.7) yields

$$\hat{\gamma}(x) = \int_{\mathbb{R}^n} \hat{\gamma}(u)h(u)e^{2\pi i (u,x)} du, \quad \text{for all } x \in \Omega, \quad (3.8)$$

where

$$h(u) = \begin{cases} 1 & \text{if } u \in \Lambda + D \\ 0 & \text{otherwise} \end{cases}.$$ 

Define $k \in L^2(\mathbb{R}^n)$ by $\hat{k} = h\hat{\gamma}$, so (3.8) asserts that $\gamma(x) = k(x)$ for almost all $x \in \Omega$. Plancherel’s theorem on $L^2(\mathbb{R}^n)$ applied to $k$, together with (3.8), gives

$$||\gamma||_2^2 \geq ||h\hat{\gamma}||_2^2 = ||k||_2^2$$

$$\geq \int_{\Omega} |k(x)|^2 dx = \int_{\Omega} |\gamma(x)|^2 dx = ||\gamma||_2^2. \quad (3.9)$$

Since Plancherel’s theorem also gives $||\gamma||_2^2 = ||\gamma||_2^2$, we must have

$$||\gamma||_2^2 = ||h\hat{\gamma}||_2^2. \quad (3.10)$$

We next show that this equality implies that $h(u) = 1$ almost everywhere on $\mathbb{R}^n$. To do this we show that $\hat{\gamma}(u) \neq 0$ a.e. in $\mathbb{R}^n$. Since $\gamma$ has compact support, the Paley-Wiener theorem states that $\hat{\gamma}(u)$ is the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$ that satisfies an exponential growth condition at infinity, see Stein and Weiss [15], Theorem 4.9. Thus $\hat{\gamma}(u)$ is real-analytic on $\mathbb{R}^n$ and is not identically zero, hence

$$Z := \{u \in \mathbb{R}^n : \hat{\gamma}(u) = 0\}$$

has Lebesgue measure zero. Together with (3.10) this yields

$$h(u) = 1 \quad \text{a.e. in } \mathbb{R}^n. \quad (3.11)$$

Thus $\Lambda + D$ covers all of $\mathbb{R}^n$ except a set of measure zero.
Finally we show that \( \Lambda + D \) covers all of \( \mathbb{R}^n \). By the well-spaced property of \( \Lambda \) and the compactness of \( D \), the set \( \Lambda + D \) is locally the union of finitely many translates of \( D \), hence \( \Lambda + D \) is closed. Thus the complement of \( \Lambda + D \) is an open set. But the complement of \( \Lambda + D \) has zero Lebesgue measure, hence it is empty, so \( \Lambda + D \) is a tiling of \( \mathbb{R}^n \).

\( \Leftarrow \). Suppose \( \Lambda + D \) tiles \( \mathbb{R}^n \). By hypothesis \( \mathcal{B}_\Lambda \) is an orthogonal set in \( L^2(\Omega) \), and to show that \( \Lambda \) is a spectrum it remains to show that it is complete in \( L^2(\Omega) \). Let \( S \) be the closed span of \( \mathcal{B}_\Lambda \) in \( L^2(\Omega) \). We will show that \( C_c^\infty(\Omega) \) is contained in \( S \). Since \( C_c^\infty(\Omega) \) is dense in \( L^2(\Omega) \) this implies \( S = L^2(\Omega) \).

For each \( \gamma \in C_c^\infty(\Omega) \) set

\[
\gamma_t(x) = e^{-2\pi i (t, x)} \gamma(x), \quad \text{for } t \in \mathbb{R}^n .
\]

Since the elements of \( \mathcal{B}_\Lambda \) are orthogonal, Bessel’s inequality gives

\[
\| \gamma_t \|^2 \geq \sum_{\lambda \in \Lambda} \frac{\langle e_\lambda, \gamma_t \rangle^2}{\| e_\lambda \|^2} = \frac{1}{m(\Omega)} \sum_{\lambda \in \Lambda} |\hat{\gamma}(\lambda + t)|^2 ,
\]

where the last series converges uniformly on compact sets by the rapid decay of \( \hat{\gamma} \) at infinity. Integrating this inequality over \( t \in D \) yields

\[
\int_D \| \gamma_t \|^2 dt \geq \frac{1}{m(\Omega)} \int_D \sum_{\lambda \in \Lambda} |\hat{\gamma}(\lambda + t)|^2 dt .
\]

