Linear Fractional Transformations of Continued Fractions with Bounded Partial Quotients

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Abstract. Let \( \theta \) be a real number with continued fraction expansion

\[
\theta = [a_0, a_1, a_2, \ldots],
\]

and let

\[
M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

be a matrix with integer entries and nonzero determinant. If \( \theta \) has bounded partial quotients, then \( \frac{a\theta + b}{c\theta + d} = [a_0', a_1', a_2', \ldots] \) also has bounded partial quotients. More precisely, if \( a_j \leq K \) for all sufficiently large \( j \), then \( a_j' \leq |\det(M)|(K + 2) \) for all sufficiently large \( j \). We also give a weaker bound valid for all \( a_j' \) with \( j \geq 1 \). The proofs use the homogeneous Diophantine approximation constant \( L_\infty(\theta) = \limsup_{q \to \infty} (q|a_\theta|)^{-1} \). We show that

\[
\frac{1}{|\det(M)|} L_\infty(\theta) \leq L_\infty \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)| L_\infty(\theta).
\]

1. Introduction.

Let \( \theta \) be a real number whose expansion as a simple continued fraction is

\[
\theta = [a_0, a_1, a_2, \ldots],
\]

and set

\[
(1.1) \quad K(\theta) := \sup_{i \geq 1} a_i,
\]

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\]

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where we adopt the convention that $K(\theta) = +\infty$ if $\theta$ is rational. We say that $\theta$ has bounded partial quotients if $K(\theta)$ is finite. We also set

$$K_\infty(\theta) := \limsup_{i \geq 1} a_i,$$

with the convention that $K_\infty(\theta) = +\infty$ if $\theta$ is rational. Certainly $K_\infty(\theta) \leq K(\theta)$, and $K_\infty(\theta)$ is finite if and only if $K(\theta)$ is finite.

A survey of results about real numbers with bounded partial quotients is given in [17]. The property of having bounded partial quotients is equivalent to $\theta$ being a badly approximable number, which is a number $\theta$ such that

$$\liminf_{q \to \infty} q||q\theta|| > 0,$$

in which $||x|| = \min(x - [x], [x] - x)$ denotes the distance from $x$ to the nearest integer and $q$ runs through integers.

This note proves two quantitative versions of the theorem that if $\theta$ has bounded partial quotients and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then $\psi = \frac{a\theta + b}{c\theta + d}$ also has bounded partial quotients.

The first result bounds $K_\infty(\frac{a\theta + b}{c\theta + d})$ in terms of $K_\infty(\theta)$ and depends only on $|\det(M)|$:

**Theorem 1.1.** Let $\theta$ have a bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then

$$\frac{1}{|\det M|} K_\infty(\theta) - 2 \leq K_\infty(\frac{a\theta + b}{c\theta + d}) \leq |\det M|(K_\infty(\theta) + 2).$$

The second result upper bounds $K(\frac{a\theta + b}{c\theta + d})$ in terms of $K(\theta)$, and depends on the entries of $M$:

**Theorem 1.2.** Let $\theta$ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then

$$K(\frac{a\theta + b}{c\theta + d}) \leq |\det(M)|(K(\theta) + 2) + |c(\theta + d)|.$$

The last term in (1.4) can be bounded in terms of the partial quotient $a_0$ of $\theta$, since

$$|\theta + d| \leq |c(a_0| + 1) + |d| \leq |ca_0| + |c| + |d|.$$

Theorem 1.2 gives no bound for the partial quotient $a_0^* := \lfloor \frac{a\theta + b}{c\theta + d} \rfloor$ of $\frac{a\theta + b}{c\theta + d}$. 

Chowla [3] proved in 1931 that $K(\frac{a}{d} \theta) < 2ad(K(\theta) + 1)^3$, a result rather weaker than Theorem 1.2.

We obtain Theorem 1.1 and Theorem 1.2 from stronger bounds that relate Diophantine approximation constants of $\theta$ and $\frac{a\theta + b}{c\theta + d}$, which appear below as Theorem 3.2 and Theorem 4.1, respectively. Theorem 3.2 is a simple consequence of a result of Cusick and Mendès France [5] concerning the Lagrange constant of $\theta$ (defined in Section 2).

The continued fraction of $\frac{a\theta + b}{c\theta + d}$ can be directly computed from that for $\theta$, as was observed in 1894 by Hurwitz [9], who gave an explicit formula for the continued fraction of $2\theta$ in terms of that of $\theta$. In 1912 Châtelet [2] gave an algorithm for computing the continued fraction of $\frac{a\theta + b}{c\theta + d}$ from that of $\theta$, and in 1947 Hall [7] also gave a method. Let $\mathcal{M}(n, \mathbb{Z})$ denote the set of $n \times n$ integer matrices. Raney [15] gave for each $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}(2, \mathbb{Z})$ with $\det(M) \neq 0$ an explicit finite automaton to compute the additive continued fraction of $\frac{a\theta + b}{c\theta + d}$ from the additive continued fraction of $\theta$.

In connection with the bound of Theorem 1.1, Davenport [6] observed that for each irrational $\theta$ and prime $p$ there exists some integer $0 \leq a < p$ such that $\theta' = \theta + \frac{a}{p}$ has infinitely many partial quotients $a_n(\theta') \geq p$. Mendès France [13] then showed that there exists some $\theta' = \theta + \frac{a}{p}$ having the property that a positive proportion of the partial quotients of $\theta'$ have $a_n(\theta') \geq p$.

Some other related results appear in Mendès France [11,12]. Basic facts on continued fractions appear in [1,8,10,18].

2. Badly Approximable Numbers

Recall that the continued fraction expansion of an irrational real number $\theta = [a_0, a_1, \ldots]$ is determined by

$$\theta = a_0 + \theta_0 , \quad 0 < \theta_0 < 1 ,$$

and for $n \geq 1$ by the recursion

$$\frac{1}{\theta_{n-1}} = a_n + \theta_n , \quad 0 < \theta_n < 1 .$$

The $n$-th complete quotient $a_n$ of $\theta$ is

$$a_n := \frac{1}{\theta_n} = [a_n, a_{n+1}, a_{n+2}, \ldots] .$$

The $n$-th convergent $\frac{p_n}{q_n}$ of $\theta$ is

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] ,$$
whose denominator is given by the recursion $q_{-1} = 0$, $q_0 = 1$, and $q_{n+1} = a_{n+1}q_n + q_{n-1}$. It is well known (see [8, §10.7]) that

$$\frac{1}{q_n|\theta - p_n|} = 1 + \frac{1}{q_n\alpha_{n+1} + q_{n-1}}.$$  

(2.1)

Since $a_{n+1} \leq \alpha_{n+1} < a_{n+1} + 1$ and $q_{n-1} \leq q_n$, this implies that

$$\frac{1}{a_{n+1} + 2} < q_n|\theta - p_n| \leq \frac{1}{a_{n+1}},$$  

(2.2)

for $n \geq 0$.

We consider the following Diophantine approximation constants. For an irrational number $\theta$ define its type $L(\theta)$ by

$$L(\theta) = \sup_{q \geq 1} (q||q\theta||)^{-1},$$

and define the homogeneous Diophantine approximation constant or Lagrange constant $L_\infty(\theta)$ of $\theta$ by

$$L_\infty(\theta) = \limsup_{q \geq 1} (q||q\theta||)^{-1}.$$  

(2.3)

We use the convention that if $\theta$ is rational, then $L(\theta) = L_\infty(\theta) = +\infty$. (N.B.: some authors study the reciprocal of what we have called the Lagrange constant.)

The best approximation properties of continued fraction convergents give

$$L(\theta) = \sup_{n \geq 0} (q_n||q_n\theta||)^{-1}$$

and

$$L_\infty(\theta) = \limsup_{n \geq 0} (q_n||q_n\theta||)^{-1}.$$  

(2.4)

The set of values taken by $L_\infty(\theta)$ over all $\theta$ is called the Lagrange spectrum [4]. It is well known that $L_\infty(\theta) \geq \sqrt{5}$ for all $\theta$. If $\theta = [a_0, a_1, a_2, \ldots]$, then another formula for $L_\infty(\theta)$ is

$$L_\infty(\theta) = \limsup_{j \to \infty} ([a_j, a_{j+1}, \ldots] + [0, a_{j-1}, a_{j-2}, \ldots, a_1]);$$  

(2.5)

see [4, p. 1].