Since \( \| \gamma_t \| = \| \gamma \| \) for all \( t \), and since \( \Lambda + D \) is a tiling, we obtain \( m(D) \| \gamma \|^2 \geq \| \hat{\gamma} \|^2 / m(\Omega) \). But \( m(D) = 1/m(\Omega) \) and \( \| \gamma \|^2 = \| \hat{\gamma} \|^2 \), so equality must hold in (3.12) for almost all \( t \):

\[
\| \gamma \|^2 = \sum_{\lambda \in \Lambda} \frac{|\langle e_\lambda, \gamma \rangle|^2}{\| e_\lambda \|^2} .
\]

Now the right side of (3.13) converges uniformly on compact sets in \( t \) to a continuous function of \( t \), and the left side is a constant, so (3.13) holds for all \( t \), including \( t = 0 \). Hence \( \gamma \in S \). \( \blacksquare \)

At first glance this proof of Theorem 3.1 appears “too good to be true” because it only uses functions \( \gamma_t(x) \) supported on a tiny part of \( \Omega \). In fact all of \( \Omega \) is used in the formula (3.6) which is required to be valid for all \( x \in \Omega \).

The proof of Theorem 3.1 yields a direct proof of Lemma 2.3. If \( D \) is an orthogonal packing set, then (3.7) holds for it, hence \( m(D)m(\Omega) \gamma(x) \) agrees with \( k(x) \) on \( \Omega \), hence

\[
m(D)m(\Omega) \| \gamma \|_2 \leq \| k \|_2 \leq \| \gamma \|_2
\]
hence (2.9) holds.

The following result is an immediate corollary of Theorem 3.1, which we state as a theorem for emphasis.

**Theorem 3.2.** Let \( \Omega \) be a regular region in \( \mathbb{R}^n \), and suppose that \( D \) is a tight orthogonal packing region for \( \Omega \). If \( \Lambda \) is a spectrum for \( \Omega \), then \( \Lambda + D \) is a tiling of \( \mathbb{R}^n \).

**Proof.** The assumption that \( D \) is a tight orthogonal packing region guarantees that \( \Lambda + D \) is a packing for all spectra \( \Lambda \), so Theorem 3.1 applies. \( \blacksquare \)

Theorem 3.2 sheds some light on Fuglede's conjecture that every spectral set \( \Omega \) tiles \( \mathbb{R}^n \).

**Definition 3.1.** A pair of regular regions \((\Omega, \hat{\Omega})\) are a **tight dual pair** if each is a tight orthogonal packing region for the other.

In §4 we show that \((A([0,1]^n), (A^T)^{-1}([0,1]^n))\) are a tight dual pair of regions. The sets \(([0,1] \cup [2,3], [0, \frac{1}{2}] \cup [\frac{3}{2}, 3])\) are a tight dual pair in \( \mathbb{R}^1 \).

If \((\Omega, \hat{\Omega})\) are a tight dual pair, then Theorem 3.1 states that if one of \((\Omega, \hat{\Omega})\) is a spectral set, say \( \Omega \), then the other set \( \hat{\Omega} \) tiles \( \mathbb{R}^n \). If \( \hat{\Omega} \) were also a spectral set (as Fuglede's conjecture implies) then Theorem 3.1 would show that \( \Omega \) tiles \( \mathbb{R}^n \). This raises the question whether the current evidence in favor of Fuglede's conjecture is mainly based on sets \( \Omega \) which are part of a tight dual pair \((\Omega, \hat{\Omega})\). At present we can only say that there are many nontrivial examples of tight dual pairs.

To clarify matters, we formulate two conjectures.