There are simple relations between these quantities and the partial quotient bounds $K(\theta)$ and $K_\infty(\theta)$, cf. [16, pp. 22–23].
Lemma 2.1. For any irrational $\theta$ with bounded partial quotients, we have

\[ K(\theta) \leq L(\theta) \leq K(\theta) + 2 \, . \tag{2.6} \]

Proof. This is immediate from (2.2) and (2.3). \qed

Lemma 2.2. For any irrational $\theta$ with bounded partial quotients

\[ K_{\infty}(\theta) \leq L_{\infty}(\theta) \leq K_{\infty}(\theta) + 2 \, . \tag{2.7} \]

Proof. This is immediate from (2.2) and (2.4). \qed

Although we do not use it in the sequel, we note that both inequalities in (2.7) can be slightly improved. Since $q_n \leq (a_n + 1)q_{n-1}$, (2.1) yields

\[ q_n \| q_n \theta \| \leq \frac{1}{\alpha_{n+1} + \frac{q_{n-1}}{q_n}} \leq \frac{1}{a_{n+1} + 1/(a_n + 1)} . \]

Since $a_n \leq K_{\infty}(\theta)$ from some point on, this and (2.4) yield

\[ L_{\infty}(\theta) \geq K_{\infty}(\theta) + \frac{1}{K_{\infty}(\theta) + 1} . \tag{2.8} \]

Next, from (2.1) we have

\[ q_n \| q_n \theta \| = \frac{q_n}{\alpha_{n+1} q_n + q_{n-1}} \]

\[ = \frac{1}{a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n}} . \]

Hence

\[ (q_n \| q_n \theta \|)^{-1} = a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n} . \]

Let $K = K_{\infty}(\theta)$. Then for all $n$ sufficiently large we have

\[ \alpha_{n+2} \geq 1 + \frac{1}{K + 1} = \frac{K + 2}{K + 1}, \]

so

\[ (q_n \| q_n \theta \|)^{-1} \leq K + \frac{K + 1}{K + 2} + 1 \]

\[ = K + 2 - \frac{1}{K + 2} . \]

We conclude that

\[ L_{\infty}(\theta) \leq K_{\infty}(\theta) + 2 - \frac{1}{K_{\infty}(\theta) + 2} . \tag{2.9} \]
3. 

LAGRANGE CONSTANTS AND PROOF OF THEOREM 1.1.

An integer matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(M) \neq 0$, acts as a linear fractional transformation on a real number $\theta$ by

\begin{equation}
M(\theta) := \frac{a\theta + b}{c\theta + d}.
\end{equation}

Note that $M_1(M_2(\theta)) = M_1M_2(\theta)$.

**Lemma 3.1.** If $M$ is an integer matrix with $\det(M) = \pm 1$, then the Lagrange constants of $\theta$ and $M(\theta)$ are related by

\[ L_\infty(M(\theta)) = L_\infty(\theta). \]

*Proof.* This is well-known, cf. [14] and [5, Lemma 1], and is deducible from (2.5). □

The main result of Cusick and Mendès France [5] yields:

**Theorem 3.2.** For any integer $m \geq 1$, let

\[ G_m = \{ M \in \mathcal{M}(2, \mathbb{Z}) : |\det(M)| = m \}. \]

Then for any irrational number $\theta$,

\begin{equation}
\sup_{M \in G_m} (L_\infty(M(\theta))) = m L_\infty(\theta).
\end{equation}

and

\begin{equation}
\inf_{M \in G_m} (L_\infty(M(\theta))) \geq \frac{1}{m} L_\infty(\theta)
\end{equation}