**Conjecture 3.1.** *(Spectral Set Duality Conjecture)* If \((\Omega, \hat{\Omega})\) is a tight dual pair of regular regions, and \( \Omega \) is a spectral set, then \( \hat{\Omega} \) is also a spectral set.

In this case Theorem 3.2 would imply that both \( \Omega \) and \( \hat{\Omega} \) tile \( \mathbb{R}^n \).

The following is a tiling analogue of the conjecture above.

**Conjecture 3.2.** *(Weak Spectral Set Conjecture)* If \((\Omega, \hat{\Omega})\) are a tight dual pair of regular regions, and one of them tiles \( \mathbb{R}^n \), then so does the other, and both \( \Omega \) and \( \hat{\Omega} \) are spectral sets.

4. **Spectra for the n-cube and Cube Tilings**

We now prove Theorem 1.2, using the results of §3.
The next two lemmas show that if $\Omega_A = A([0, 1]^n)$ is an affine image of the $n$-cube, with $A \in GL(n, \mathbb{R})$, then
\[
D := (A^T)^{-1}([0, 1]^n)
\]
is a tight orthogonal packing region for $\Omega_A$.

**Lemma 4.1.** $B_A := \{e^{2\pi i (\lambda, x)} : \lambda \in \Lambda \}$ gives a set of orthogonal functions in $L^2([0, 1]^n)$ if and only if for any distinct $\lambda, \mu \in \Lambda$,
\[
\lambda_j - \mu_j \in \mathbb{Z} \setminus \{0\} \quad \text{for some} \quad j, \quad 1 \leq j \leq n.
\]

**Proof.** For $\Omega = [0, 1]^n$ and $u \in \mathbb{R}^n$,
\[
\hat{\chi}_\Omega(u) = \int_{[0, 1]^n} e^{-2\pi i (u, x)} dx = \prod_{j=1}^{n} h_0(u_j),
\]
where $h_0(\omega) := (1 - e^{-2\pi i \omega})/(2\pi i \omega)$, $\omega \in \mathbb{R}$, and $h_0(0) := 1$. Note that $h_0(\omega) = 0$ if and only if $\omega \in \mathbb{Z} \setminus \{0\}$. Hence $\hat{\chi}_\Omega(u) = 0$ if and only if $u_j \in \mathbb{Z} \setminus \{0\}$ for some $j$, $1 \leq j \leq n$. The lemma now follows immediately from Lemma 2.1. ■

**Lemma 4.2.** Let $A \in GL(n, \mathbb{R})$ and $\Omega = A([0, 1]^n)$. Then $D = (A^T)^{-1}([0, 1]^n)$ is a tight orthogonal packing region for $\Omega$.

**Proof.** By Lemma 2.4 we need only check this for $\Omega = [0, 1]^n$. Lemma 4.1 implies that $D = [0, 1]^n$ is an orthogonal packing region for $\Omega$, and it is clearly tight. ■

We will also use the following basic result of Keller [7].

**Proposition 4.1.** (Keller) If $\Lambda + [0, 1]^n$ is a tiling of $\mathbb{R}^n$, then each $\lambda, \lambda' \in \Lambda$ has
\[
\lambda_i - \lambda'_i \in \mathbb{Z} \setminus \{0\} \quad \text{for some} \quad i, \quad 1 \leq i \leq n.
\]

**Proof.** This result was proved by Keller [7] in 1930. A detailed proof appears in Perron [14], Satz 9. ■

**Proof of Theorem 1.2.** (i) $\Rightarrow$ (ii). Suppose that $\Lambda$ is a spectrum for $D = A([0, 1]^n)$.

Lemma 4.2 gives that $D = (A^T)^{-1}([0, 1]^n)$ is a tight orthogonal packing set for $\Omega$. By Theorem 3.2 $\Lambda + D$ is a tiling of $\mathbb{R}^n$. 

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(ii) $\Rightarrow$ (i). It suffices to prove this direction for the $n$-cube $\Omega = [0, 1]^n$, since the general case follows by a linear change of variables. We take $D = [0, 1]^n$, so $m(\Omega)m(D) = 1$. Let $\Lambda + D$ be a cube-tiling. Now Proposition 4.1 shows that $\mathcal{B}_\Lambda$ is an orthogonal set in $L^2([0, 1]^n)$, by the criterion of Lemma 2.4. The hypotheses of Theorem 3.1 hold, and we conclude that $\Lambda$ is a spectrum because $\Lambda + D$ is a tiling.
Appendix A. Extending Cube Packings to Cube Tilings

This appendix addresses the problem of when a cube packing in \( \mathbb{R}^n \) can be extended to a cube tiling by adding extra cubes.

**Definition A.1.** A cube packing \( \Lambda + [0, 1]^n \) is *orthogonal* if for distinct \( \lambda, \mu \in \Lambda \),

\[
\lambda_j - \mu_j \in \mathbb{Z} \setminus \{0\} \quad \text{for some } j, \quad 1 \leq j \leq n. \tag{A.1}
\]

Keller’s theorem (Proposition 4.1) shows that a necessary condition for a cube packing to be extendible to a cube tiling is that it be orthogonal. A natural question is: Can every orthogonal cube packing in \( \mathbb{R}^n \) be completed to a cube tiling of \( \mathbb{R}^n \)? The answer is “yes” in dimensions 1 and 2, as can be easily checked. However, we show that it is “no” in dimensions 3 and above.

**Theorem A.1.** *In each dimension \( n \geq 3 \) there is an orthogonal cube packing that does not extend to a cube tiling of \( \mathbb{R}^n \).*

**Proof.** In dimension 3, consider the set of four cubes \( \{v^{(i)} + [0, 1]^3 : 1 \leq i \leq 4\} \) in \( \mathbb{R}^4 \), given by

\[
\begin{align*}
v^{(1)} &= \left( -1, \ 0, \ -\frac{1}{2} \right) \\
v^{(2)} &= \left( -\frac{1}{2}, \ -1, \ 0 \right) \\
v^{(3)} &= \left( 0, \ -\frac{1}{2}, \ -1 \right) \\
v^{(4)} &= \left( \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2} \right)
\end{align*}
\]

The orthogonality condition (4.2) is easily verified. The cubes corresponding to \( v^{(1)} \) through \( v^{(3)} \) contain \( (0,0,0) \) on their boundary and create a corner \( (0,0,0) \). Any cube tiling that extended \( \{v^{(i)} + [0, 1]^3 : 1 \leq i \leq 3\} \) would have to fill this corner by including the cube \([0,1]^3\). However \([0,1]^3\) has nonempty interior in common with \( v^{(4)} + [0,1]^3\).

This construction easily generalizes to \( \mathbb{R}^n \) for \( n \geq 3 \). \( \blacksquare \)
Appendix B. Commuting Self-Adjoint Partial Differential Operators

B. Fuglede [2] studied the problem of finding commuting self-adjoint extensions of the operators \(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}\) to suitable regions in \(L^2(\Omega)\). Note that each operator \(-i\frac{\partial}{\partial x_i}\) is a “Dirac operator” in the sense that it is the “square root” of the “Laplace operator” \(\frac{\partial^2}{\partial x_i^2}\).

Definition B.1 A Nikodym region \(\Omega\) in \(\mathbb{R}^n\) is an open set such that every distribution \(u\) on \(\Omega\) such that \(D_j u \in L^2(\Omega)\) for \(1 \leq j \leq n\) necessarily has \(u \in L^2(\Omega)\).

Any bounded open subset of \(\mathbb{R}^n\) of finite measure which is star-shaped with respect to some interior point is a Nikodym region [1, p. 332]. Thus the open unit cube \((0, 1)^n\) is a Nikodym region.