*Proof.* Theorem 1 of [5] states that

\begin{equation}
\max_{\begin{subarray}{l} a,b,d \\ ad = m \\ 0 \leq b < d \end{subarray}} \left( L_\infty \left( a\theta + b \over d \right) \right) = m L_\infty(\theta).
\end{equation}

Let $GL(2, \mathbb{Z})$ denote the group of $2 \times 2$ integer matrices with determinant $\pm 1$. We need only observe that for any $M$ in $G_m$ there exists some $\tilde{M} \in GL(2, \mathbb{Z})$ such that $\tilde{M}M = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ with $d'd' = m$ and $0 \leq b' < d'$. For if so, and $\psi = \frac{a\theta + b}{c\theta + d}$, then Lemma 3.1 gives

\[ L_\infty(\psi) = L_\infty(\tilde{M}(\psi)) = L_\infty(\tilde{M}M(\theta)) = L_\infty \left( \frac{a'\theta + b'}{d'} \right), \]
whence (3.4) implies (3.2). To construct \( \tilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), we must have
\[
Ca + Dc = 0.
\]
Take \( C = \frac{\text{gcd}(a,c)}{a} \) and \( D = -\frac{\text{gcd}(a,c)}{a} \). Then \( \text{gcd}(C,D) = 1 \), so we may complete this row to a matrix \( \tilde{M} \in GL(2,\mathbb{Z}) \). Multiplying this by a suitable matrix
\[
\begin{bmatrix}
\pm 1 & c \\
0 & \pm 1
\end{bmatrix}
\]
yields the desired \( \tilde{M} \).

The lower bound (3.3) follows from the upper bound (3.2). We use the adjoint matrix
\[
M' = \text{adj}(M) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix},
\]
which has \( M'M = \det(M)I = mI \) and \( \det(M') = \det(M) \). If \( \theta' = M(\theta) \), then
\[
M'(\theta') = M'(M(\theta)) = M'M(\theta) = \theta.
\]
We prove by contradiction. Suppose (3.3) were false, so that for some \( M \in G_m \) and some \( \theta \) we have
\[
L_{\infty}(M(\theta)) < \frac{1}{m}L_{\infty}(\theta).
\]
This states that
\[
mL_{\infty}(\theta') < L_{\infty}(M'(\theta')),
\]
which contradicts (3.2) for \( \theta' \), since \( \det(M') = \det(M) = m \). \( \square \)

**Remark.** The lower bound (3.3) holds with equality for some values of \( \theta \) and not for other values. If for given \( \theta \) we choose an \( M \in G_m \) which gives equality in (3.2), so that \( L_{\infty}(M(\theta)) = mL_{\infty}(\theta) \), then equality holds in (3.3) for \( \theta' = \text{adj}(M)(\theta) \). However, if \( L_{\infty}(\theta) = \sqrt{5} \), as occurs for \( \theta = \frac{1+i\sqrt{5}}{2} \), then \( L_{\infty}(M(\theta)) \geq L_{\infty}(\theta) \) for all \( M \); hence (3.3) does not hold with equality when \( m \geq 2 \).

**Proof of Theorem 1.1.** Theorem 3.2 gives \( L_{\infty}(M(\theta)) \leq \det(M)L_{\infty}(\theta) \). Now apply Lemma 2.2 twice to get
\[
K_{\infty}(M(\theta)) \leq L_{\infty}(M(\theta))
\]
\[
\leq |\det(M)|L_{\infty}(\theta)
\]
\[
\leq |\det(M)|(K_{\infty}(\theta) + 2).
\]
(3.5)

To obtain the lower bound, we use the adjoint \( M' = \text{adj}(M) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \), and apply (3.5) with \( M' \) and \( \theta' = M(\theta) \) to obtain
\[
K_{\infty}(\theta) = K_{\infty}(M'(M(\theta))) \leq |\det(M')|(K_{\infty}(M(\theta))) + 2.
\]
Since \( |\det(M)| = |\det(M')| \), this yields
\[
K_{\infty}(M(\theta)) \geq \frac{1}{|\det(M)|}K_{\infty}(\theta) - 2.
\]
\( \square \)
4. Numbers of Bounded Type and Proof of Theorem 1.2

Recall that the type $L(\theta)$ of $\theta$ is the smallest real number such that $q \|q\theta\| \geq \frac{1}{L(\theta)}$ for all $q \geq 1$.

**Theorem 4.1.** Let $\theta$ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then

\begin{equation}
L\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det(M)|L(\theta) + |e(c\theta + d)|.
\end{equation}

**Proof.** Set $\psi = \frac{a\theta + b}{c\theta + d}$. Suppose first that $c = 0$ so that $|\det(M)| = |ad| > 0$. Then $L(\psi) \geq \frac{1}{x}$, where

\begin{equation}
x := q\|q\psi\| = q||q\left(\frac{a\theta + b}{d}\right)|| = q\left|q\left(\frac{a\theta + b}{d}\right) - p\right|.
\end{equation}

We have

\begin{equation}
|ad|x = |aq| |aq\theta + (bp - dp)|
\end{equation}

\begin{equation}
\geq |aq| |aq\theta| \geq \frac{1}{L(\theta)}.
\end{equation}

For any $\epsilon > 0$ we may choose $q$ in (4.2) so that $\frac{1}{x} \geq L(\psi) - \epsilon$. Then

\begin{equation}
|\det(M)|L(\theta) = |ad||L(\theta)| \geq \frac{1}{x} \geq L(\psi) - \epsilon.
\end{equation}

Letting $\epsilon \to 0$ yields (4.1) when $c = 0$.

Suppose now that $c \neq 0$. Again $L(\psi) \geq \frac{1}{x}$, where

\begin{equation}
x := q\|q\psi\| = q\left|q\left(\frac{a\theta + b}{c\theta + d}\right) - p\right|.
\end{equation}

We have

\begin{equation}
|c\theta + d|x = q|(qa - pc)\theta - (pd - qb)|,
\end{equation}

so that

\begin{equation}
|c\theta + d| \left|\frac{qa - pc}{q}\right| x = |qa - pc| |(qa - pc)\theta - (pd - qb)|
\end{equation}

\begin{equation}
\geq |qa - pc| |(qa - pc)\theta|.
\end{equation}
We first treat the case $qa - pc = 0$. Now
\[
\begin{bmatrix}
a & -c \\
-b & d
\end{bmatrix}
\begin{bmatrix}
q \\
p
\end{bmatrix}
= \begin{bmatrix}
qa - pc \\
pd - qb
\end{bmatrix} \neq \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]
since $\det \begin{bmatrix}
a & -c \\
-b & d
\end{bmatrix} = \det(M) \neq 0$. Thus if $qa - pc = 0$ then $|pd - qb| \geq 1$, hence (4.5) gives
\[
|c\theta + d|x = q|pd - qb| \geq 1.
\]
This yields
\[
L(\psi) - \epsilon \leq \frac{1}{x} \leq |c\theta + d| \leq |c(c\theta + d)|.
\]
since $c \neq 0$. If we can let $\epsilon \to 0$ or else $\epsilon = 0$ for values with $qa - pc = 0$, then (4.1) follows from (4.8).
Now suppose that $c \neq 0$ and $qa - pc \neq 0$. From the definition of $L(\theta)$ we have
\[
|qa - pc| \|qa - pc\theta\| \geq \frac{1}{L(\theta)}.
\]
Given $\epsilon > 0$, we may choose $q$ so that $\frac{1}{x} \geq L(\psi) - \epsilon$, and we obtain from (4.6) and (4.9) that
\[
|c\theta + d| \left| \frac{qa - pc}{q} \right| L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon.
\]
Using the bound
\[
\left|q \left( \frac{a}{c} \right) - p \right| \leq \left|q \left( \frac{a\theta + b}{c\theta + d} \right) - q \left( \frac{a}{c} \right) \right| + \left|q \left( \frac{a\theta + b}{c\theta + d} \right) - p \right|,
\]
and noting that first term on the right side is equal to $q|\det(M)|\frac{1}{|c(c\theta + d)|}$ while the second term is $|q\psi| = \frac{\epsilon}{q}$, we obtain
\[
\left| \frac{qa - pc}{c} \right| \leq q|\det(M)|\frac{1}{|c(c\theta + d)|} + \frac{x}{q}.
\]
Multiplying this by $\frac{c}{q}$ and applying it to the left side of (4.10) yields
\[
L \left( \frac{a\theta + b}{c\theta + d} \right) - \epsilon \leq |\det(M)||L(\theta)| + |c(c\theta + d)| \frac{xL(\theta)}{q^2}.
\]
We now claim that
\[(4.12) \quad L \left( \frac{a \theta + b}{c \theta + d} \right) \leq |\det(M)|L(\theta) + |c(\theta + d)|.\]
always holds. This is immediate if \(L(\theta) \geq L(\psi)\). Now suppose \(L(\theta) < L(\psi)\). Suppose we can let \(\epsilon \to 0\) or else \(\epsilon = 0\) through values with \(qa - pc \neq 0\). Now the ratio \(\frac{L(\theta)}{L(\psi)}\) becomes \(\leq 1\) in the limit, and since \(q \geq 1\), (4.12) follows from (4.11). This completes the case \(c \neq 0\). \(\Box\).