Let \(D_j\) denote the operator \(\frac{\partial}{\partial x_j}\) extended to its maximal domain in \(L^2(\Omega)\), given by

\[
\text{dom}(D_j) := \{ u \in L^2(\Omega) : D_j u \in L^2(\Omega) \},
\]

where \(D_j\) acts in the sense of distributions on \(L^2(\Omega)\). Fuglede [2, Theorem 1], proved the following\(^3\) result.

Theorem B.1. (Fuglede) Suppose that \(\Omega \subset \mathbb{R}^n\) is a Nikodym region.

(i) Let \(H = (H_1, \ldots, H_n)\) denote a commuting family (if any) of self-adjoint restrictions \(H_j\) of \(D_j\) on \(L^2(\Omega)\). Then \(H\) has a discrete joint spectrum \(\sigma(H)\) in which each point \(2\pi \lambda \in \sigma(H)\) is a simple eigenvalue with eigenspace \(\mathbb{C}e_{\lambda}\), and if \(\Lambda = \frac{1}{2\pi}\sigma(H)\) then \(B_{\Lambda} = \{ e_{\lambda} : \lambda \in \Lambda \}\) is an orthogonal basis of \(L^2(\Omega)\).

(ii) Conversely, let \(\Lambda\) be a subset (if any) of \(\mathbb{R}^n\) such that \(\{ e_{\lambda} : \lambda \in \Lambda \}\) is an orthogonal basis for \(L^2(\Omega)\). Then there exists a unique commuting family \(H = (H_1, \ldots, H_n)\) of self-adjoint restrictions \(H_j\) of \(D_j\) on \(L^2(\Omega)\) such that \(\{ e_{\lambda} : \lambda \in \Lambda \} \subset \text{dom}(H)\), or equivalently that \(\Lambda = \frac{1}{2\pi}\sigma(H)\).

We apply this theorem to the special case where \(\Omega = [0, 1]^n\) is the \(n\)-cube. Theorem 1.1 classified all orthogonal bases of exponentials for \(\Omega\), and the result above shows that there is a unique commuting family \(H_{\Lambda}\) associated to each cube tiling \(\Lambda\). Can one give a precise description of \(H_{\Lambda}\) in terms of the data \(\Lambda\)?

\(^3\)Note that our exponential \(e_{\lambda}\) corresponds to Fuglede’s exponential \(e^{2\pi i \lambda}\).
The self-adjoint extensions of $-i\frac{\partial}{\partial x_j}$ acting on $C^\infty([0,1]^n)$ inside the Hilbert space $L^2([0,1]^n)$ may be thought of as being specified by boundary conditions; this is described in Jorgensen and Pedersen [6, Lemma 3.1]. The boundary conditions for $-i\frac{\partial}{\partial x_j}$ are imposed on the two opposite $(n-1)$-faces of the cube $H_j^{(0)}$ and $H_j^{(1)}$ given by

$$H_j^{(k)} := \{ x \in [0,1] : x_j = k \} \text{ for } k = 0, 1 . \quad (B.2)$$

Each self-adjoint extension $V_j$ of $-i\frac{\partial}{\partial x_j}$ corresponds to a partial isometry

$$U_{V_j} : \mathcal{D}_+^{(j)} \longrightarrow \mathcal{D}_-^{(j)} ,$$

in which $\mathcal{D}_+^{(j)} \subseteq L^2(H_j^{(0)})$ and $\mathcal{D}_-^{(j)} \subseteq L^2(H_j^{(1)})$ are suitable dense subspaces. Can the boundary condition operators $(U_{V_1}, \ldots, U_{V_n})$ for $H = H_\Lambda$ be explicitly constructed for a tiling $\Lambda^2$?

As one example, consider the translated Fourier basis $\Lambda = \mathbb{Z}+t$. If we identify each $L_2(H_j^{(i)})$ with $L^2([0,1]^{n-1})$ in the obvious way, then the corresponding boundary conditions are given by

$$U_{V_j}(f) = e^{2\pi it_j}f, \quad 1 \leq j \leq n . \quad (B.3)$$

Here the domain and range of $U_{V_j}$ are all of $L^2([0,1]^{n-1})$.
References