**Proof of Theorem 1.2.** Applying Theorem 4.1 and Lemma 2.1 gives
\[
K \left( \frac{a \theta + b}{c \theta + d} \right) \leq L \left( \frac{a \theta + b}{c \theta + d} \right) \leq |\det(M)|L(\theta) + |c(\theta + d)| \\
\leq |\det(M)|(K(\theta) + 2) + |c(\theta + d)| ,
\]
which is the desired bound. \(\Box\)

**Remarks.** (1.) The proof method of Theorem 4.1 can also be used to directly prove the bounds
\[(4.14) \quad \frac{1}{|\det(M)|}L_{\infty}(\theta) \leq L_{\infty}(M(\theta)) \leq |\det(M)|L_{\infty}(\theta),\]
of Theorem 3.2, from which Theorem 4.1 can be easily deduced. The lower bound in (4.14) follows from the upper bound as in the proof of Theorem 3.2. We sketch a proof of the upper bound in (4.14) for the case \(\psi = \frac{a \theta + b}{c \theta + d}\) with \(c \neq 0\). For any \(\epsilon^* > 0\) and all sufficiently large \(q^* \geq q^*(\epsilon^*)\), we have
\[(4.15) \quad q^*q^* \geq \frac{1}{L_{\infty}(\theta) + \epsilon^*} .\]
We choose \(q = q_n(\psi)\) for sufficiently large \(n\), and note that
\[q^* = |q_n(\psi)a - p_n(\psi)c| \to \infty\]
as \(n \to \infty\), since \(\psi\) is irrational. We can then replace (4.9) by (4.15), and then deduce (4.11) with \(L(\theta)\) replaced by \(L_{\infty}(\theta) + \epsilon^*\). Letting \(q \to \infty\), \(\epsilon \to 0\) and \(\epsilon^* \to 0\) in that order yields the upper bound in (4.14).

(2.) For a given matrix \(M\) consider the set of attainable ratios
\[(4.16) \quad \mathcal{V}(M) := \left\{ \frac{L_{\infty}(M\theta)}{L_{\infty}(\theta)} : \theta \text{ has bounded partial quotients} \right\} .\]
By Lemma 3.1 the set \(\mathcal{V}(M)\) depends only on its \(SL(2,\mathbb{Z})\)-double coset
\[\mathcal{M} = \{ N_1 M N_2 : N_1, N_2 \in SL(2,\mathbb{Z}) \} .\]
Theorem 3.2 shows that

\[(4.17) \quad \mathcal{V}(M) \subseteq \left[ \frac{1}{\det(M)}, |\det(M)| \right].\]

It is an interesting open problem to determine the set \(\mathcal{V}(M)\). Both \(\det(M)\) and \(\frac{1}{\det(M)}\) lie in \(\mathcal{V}(M)\), as follows from Theorem 3.2 and the remark following it.

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